HYPERBOLIC POLYNOMIAL DIFFEOMORPHISMS OF $\mathbb{C}^2$

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ABSTRACT. In this paper we develop a general framework for verifying hyperbolicity of holomorphic dynamical systems in $\mathbb{C}^2$. This framework in particular enables us to construct the first example of a hyperbolic (cubic) Hénon map of $\mathbb{C}^2$ which cannot be topologically conjugate on its Julia set to a small perturbation of any expanding polynomial in one variable. Key ideas in its proof are: the Poincaré box which is a building block to apply our criterion for hyperbolicity, an operation called fusion to produce essentially two–dimensional dynamics from two polynomials in one variable, and rigorous computation techniques by using interval arithmetic. Some applications to the analysis of parameter loci for the Hénon family in $\mathbb{R}^2$ are also given.

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Date: *Version July 7, 2005.* 2000 MSC: Primary 37F15, Secondary 30F45, 37B10, 37F20, 65G30.

1. Introduction and Main Results

Hyperbolic polynomial diffeomorphisms of $\mathbb{C}^2$ have been extensively studied, e.g., from the viewpoint of Axiom A theory by [BS$\mathbb{C}$1] and the combinatorial point of view by [BS$\mathbb{C}$7]. Here, a polynomial diffeomorphism f of $\mathbb{C}^2$ is said to be *hyperbolic* if its Julia set is a hyperbolic set for f (see Subsection 2.1 for the definition of the *Julia set* J of f). In [HO2, FS, BS$\mathbb{C}$3] it has been shown that a sufficiently small perturbation of any expanding polynomial in one variable inside the *generalized Hénon family*:

$$f_{p,b} : (x, y) \longmapsto (p(x) - by, x)$$

is hyperbolic, i.e. for any expanding polynomial $p(x)$ there exists a small $b_* > 0$ such that $\{0 < |b| < b_*\}$ is contained in $\mathcal{H}_p \equiv \{b \in \mathbb{C}^\times : f_{p,b} \text{ is hyperbolic}\}$. However, this is so far the only known example of a polynomial diffeomorphism of $\mathbb{C}^2$ which is rigorously shown to be hyperbolic. Moreover, the dynamics of such $f_{p,b}$ is conjugate to the projective limit of p on its Julia set [HO2]. It is thus still not known if there exists a hyperbolic polynomial diffeomorphism with essentially two–dimensional dynamics.

The purpose of this article is to develop a general framework for verifying hyperbolicity of holomorphic dynamical systems in $\mathbb{C}^2$. This framework in particular enables us to construct the first example of a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$ which can not be obtained in the way described above. Consider a cubic complex Hénon map:

$$f_{a,b} : (x, y) \longmapsto (-x^3 + a - by, x)$$

with $(a, b) = (-1.35, 0.2)$.

Theorem A. *The cubic complex Hénon map above is hyperbolic but is not topologically conjugate on J to a small perturbation of any expanding polynomial in one variable.*

The method for proving the hyperbolicity of $f_{a,b}$ in Theorem A also enables us to analyze topology and combinatorics of the Julia set. In particular, in Theorem 4.23 it is shown that the Julia set is obtained by gluing two solenoids and uncountably many topological circles, and by adding Cantor sets and finite points to them. Moreover, these pieces are identified only inside the stable manifold of the saddle fixed point in the third quadrant of $\mathbb{R}^2$ and this identification is at most two to one. We also obtain a necessary condition for the pieces to be glued in terms of symbolic dynamics.

The Julia set of the map in Theorem A is not connected, thus it would be interesting to find a connected example to apply results in [BS$\mathbb{C}$7]. Oliva [Ol] found some examples of complex Hénon maps whose Julia sets seem hyperbolic and connected, and can not be obtained by small perturbation of expanding one–dimensional maps.

The proof of Theorem A relies on the combination of some analytic tools from complex analysis, a combinatorial idea called the *fusion*, and rigorous numerical computation by using interval arithmetic. The analytic and the combinatorial parts behind the proof allow us to show the next theorem without computer assistance. Given an expanding polynomial map $p(x)$, let $\mathcal{H}_p^0$ be the connected component of $\mathcal{H}_p$ containing the small punctured disk $\{0 < |b| < b_*\}$ in the small perturbation result above.

Theorem B. *For any $0 < \delta < 1/2$ there exists an expanding polynomial $p_0(x)$ so that $\{\delta < |b| < 1 - \delta\} \subset \mathcal{H}_{p_0} \setminus \mathcal{H}^0_{p_0}$ (see Figure 1 below).*

We also see in Theorem 6.1 that if f_{p_0,b_0} with $\delta < |b_0| < 1 - \delta$ as in Theorem B is conjugate to a small perturbation of some expanding polynomial $q(x)$, then q should be conjugate to p_0. Thus, once f_{p_0,b_0} is shown not to be conjugate to a small perturbation of p_0, it follows that f_{p_0,b_0} is the first example of a polynomial diffeomorphism of $\mathbb{C}^2$ with essentially two–dimensional dynamics which is proved to be hyperbolic *without* computer assistance.

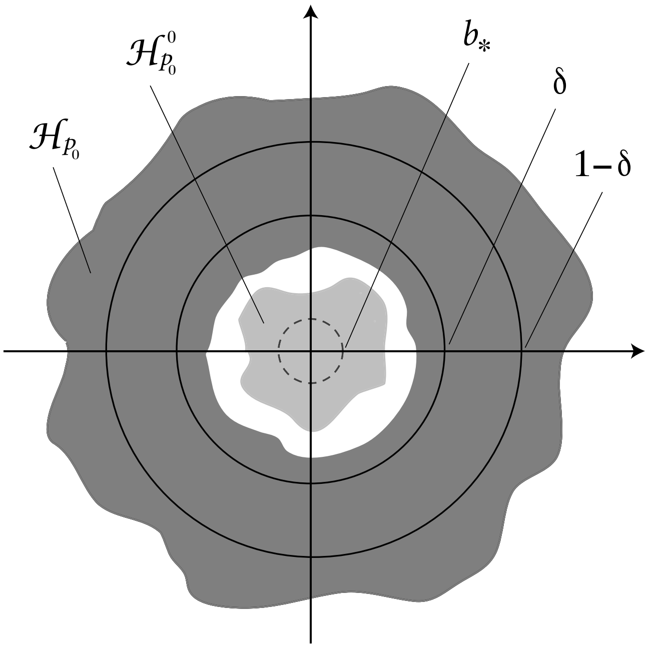

Figure 1. The b–plane through the polynomial $p_0(x)$.

To prove Theorems A and B, we first establish several topological criteria which imply hyperbolicity of a polynomial diffeomorphism f. Let A_x and A_y be bounded domains in $\mathbb{C}$. Then, two kinds of cone fields called the *horizontal/vertical Poincaré cone fields* on $\mathcal{A} = A_x \times A_y \subset \mathbb{C}^2$ can be defined in terms of the "slope" with respect to the Poincaré metrics in A_x and A_y. In our central claim Theorem 2.14 it is shown that two topological conditions for $f : \mathcal{A} \cap f^{-1}(\mathcal{A}) \to \mathcal{A}$ called the *crossed mapping condition* and the *no–tangency condition* imply the expansion/contraction of the horizontal/vertical Poincaré cone fields. We will also see in Corollaries 2.17 and 2.18 that these two conditions can be restated by more checkable ones called the *boundary compatibility condition* and the *off–criticality condition* respectively. The product set $\mathcal{A} = A_x \times A_y$ equipped with the horizontal and vertical Poincaré cone fields will be a building block of our construction throughout this article, and is called a *Poincaré box*.

The combinatorial idea to construct new types of hyperbolic generalized Hénon maps as in Theorems A and B is to make a *fusion* of two different polynomials in one variable. Let us put $\Delta_x(x_0; r) = \{x \in \mathbb{C} : |x - x_0| < r\}$ and take some $R > 0$. For $i = 1, 2$, we choose $y_i \in \Delta_y(0; R)$ with $y_1 \neq y_2$. Take a small $\varepsilon > 0$ so that the bidisks $\mathcal{A}_i = \Delta_x(0; R) \times \Delta_y(y_i; \varepsilon)$ become disjoint. Since $\varepsilon > 0$ is small, we see that $f_{p,b}|_{\mathcal{A}_i}(x, y) \approx (p_i(x), x)$, where $p_i(x) = p(x) - by_i$. In this way, the generalized Hénon map $f_{p,b}$ restricted to $\mathcal{A}_1 \cup \mathcal{A}_2$ can be viewed as a fusion of two polynomials $p_1(x)$ and $p_2(x)$ in one variable. Notice that we are not assuming $|b|$ being small, so the constant $p_1(x) - p_2(x) = b(y_2 - y_1)$ is not necessarily close to zero and thus p_1 and p_2 may be combinatorially different. Now, our task is to find a polynomial $p(x)$, a constant $b \in \mathbb{C}$ and Poincaré boxes $\mathcal{A}_i$ so that $f_{p,b} : \mathcal{A}_1 \cup \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ [1] satisfies the hyperbolicity criterion. This can be done since $f_{p,b}|_{\mathcal{A}_i}$ is close to $(p_i(x), x)$ and $p_i(x)$ is chosen to be expanding. Moreover, since $p_1(x)$ and $p_2(x)$ are combinatorially different, we are able to show that the map constructed in Theorem A is not conjugate to a small perturbation of any expanding polynomial in one variable, and that any continuous one–parameter family $\{f_{p_0,b_\mu}\}_{\mu\in[0,1]}$ in the b–plane connecting f_{p_0,b_0} with $\delta < |b_0| < 1 - \delta$ constructed in Theorem B and a small perturbation f_{p_0,b_1} of $p_0(x)$ must experience bifurcation at some $\mu_0 \in (0, 1)$. To this end, we decompose the Julia sets in Theorems A and B into the connected components by using symbolic dynamics and analyze their topology.

Another by–product of Theorem 2.14 is explicit lower estimates on the size of $\mathcal{H}^0_p$ for various polynomials p. As an illustration we give the following result when p is a quadratic polynomial $p(x) = x^2 + c$, i.e. we consider the (quadratic) *Hénon family*:

$$f_{c,b} : (x, y) \longmapsto (x^2 + c - by, x),$$

where $b \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ are complex parameters.

Theorem C. *If (c, b) satisfies either*

(i) $|c| > 2(1 + |b|)^2$ *(a hyperbolic horseshoe case),*
(ii) $c = 0$ *and* $|b| < (\sqrt{2} - 1)/2$ *(an attractive fixed point case) or*
(iii) $c = -1$ *and* $|b| < 0.02$ *(an attractive cycle of period two case),*

then the complex Hénon map $f_{c,b}$ is hyperbolic on J.

We note that Hubbard and Oberste-Vorth [Ob] has obtained a weaker estimate to (i) in Theorem C and Ueda [MNTU] has obtained the same bound as in (i). Confer also [Hr] where some particular parameters slightly outside our estimates (i) and (ii) are shown to be hyperbolic, but her method could not verify hyperbolicity for the case (iii). Topology of the Julia sets of the hyperbolic complex Hénon maps in Theorem C will be studied through the framework developed in this article in terms of the projective limits of $p(x) = x^2 + c$ in a separated paper [IS] (see also the end of the proof of Theorem 6.1, where a part of its idea is shown for complex one–dimensional systems).

[1]When we write $f : X \to Y$, this does *not* necessarily mean $f(X) \subset Y$. Rather than that, we are interested in relative position of $f(X)$ with respect to Y (see the conditions presented in Subsection 2.3 for more details).

When the parameters b and c are real, the Hénon map $f_{c,b}$ becomes a polynomial automorphism of $\mathbb{R}^2$. It is known [FM] that $0 \leq h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2}) \leq \log 2$. Thus, it would be interesting to investigate the shape of the *maximal entropy locus*:

$$\mathcal{M} \equiv \big\{(c,b) \in \mathbb{R} \times \mathbb{R}^\times : h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2}) = \log 2\big\}$$

and the *hyperbolic horseshoe locus*:

$$\mathcal{H} \equiv \big\{(c,b) \in \mathbb{R} \times \mathbb{R}^\times : f_{c,b}|_{\mathbb{R}^2} \text{ is a hyperbolic horseshoe}\big\}.$$

Note that $\overline{\mathcal{H}} \subset \mathcal{M}$, $\mathcal{M}$ is closed and $\mathcal{H}$ is open (see, e.g. [M]). Our method in this article also provides a way to compute these two loci quite accurately.

Theorem D. *Below in the list, for each b chosen, (i) c_1 is a value of c such that the Hénon map $f_{c,b}$ is rigorously shown to be a hyperbolic horseshoe on $\mathbb{R}^2$ for all $c \leq c_1$, (ii) c_2 is a value of c such that $h_{\mathrm{top}}(f_{c,b}|_{\mathbf{R}^2}) < \log 2$ is rigorously shown for all $c \geq c_2$.*

b	c_1	c_{t}	c_2
1.000	−5.900	−5.700	−5.699
0.900	−5.320	−5.151	−5.149
0.800	−4.800	−4.644	−4.642
0.700	−4.310	−4.179	−4.176
0.600	−3.860	−3.755	−3.752
0.500	−3.450	−3.372	−3.368
0.400	−3.090	−3.028	−3.025
0.300	−2.760	−2.722	−2.716
0.200	−2.480	−2.451	NA
0.100	−2.240	−2.212	NA
0.000	———	−2.000	———
−0.100	−2.280	−2.244	NA
−0.200	−2.570	−2.525	NA
−0.300	−2.900	−2.845	NA
−0.400	−3.280	−3.204	−3.171
−0.500	−3.700	−3.603	−3.567
−0.600	−4.160	−4.042	−4.002
−0.700	−4.640	−4.521	−4.457
−0.800	−5.170	−5.040	−4.960
−0.900	−5.740	−5.599	−5.488
−1.000	−6.380	−6.199	−6.049

The value c_{t} in the list is an *approximate* value of c so that $f_{c,b}$ has the first tangency, thus (c_{t}, b) seems to approximate the boundary of the two loci. It is obtained by using the programs `PlanarIterations` and `FractalAsm` [DC]. Due to some technical difficulties, we are not able to give bounds for c_2 when $-0.4 < b < 0.3$ (see Remark 5.9 for details). Confer also the bifurcation diagram of Mira et al [EM], where the boundary of these loci has been implicitly figured out.

We hope that Theorem D will be an indispensable step to prove the following

Conjecture 1. *There exists a piecewise real analytic, piecewise monotone function* $c_{\mathrm{crit}} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *with two monotone pieces from the* b*-axis to the* c*-axis of the parameter space for the Hénon family* $f_{c,b}$ *on* $\mathbb{R}^2$ *with the following properties:*

(i) $c < c_{\mathrm{crit}}(b)$ *iff* $f_{c,b}|_{\mathbb{R}^2}$ *is a hyperbolic horseshoe.*

(ii) $c > c_{\mathrm{crit}}(b)$ *iff* $h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2}) < \log 2$.

(iii) *When* $b > 0$*, we have* $c = c_{\mathrm{crit}}(b)$ *iff* $f_{c,b}|_K$ *is topologically conjugate to the factor* $\sigma/\!\sim$ *of the shift map* σ *on* $\{0,1\}^{\mathbb{Z}}/\!\sim$*, where* $\sim$ *is given by*

$$\sigma^n(\cdots 111100.01111\cdots) \sim \sigma^n(\cdots 111101.01111\cdots)$$

for $n \in \mathbb{Z}$*. Moreover,* $f_{c,b}|_{\mathbb{R}^2}$ *has exactly one orbit of homoclinic tangencies between the stable and unstable manifolds of the saddle fixed point in the first quadrant.*

When $b < 0$*, we have* $c = c_{\mathrm{crit}}(b)$ *iff* $f_{c,b}|_K$ *is topologically conjugate to the factor* $\sigma/\!\sim$ *of the shift map* σ *on* $\{0,1\}^{\mathbb{Z}}/\!\sim$*, where* $\sim$ *is given by*

$$\sigma^n(\cdots 000010.01111\cdots) \sim \sigma^n(\cdots 000011.01111\cdots)$$

for $n \in \mathbb{Z}$*. Moreover,* $f_{c,b}|_{\mathbb{R}^2}$ *has exactly one orbit of heteroclinic tangencies between the stable manifold of the saddle fixed point in the first quadrant and the unstable manifold of the saddle fixed point in the third quadrant.*

This conjecture in particular implies that $\mathcal{H}$ and $\mathcal{M}$ are connected (by adding the lines $\{(c,b) \in \mathbb{R}\times\mathbb{R}^\times : b = 0, c < -2\}$ and $\{(c,b) \in \mathbb{R}\times\mathbb{R}^\times : b = 0, c \leq -2\}$ to them respectively) and simply connected, $\overline{\mathcal{H}} = \mathcal{M}$, and $\partial\mathcal{H} = \partial\mathcal{M}$.

Notice that an affirmative answer to this conjecture in the case $|b| < 0.08$ has been recently provided in the paper [BSℝ2] through the analysis of some complex one–dimensional dynamics (see also [CLR]). Theorem D of the current article can be seen as a sharpened version of Theorem 1.1 and Theorem 1.2 of [BSℝ2] which were important steps in their proof. Moreover, we have also already obtained a key claim for the cases $b = 1$, 0.7, 0.5, 0.3, etc which corresponds to the crossed mapping condition near the first tangency parameter c_{t} as in Proposition 2.2 and Corollary 2.3 of [BSℝ2], but with a completely different choice of bidisks from the ones in their paper.

A corresponding claim for a family of piecewise affine homeomorphisms of $\mathbb{R}^2$ called the *Lozi family*:

$$\mathcal{L}_{a,b} : (x,y) \longmapsto (1 - a|x| + by, x)$$

has been established in [I1], where the boundaries of the two loci are shown to be the graph of a piecewise algebraic function from the b-axis to the a-axis in the parameter space. It has been also proved that the topological entropy and the bifurcations are monotone near the boundary of the loci [I2]. It might be interesting to compare this monotonicity result to the anti–monotonicity theorem of Kan et al [KKY] which claims that one can find both infinitely many orbit–creation and orbit–annihilation parameter values in an arbitrary neighborhood of a non-degenerate homoclinic tangency of a one–parameter family of dissipative C^3-diffeomorphisms of the plane. In fact, contrary to the monotonicity of the Lozi family, such anti–monotonicity phenomena is shown to be inevitable in the Hénon family near the boundary of the loci when $|b| < 0.08$ [BS$\mathbb{R}$2].

The plan of this paper is as follows: in Section 2, we recall some basic facts and present a general framework for verifying hyperbolicity of biholomorphic dynamics in complex dimension two. The fundamental claim is Theorem 2.14, where the expansion of the horizontal Poincaré cone field is shown to be equivalent to some topological conditions. More checkable criteria are presented in Corollaries 2.17 and 2.18, and Theorem C is proved as a consequence of them. In Section 3, a detailed topological model of fusion is given. This model is realized as an actual generalized Hénon map and shown to be hyperbolic by constructing a polynomial in one variable whose Julia set has some special geometric properties. The next section is dedicated to the proof of Theorem A. For this, we have to treat the case where several Poincaré boxes have overlaps. A problem then is to define a new cone field on the overlaps which maintains its invariance and expansion/contraction. This section begins with a general treatment of this problem. Some techniques from interval arithmetic will be explained in Subsection 4.3. In the next subsection, we construct a topological model of the map in Theorem A inspired by the idea of fusion and verify its hyperbolicity with the help of interval arithmetic. To do this, 10 programs written in C++ with an interval arithmetic software called PROFIL (Programmer's Runtime Optimized Fast Interval Library) [P] are used. The discussions above combined with the idea of fusion give a proof of Theorem A. Section 5 consists of the proof of Theorem D. To get the estimates c_1 in Theorem D, we use our criteria for hyperbolicity together with rigorous computation and the notion of a *projective bidisk* which fits to the trellis formed by stable and unstable manifolds of $f_{c,b}$ in $\mathbb{R}^2$. An algorithm to verify if a given Hénon map on $\mathbb{R}^2$ has maximal entropy is also given by using some ideas from the pluripotential theory. In the last section, some conjectures and open problems related to the subject of this article are proposed.

Acknowledgment. The content presented in Section 2 of this paper is based on my joint work with John Smillie which was initiated while I was at Cornell University during the academic year 1999–2000. I would like to thank him for many stimulating discussion and inspiring suggestions throughout this work, and thank the Department of Mathematics in Cornell University for the hospitality. I am also grateful to Eric Bedford, John H. Hubbard and Tetsuo Ueda for their fruitful comments and encouragement. My colleague in Kyushu, Kaori Nagatou, prepared me a nice environment of interval arithmetic, whom I thank so much. Part of this manuscript is written during my participation to the year–long project "Complex Dynamics" in 2003 at the Research Institute for Mathematical Sciences in Kyoto University and to the trimester "Analyse Complexe et Applications" in 2004 at l'Institut Henri Poincaré in Paris. I appreciate the hospitality of the both Institutes and thank Mitsuhiro Shishikura and Nessim Sibony, the chief organizers of these projects.

Note. After the preparation of this article, the author received a preprint [BS] which contains a result similar to Theorem 4.23 of this article for a different kind of map from the one in Theoreom A but without proof.

2. Topological Criteria for the Hyperbolicity

In this section several criteria for verifying hyperbolicity of holomorphic dynamics in $\mathbb{C}^2$ are established. In Subsection 2.1 we collect some preliminary results which will be used later. Our hyperbolicity criteria are Theorem 2.14, Corollaries 2.17 and 2.18 in the next subsection. As an immediate consequence of them, Theorem C is obtained in Subsection 2.3.

2.1. **Preliminary results.** Let f be a polynomial diffeomorphism of $\mathbb{C}^2$. It is known by a result of Friedland and Milnor [FM] that f is conjugate to either (i) an affine map, (ii) an elementary map, or (iii) the composition of finitely many generalized complex Hénon maps. Since the affine maps and the elementary maps do not present dynamically interesting behavior, we will hereafter focus only on a map in the class (iii), i.e. a map of the form $f = f_{p_1,b_1} \circ \cdots \circ f_{p_k,b_k}$ throughout this article. The product $d \equiv \deg p_1 \cdots \deg p_k$ is called the *(algebraic) degree* of f. Note also that we have $b \equiv \det(Df) = \det(Df_{p_1,b_1}) \cdots \det(Df_{p_k,b_k}) = b_1 \cdots b_k$.

For a polynomial diffeomorphism f, let us define

$$K^{\pm} = K^{\pm}_f \equiv \bigl\{(x, y) \in \mathbb{C}^2 : \{f^{\pm n}(x, y)\}_{n>0} \text{ is bounded in } \mathbb{C}^2\bigr\},$$

i.e. K^+ (resp. K^-) is the set of points whose forward (resp. backward) orbits are bounded in $\mathbb{C}^2$. We also put $K \equiv K^+ \cap K^-$ and $J^{\pm} \equiv \partial K^{\pm}$. The *Julia set* of f is defined as $J = J_f \equiv J^+ \cap J^-$ (see [HO1]). Obviously these sets are invariant by f.

Hereafter, we will often consider two different spaces $\mathcal{A}^* \subset \mathbb{C}^2$ where $* = \mathfrak{D}$ or $\mathfrak{R}$, and consider a polynomial diffeomorphism $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ (again notice that this does not necessarily mean $f(\mathcal{A}^{\mathfrak{D}}) \subset \mathcal{A}^{\mathfrak{R}}$). Here, $\mathfrak{D}$ signifies the domain and $\mathfrak{R}$ signifies the range of f.

A subset of $T_p\mathbb{C}^2$ is called a *cone* if it can be expressed as the union of complex lines through the origin of $T_p\mathbb{C}^2$. Let $\{C^*_p\}_{p\in\mathcal{A}^*}$ ($* = \mathfrak{D}, \mathfrak{R}$) be two cone fields in $T_p\mathbb{C}^2$ over $\mathcal{A}^*$ and $\|\cdot\|_*$ be metrics in C^*_p.

Definition 2.1 (**Pair of Expanding/Contracting Cone Fields).** *We say that* $(\{C^{\mathfrak{D}}_p\}_{p\in\mathcal{A}^{\mathfrak{D}}}, \|\cdot\|_{\mathfrak{D}})$ *and* $(\{C^{\mathfrak{R}}_p\}_{p\in\mathcal{A}^{\mathfrak{R}}}, \|\cdot\|_{\mathfrak{R}})$ *form a pair of weakly expanding cone fields for* f *(or,* f *weakly expands the pair of cone fields) if there exists a constant* $\lambda \geq 1$ *so that*

$$Df(C^{\mathfrak{D}}_p) \subset C^{\mathfrak{R}}_{f(p)} \quad \textit{and} \quad \lambda\|v\|_{\mathfrak{D}} \leq \|Df(v)\|_{\mathfrak{R}}$$

hold for all $p \in \mathcal{A}^{\mathfrak{D}} \cap f^{-1}(\mathcal{A}^{\mathfrak{R}})$ *and all* $v \in C^{\mathfrak{D}}_p$*. When we can take* $\lambda > 1$ *uniformly with respect to* p *and* v*, we call the cone fields a pair of expanding cone fields for* f *(or,* f *expands the pair of cone fields). Similarly, a pair of (weakly) contracting cone fields for* f *is defined as a pair of (weakly) expanding cone fields for* f^{-1}*.*

In particular, if $\mathcal{A} \equiv \mathcal{A}^{\mathfrak{D}} = \mathcal{A}^{\mathfrak{R}}$, $\|\cdot\| \equiv \|\cdot\|_{\mathfrak{D}} = \|\cdot\|_{\mathfrak{R}}$ and $C^u_p \equiv C^{\mathfrak{D}}_p = C^{\mathfrak{R}}_p$ for all $p \in \mathcal{A} \cap f^{-1}(\mathcal{A})$ and the above condition holds, then we say $(\{C^u_p\}_{p\in\mathcal{A}}, \|\cdot\|)$ forms an *(weakly) expanding cone field* (or, f *(weakly) expands the cone field*). Similarly, the notion of *(weakly) contracting cone field* (or, f *(weakly) contracts the cone field*) can be defined.

The next claim tells that, to prove hyperbolicity, it is sufficient to construct some expanding/contracting cone fields.

Lemma 2.2. *If $f : \mathcal{A} \to \mathcal{A}$ has both non–empty expanding/contracting cone fields $\{C_p^{u/s}\}_{p\in\mathcal{A}}$, then f is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A})$.*

Proof. Let us put

$$E_p^u \equiv \bigcap_{n\geq 0} Df^n(C^u_{f^{-n}(p)}) \quad \text{and} \quad E_p^s \equiv \bigcap_{n\geq 0} Df^{-n}(C^s_{f^n(p)}).$$

Because $C_p^{u/s}$ is a non–empty cone, so is $E_p^{u/s}$, and thus it is the union of complex lines through the origin of $T_p\mathbb{C}^2$. By replacing f by f^{-1} if necessary, we may assume that $|b| \leq 1$. Let us put $M_p \equiv (1/\sqrt{|b|})(Df)_p$, and define $M_p^{(n)} \equiv M_{f^{n-1}(p)} \cdots M_p$ and $M_p^{(-n)} \equiv M^{-1}_{f^{-n}(p)} \cdots M^{-1}_{f^{-1}(p)}$ for $n \geq 1$. Then, $|\det M_p^{(n)}| = 1$. Because $|b| \leq 1$, $M_p^{-1} = \sqrt{|b|}(Df)_p^{-1}$ is contracting on E_p^u. Suppose that E_p^u contains two distinct complex lines. Then every vector $v \in T_p\mathbb{C}^2$ is expressed as a linear combination of two vectors in E_p^u. Thus, $\|M_p^{(-n)}v\|$ decreases exponentially to zero, which contradicts to the fact that $|\det M_p^{(-n)}| = 1$. So E_p^u is a vector space of dimension one over $\mathbb{C}$.

Again, suppose that E_p^s contains two distinct complex lines. Then, in the same way, we see that $\|(Df)_{f^n(p)} \cdots (Df)_p v\|$ decreases exponentially to zero for any $v \in T_p\mathbb{C}^2$, which contradicts to the existence of the expanding subspace E_p^u. So E_p^s forms a vector space of dimension one over $\mathbb{C}$.

Due to the contraction/expansion along $E_p^{s/u}$, we see that $E_p^s \cap E_p^u = \{0\}$. This means that $E_p^u \oplus E_p^s = T_p\mathbb{C}^2$. Thus we are done. Q.E.D.

On the hyperbolicity of the polynomial diffeomorphisms of $\mathbb{C}^2$, the following fact is known (see [BS$\mathbb{C}$1], Lemma 5.5 and Theorem 5.6).

Lemma 2.3. *f is hyperbolic on J iff so is on its nonwandering set iff so is on its chain recurrent set iff so is on K.*

Thanks to this fact, one may simply say that a polynomial diffeomorphism f is *hyperbolic* when one of the four sets in the above lemma is a hyperbolic set. In what follows, we thus prove hyperbolicity of some f on its Julia set J.

2.2. **Poincaré boxes.** Let A_x and A_y be bounded regions in $\mathbb{C}$. Define $\mathcal{A} = A_x \times A_y$, and let $\pi_x : \mathcal{A} \to A_x$ and $\pi_y : \mathcal{A} \to A_y$ be two projections. Below, we will define several types of cone fields on $\mathcal{A}$. The first one (to which we do not equip a metric) looks the most general cone field among those.

Definition 2.4 (Horizontal/Vertical Cone Fields). *A cone field on $\mathcal{A}$ is called a horizontal cone field if each cone contains the horizontal direction but not the vertical direction. A vertical cone field can be defined similarly.*

Next, a very specific cone field is defined in terms of Poincaré metrics. Let $|\cdot|_D$ be the Poincaré metric in a bounded domain $D \subset \mathbb{C}$. Define a cone field in terms of the "slope" with respect to the Poincaré metrics in A_x and A_y as

$$C_p^h \equiv \big\{v = (v_x, v_y) \in T_p\mathcal{A} : |v_x|_{A_x} \geq |v_y|_{A_y}\big\}.$$

A metric in this cone is given by $\|v\|_h \equiv |D\pi_x(v)|_{A_x}$.

Definition 2.5 (**Poincaré Cone Fields**). *We call* $(\{C_p^h\}_{p\in\mathcal{A}}, \|\cdot\|_h)$ *the horizontal Poincaré cone field. The vertical Poincaré cone field* $(\{C_p^v\}_{p\in\mathcal{A}}, \|\cdot\|_v)$ *can be defined similarly.*

Finally we define the third type of cone fields which will be useful in the proof of our central claim for hyperbolicity. To do this, let us prepare some notations here. Given $x_0 \in \mathbb{C}$ and $r > 0$, we set $\Delta(x_0; r) \equiv \{x \in \mathbb{C} : |x-x_0| < r\}$. Let $\Delta = \Delta_x = \Delta_y \equiv \Delta(0; 1)$ be unit disks and let $\mathcal{D} = \Delta_x \times \Delta_y$ be a unit bidisk.

Let $\Delta_x = \widetilde{A_x}$ be the universal covering space of A_x and $\tau_x : \Delta_x \to A_x$ be the natural projection. It then follows that $(\tau_x, \tau_y) : \mathcal{D} \to \mathcal{A}$ gives the universal covering of $\mathcal{A}$. Consider a holomorphic map $\phi : \Delta \to \mathcal{A}$. Since Δ is simply connected, there is a lift $\widetilde{\phi} : \Delta \to \mathcal{D}$ of ϕ. We say a holomorphic map ϕ is *of degree* k if $\pi_x \circ \widetilde{\phi} : \Delta \to \Delta_x$ is proper of degree k. When $\phi : \Delta \to \mathcal{A}$ is of degree k, its image $\phi(\Delta)$ is called a *degree* k *disk*. Note that these notions are independent of the choice of the lift $\widetilde{\phi}$.

Now, take $p \in \mathcal{A}$. We will define a cone $\widehat{C}_p^h$ at p in terms of degree one disks. To do this, choose any $q \in (\tau_x, \tau_y)^{-1}(p) \subset \mathcal{D}$ and define

$$\widetilde{C}_q^h \equiv \big\{v_q \in T_q\mathcal{D} : v_q = D\widetilde{\phi}(w) \text{ for a degree one } \phi \text{ with } \widetilde{\phi}(z) = q \text{ and } w \in T_z\Delta\big\}$$

and put $\widehat{C}_p^h \equiv D(\tau_x, \tau_y)(\widetilde{C}_q^h)$.

Lemma 2.6. *The cone* $\widehat{C}_p^h$ *is independent of the choice of* $q \in (\tau_x, \tau_y)^{-1}(p)$.

Proof. Take any two points q and q' in $(\tau_x, \tau_y)^{-1}(p)$. Then, there exist two conformal automorphisms γ_x of $\widetilde{A_x}$ and γ_y of $\widetilde{A_y}$ such that $q' = (\gamma_x, \gamma_y)(q)$. If $\widetilde{\phi} : \Delta \to \mathcal{D}$ is of degree one, then so is $(\gamma_x, \gamma_y) \circ \widetilde{\phi}$. Thus, $\widetilde{C}_{q'}^h \supset D(\gamma_x, \gamma_y)(\widetilde{C}_q^h)$. Since $(\tau_x, \tau_y) \circ (\gamma_x, \gamma_y) = (\tau_x, \tau_y)$, we see that $D(\tau_x, \tau_y)(\widetilde{C}_{q'}^h) \supset D(\tau_x, \tau_y) \circ D(\gamma_x, \gamma_y)(\widetilde{C}_q^h) = D(\tau_x, \tau_y)(\widetilde{C}_q^h)$. This proves the claim. Q.E.D.

For each element $v_p \in \widehat{C}_p^h$, we take $q \in (\tau_x, \tau_y)^{-1}(p)$ and $v_q \in \widetilde{C}_q^h$ so that $v_p = D(\tau_x, \tau_y)(v_q)$. Let us define the metric:

$$|||v_q|||_h \equiv \sup\big\{|w|_\Delta : v_q = D\widetilde{\phi}(w) \text{ for a degree one } \phi \text{ with } \widetilde{\phi}(z) = q \text{ and } w \in T_z\Delta\big\}$$

and put $|||v_p|||_h \equiv |||v_q|||_h$. This definition is again independent of the choice of q.

Definition 2.7 (**Degree One Cone Fields**). *We call* $(\{\widehat{C}_p^h\}_{p\in\mathcal{A}}, |||\cdot|||_h)$ *the horizontal degree one cone field. The vertical degree one cone field* $(\{\widehat{C}_p^v\}_{p\in\mathcal{A}}, |||\cdot|||_v)$ *can be defined similarly.*

In fact we have

Lemma 2.8. *These two types of cones coincide, i.e.* $C^h_p = \widehat{C}^h_p$ *and* $C^v_p = \widehat{C}^v_p$.

Proof. First consider the case $\mathcal{A} = B_x \times B_y$, where B_x and B_y are bounded open topological disks. One may assume that $\mathcal{B} = \Delta_x \times \Delta_y$ and $p = (0, 0)$. Then, the line which passes through $(0, 0)$ and tangents to any $(v_x, v_y) \in C^h_p$ can be expressed as the graph of a holomorphic map from Δ_x to Δ_y because $|v_x|_{\Delta_x} \geq |v_y|_{\Delta_y}$. So we get $(v_x, v_y) \in \widehat{C}^h_p$, and thus $\widehat{C}^h_p \supset C^h_p$.

Conversely, take $v_p = (v_x, v_y) \in \widehat{C}^h_p$ with $v_p = D\phi(w)$. Since $\deg(\phi) = 1$, one can define a holomorphic map $\widehat{\phi} : \Delta_x \to \Delta_y$ such that the image of ϕ coincides with the graph of $\widehat{\phi}$ by putting $\widehat{\phi}(x) \equiv \pi_y(\phi(\Delta) \cap \pi_x^{-1}(x))$. Then, $D\widehat{\phi}(v_x) = v_y$. By Schwarz–Pick Lemma it follows that $|v_x|_{\Delta_x} \geq |v_y|_{\Delta_y}$. Thus, $\widehat{C}^h_p \subset C^h_p$.

Now, the claim for the general case easily follows from the fact that the covering maps τ_x and τ_y are local isometries and the definition $\widehat{C}^h_p \equiv D(\tau_x, \tau_y)(\widetilde{C}^h_q)$. The proof for the vertical cone fields is similar. So, we are done. Q.E.D.

The next lemma relates the two metrics in the definitions of the cone fields.

Lemma 2.9. *We have* $|||v_p|||_h = \|v_p\|_h$ *and* $|||v_p|||_v = \|v_p\|_v$.

Proof. Our task is to prove that $|||v_p|||_h = |D\pi_x(v_p)|_{A_x}$. Let $v_q \in D(\tau_x, \tau_y)^{-1}(v_p)$. The covering map τ_x is a local isometry with respect to the Poincaré metrics, so $|D\pi_x(v_q)|_{\widetilde{A_x}} = |D(\tau_x \circ \pi_x)(v_q)|_{A_x} = |D(\pi_x \circ (\tau_x, \tau_y))(v_q)|_{A_x} = |D\pi_x(v_p)|_{A_x}$. Thus, it is sufficient to show that $|||v_q|||_h = |D\pi_x(v_q)|_{\widetilde{A_x}}$. Let $\widetilde{\phi}$ be a map of degree one and $w \in T_z\Delta$ such that $D\widetilde{\phi}(w) = v_q$. Because $\pi_x \circ \widetilde{\phi}(\Delta) = \widetilde{A_x}$ and $\pi_x \circ \widetilde{\phi} : \Delta \to \widetilde{A_x}$ is isometric, it follows that $|D\pi_x(v_q)|_{\widetilde{A_x}} = |D(\pi_x \circ \widetilde{\phi})(w)|_{\widetilde{A_x}} = |D(\pi_x \circ \widetilde{\phi})(w)|_{\pi_x \circ \widetilde{\phi}(\Delta)} = |w|_\Delta$. This is true for any ϕ of degree one and any w, thus $|D\pi_x(v_q)|_{\widetilde{A_x}} = |||v_q|||_h$ as required. The proof for the vertical norm is similar. Q.E.D.

Thus, we have the following consequence which will be essential later.

Corollary 2.10. *The horizontal (resp. vertical) degree one cone field and the horizontal (resp. vertical) Poincaré cone field are identical including their metrics.*

Example. When $\mathcal{A} = \Delta_x(0; R_x) \times \Delta_y(0; R_y)$, the following explicit expression of the cone at each point $p = (x, y) \in \mathcal{A}$ can be obtained:

$$C^h_p = \left\{ (v_x, v_y) \in T_p\mathcal{A} : |v_y|_E \leq \frac{R_y^2 - |y|^2}{R_x^2 - |x|^2} |v_x|_E \right\},$$

where $|v|_E$ is the Euclidean metric in $T_p\mathcal{A}$. (End of Example.)

Definition 2.11 (Poincaré Boxes). *A product set* $\mathcal{A} = A_x \times A_y$ *equipped with the horizontal/vertical Poincaré cone fields* $(\{C^{h/v}_p\}_{p \in \mathcal{A}}, \| \cdot \|_{h/v})$ *is called a Poincaré box.*

A Poincaré box will be a building block for verifying hyperbolicity of polynomial diffeomorphisms throughout this article.

2.3. **Hyperbolicity criteria.** In this subsection, we present several criteria for hyperbolicity of holomorphic dynamics in $\mathbb{C}^2$ in several forms. To state them, some topological conditions for $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ which imply the expansion of several pairs of cone fields defined in Subsection 2.2 will be employed.

Let $\mathcal{A}^* = A_x^* \times A_y^*$ $(* = \mathfrak{D}, \mathfrak{R})$ be two Poincaré boxes, $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ be a holomorphic injection and $\iota : \mathcal{A}^{\mathfrak{D}} \cap f^{-1}(\mathcal{A}^{\mathfrak{R}}) \to \mathcal{A}^{\mathfrak{D}}$ be the inclusion map. We first define the following notion which extends a similar one in [HO2] (*cf.* Definition 2.15 below).

Definition 2.12 (**Crossed Mapping Condition**). *We say that $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ satisfies the crossed mapping condition (CMC) of degree d if*

$$\rho_f \equiv (\pi_x^{\mathfrak{R}} \circ f, \pi_y^{\mathfrak{D}} \circ \iota) : \iota^{-1}(\mathcal{A}^{\mathfrak{D}}) \cap f^{-1}(\mathcal{A}^{\mathfrak{R}}) \longrightarrow A_x^{\mathfrak{R}} \times A_y^{\mathfrak{D}}$$

is proprer of degree d.

Let $\mathcal{F}_h^{\mathfrak{D}} = \{A_x^{\mathfrak{D}}(y)\}_{y \in A_y^{\mathfrak{D}}}$ be the horizontal foliation of the domain $\mathcal{A}^{\mathfrak{D}}$ with the leaves $A_x^{\mathfrak{D}}(y) = A_x^{\mathfrak{D}} \times \{y\}$, and let $\mathcal{F}_v^{\mathfrak{R}} = \{A_y^{\mathfrak{R}}(x)\}_{x \in A_x^{\mathfrak{R}}}$ be the vertical foliation of the range $\mathcal{A}^{\mathfrak{R}}$ with the leaves $A_y^{\mathfrak{R}}(x) = \{x\} \times A_y^{\mathfrak{R}}$.

Definition 2.13 (**No–Tangency Condition**). *We say that $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ satisfies the no–tangency condition (NTC) if $f(\mathcal{F}_h^{\mathfrak{D}})$ and $\mathcal{F}_v^{\mathfrak{R}}$ have no tangencies. Similarly we say that $f^{-1} : \mathcal{A}^{\mathfrak{R}} \to \mathcal{A}^{\mathfrak{D}}$ satisfies the no–tangency condition if $\mathcal{F}_h^{\mathfrak{D}}$ and $f^{-1}(\mathcal{F}_v^{\mathfrak{R}})$ have no tangencies.*

Notice that we do not exchange h and v of the foliations in the definition of the non–tangency condition for f^{-1}. Hence, f satisfies the (NTC) iff so does f^{-1}.

Example. Given a polynomial diffeomorphism f, choose a sufficiently large $R > 0$. Put $\mathcal{D}_R = \Delta_x(0; R) \times \Delta_y(0; R)$, $V^+ = V_R^+ \equiv \{(x, y) \in \mathbb{C}^2 : |x| \geq R, |x| \geq |y|\}$ and $V^- = V_R^- \equiv \{(x, y) \in \mathbb{C}^2 : |y| \geq R, |y| \geq |x|\}$. Then, f induces a homomorphism:

$$f_* : H_2(\mathcal{D}_R \cup V^+, V^+) \longrightarrow H_2(\mathcal{D}_R \cup V^+, V^+)$$

on the two–dimensional relative homology group. Since $H_2(\mathcal{D}_R \cup V^+, V^+) = \mathbb{Z}$, one can define the *(topological) degree* of f to be $f_*(1)$. It is easy to see that the topological degree of f is equal to the algebraic degree d of f.

Consider $f : \mathcal{D}_R \to \mathcal{D}_R$ and $\rho_f : \mathcal{D}_R \cap f^{-1}(\mathcal{D}_R) \to \mathcal{D}_R$. Given $(x, y) \in \mathcal{D}_R$, the set $f(\rho^{-1}(x, y))$ is equal to $f(D_x(y)) \cap D_y(x)$, where we write $D_x(y) = \Delta_x(0; R) \times \{y\}$ and $D_y(x) = \{x\} \times \Delta_y(0; R)$. Since $f(V^+) \subset V^+$ and $f^{-1}(V^-) \subset V^-$ hold, the number $\mathrm{card}\,(f(D_x(y)) \cap D_y(x))$ can be counted by the number of times $\pi_x \circ f(\partial D_x(y))$ rounds around $\Delta_x(0; R)$ by the Argument Principle. This is also equal to the topological degree of f, so it follows that $\mathrm{card}\,(f(D_x(y)) \cap D_y(x)) = d$ counted with multiplicity for all $(x, y) \in \mathcal{D}_R$. Thus, $f : \mathcal{D}_R \to \mathcal{D}_R$ satisfies the (CMC). Notice that $f : \mathcal{D}_R \to \mathcal{D}_R$ satisfies the (NTC) if and only if $\mathrm{card}\,(f(D_x(y)) \cap D_y(x)) = d$ counted without multiplicity for all $(x, y) \in \mathcal{D}_R$. (End of Example.)

Now, the central claim for verifying hyperbolicity is stated as

Theorem 2.14 (Equivalent Conditions). *Assume that $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ satisfies the crossed mapping condition of degree d. Then, the following are equivalent:*

(i) *f preserves some pair of horizontal cone fields,*
(ii) *f^{-1} preserves some pair of vertical cone fields,*
(iii) *f weakly expands the pair of the horizontal Poincaré cone fields,*
(iv) *f^{-1} weakly expands the pair of the vertical Poincaré cone fields,*
(v) *f satisfies the no–tangency condition,*
(vi) *f^{-1} satisfies the no–tangency condition.*

Moreover, when $\mathcal{A}^{\mathfrak{D}} = \mathcal{A}^{\mathfrak{R}} = \mathcal{B} = B_x \times B_y$, where B_x and B_y are bounded open topological disks in $\mathbb{C}$, then any of the six conditions (i) to (vi) above is equivalent to the following:

(vii) *$\mathcal{B} \cap f^{-1}(\mathcal{B})$ has d connected components.*

Proof. We will show "(v) $\Leftrightarrow$ (vi)", "(v) $\Leftrightarrow$ (vii)", "(iii) $\Rightarrow$ (i) $\Rightarrow$ (v) $\Rightarrow$ (iii)" and "(iv) $\Rightarrow$ (ii) $\Rightarrow$ (vi) $\Rightarrow$ (iv)". However, the proofs for the last two cycles of implications are logically identical (just interchange f and f^{-1}, and the horizontal and the vertical directions), so it is sufficient to prove that "(v) $\Leftrightarrow$ (vi)", "(iii) $\Rightarrow$ (i) $\Rightarrow$ (v) $\Rightarrow$ (iii)" and "(v) $\Leftrightarrow$ (vii)".

Step 1: (v) $\Leftrightarrow$ (vi). This immediately follows from the fact that f is a diffeomorphism and the definition of the no–tangency conditions for f and f^{-1}.

Step 2: (iii) $\Rightarrow$ (i). This is trivial since the horizontal Poincaré cone field is a horizontal cone field.

Step 3: (i) $\Rightarrow$ (v). If f does not satisfy the no–tangency condition, then there exists a point $p \in \iota^{-1}(\mathcal{A}^{\mathfrak{D}}) \cap f^{-1}(\mathcal{A}^{\mathfrak{R}})$ such that any horizontal vector in $T_p\mathbb{C}^2$ is mapped to a vertical vector in $T_{f(p)}\mathbb{C}^2$. This contradicts to (i).

Step 4: (v) $\Rightarrow$ (iii). For simplicity of the presentation, we drop $\mathfrak{D}$ from $\pi_x^{\mathfrak{D}}$ and $\mathfrak{R}$ from $\pi_x^{\mathfrak{R}}$, and write $\pi_x^{\mathfrak{D}} = \pi_x^{\mathfrak{R}} = \pi_x$. Take a point $p \in \iota^{-1}(\mathcal{A}^{\mathfrak{D}}) \cap f^{-1}(\mathcal{A}^{\mathfrak{R}})$ and a vector $v \in C_p^h$. Since $C_p^h = \widehat{C}_p^h$ by Lemma 2.8, there is a degree one disk D through p tangent to v. Let $V \equiv \iota^{-1}(D) = D \cap f^{-1}(\mathcal{A}^{\mathfrak{R}})$ and consider its universal covering $\tau : \widetilde{V} \to V$. By the (NTC), $\pi_x \circ f|_V$ does not have branch points. By the (CMC), $\pi_x \circ f|_V : V \to A_x^{\mathfrak{R}}$ is proper of degree d. Thus, $\pi_x \circ f|_V : V \to A_x^{\mathfrak{R}}$ is a (unbranched) covering and so is $\pi_x \circ f \circ \tau : \widetilde{V} \to A_x^{\mathfrak{R}}$. Since $\widetilde{V}$ is simply connected, there exists a lift $\widetilde{f \circ \tau} : \widetilde{V} \to \widetilde{A_x^{\mathfrak{R}}} \times \widetilde{A_y^{\mathfrak{R}}}$ of $f \circ \tau : \widetilde{V} \to \mathcal{A}^{\mathfrak{R}}$ to the bidisk $\widetilde{A_x^{\mathfrak{R}}} \times \widetilde{A_y^{\mathfrak{R}}}$. It then follows that the degree of the disk $\widetilde{f \circ \tau}(\widetilde{V})$ becomes one. Thus, $f \circ \tau(\widetilde{V})$ is a degree one disk in $\mathcal{A}^{\mathfrak{R}}$ and tangent to $Df(v)$. This shows that $Df(\widehat{C}_p^h) \subset \widehat{C}_{f(p)}^h$ and thus $Df(C_p^h) \subset C_{f(p)}^h$ by Lemma 2.8.

Next we prove the weak expansion of the cone fields. Since $\iota : V \to D$ is the inclusion, there exists $\lambda \geq 1$ such that $|v|_{\pi_x(V)} \geq \lambda |v|_{\pi_x(D)}$ holds. On the other hand, because $\pi_x \circ f : V \to A_x^{\mathfrak{R}}$ is an isometry, we see that $|v|_{\pi_x(V)} = |D(\pi_x \circ f)(v)|_{\pi_x \circ f(V)} = |D\pi_x(Df(v))|_{A_x^{\mathfrak{R}}} = \|Df(v)\|_h$. So one gets $\|Df(v)\|_h \geq \lambda |v|_{\pi_x(D)}$. Since this holds for any degree one disk D through p tangent to v, it follows that $\|Df(v)\|_h \geq \lambda \||v\||_h$. By Lemma 2.9, we conclude $\|Df(v)\|_h \geq \lambda \|v\|_h$.

Step 5: (vii) $\Rightarrow$ (v). First notice that f satisfies the (NTC) if and only if ρ_f is unbranched.

Let $\mathcal{C}_\rho$ be the branch locus of ρ_f and let $\mathcal{B} = B_x \times B_y$. We claim that, if $\mathcal{B} \cap f^{-1}(\mathcal{B})$ has precisely d connected components, then $\mathcal{C}_\rho$ is empty. Let $\mathcal{B}^{(i)}$ be the components of $\mathcal{B} \cap f^{-1}(\mathcal{B})$. If there is $1 \leq i_0 \leq d$ such that $\deg(\rho_f|_{\mathcal{B}^{(i_0)}}) > 1$, then

$$\deg \rho_f = \sum_{i=1}^{d} \deg(\rho_f|_{\mathcal{B}^{(i)}}) \geq d + 1 > d$$

which is a contradiction. Thus, it follows that $\deg(\rho_f|_{\mathcal{B}^{(i)}}) = 1$ for all $1 \leq i \leq d$, i.e. $\rho_f|_{\mathcal{B}^{(i)}}$ is a holomorphic injection. By a standard fact in several complex variables (see, for example, page 31 of [G]), one sees that $\rho_f|_{\mathcal{B}^{(i)}}$ is biholomorphic. Consequently $\rho_f|_{\mathcal{B}^{(i)}}$ can not have branch points.

Step 6: (v) $\Rightarrow$ (vii). The claim follows from the fact that $\mathcal{B}$ is simply connected.

This finishes the proof of Theorem 2.14. Q.E.D.

In what follows we restate Theorem 2.14 in a more checkable way. To do this, given two subsets V and W of $\mathbb{C}$ let us write $\partial_v(V \times W) = \partial V \times W$ and $\partial_h(V \times W) = V \times \partial W$. For $* = \mathfrak{D}, \mathfrak{R}$, let B_x^* be bounded open topological disks in $\mathbb{C}$ and let $\{H_{x,j}^*\}_{j=1}^{N_x^*}$ be a family of finitely many mutually disjoint closed topological disks which are contained in B_x^*. Put $A_x^* = B_x^* \setminus \bigcup_{1 \leq j \leq N_x^*} H_{x,j}^*$. Similarly we define B_y^* and A_y^*, and put $\mathcal{A}^* = A_x^* \times A_y^*$. The projections π_x^* and π_y^* can be also defined.

Definition 2.15 (Boundary Compatibility Condition). *We say that $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ satisfies the boundary compatibility condition (BCC) if*

(i) $\mathrm{dist}(\pi_x^{\mathfrak{R}} \circ f(\partial_v \mathcal{A}^{\mathfrak{D}}), A_x^{\mathfrak{R}}) > 0$ *and*
(ii) $\mathrm{dist}(\pi_y^{\mathfrak{D}} \circ f^{-1}(\partial_h \mathcal{A}^{\mathfrak{R}}), A_y^{\mathfrak{D}}) > 0$

hold, where $\mathrm{dist}(\cdot, \cdot)$ *means the Euclidean distance between two sets in* $\mathbb{C}$.

Note that if $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ satisfies the boundary compatibility condition, then $\mathrm{dist}\,(\pi_x^{\mathfrak{D}}(\partial_v \mathcal{A}^{\mathfrak{D}}), \pi_x^{\mathfrak{D}}(f^{-1}(\mathcal{A}^{\mathfrak{R}}) \cap \mathcal{A}^{\mathfrak{D}})) > 0$.

Let us define

$$\mathcal{C} = \mathcal{C}_f \equiv \bigcup_{y \in A_y^{\mathfrak{D}}} \left\{\text{critical points of } \pi_x^{\mathfrak{R}} \circ f : B_x^{\mathfrak{D}} \times \{y\} \to A_x^{\mathfrak{R}}\right\},$$

(here, $B_x^{\mathfrak{D}}$ can be replaced by $A_x^{\mathfrak{D}}$) and call it the *dynamical critical set* of f.

Definition 2.16 (Off–Criticality Condition). *We say that $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ satisfies the off–criticality condition (OCC) if*

$$\mathrm{dist}(\pi_x^{\mathfrak{R}} \circ f(\mathcal{C}_f), A_x^{\mathfrak{R}}) > 0$$

holds.

A more useful form of Theorem 2.14 is expressed as

Corollary 2.17 (**Hyperbolicity Criterion I).** *If $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ satisfies the (BCC) and the (OCC), then f expands the pair of the horizontal Poincaré cone fields and contracts the pair of the vertical Poincaré cone fields. In particular, if $\mathcal{A}^{\mathfrak{D}} = \mathcal{A}^{\mathfrak{R}} = \mathcal{A}$ and $f : \mathcal{A} \to \mathcal{A}$ satisfies the (BCC) and the (OCC), then f is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A})$.*

Proof. It is fairly easy to see that the (OCC) implies the (NTC) since $B_x^{\mathfrak{D}} \times \{y\} \supset A_x^{\mathfrak{D}}(y)$. The condition (i) in the (BCC) implies that the number of intersections $f(A_x^{\mathfrak{D}}(y)) \cap (\{x\} \times \mathbb{C})$ counted with multiplicity is independent of the choice of $(x, y) \in A_x^{\mathfrak{R}} \times A_y^{\mathfrak{D}}$ by the Argument Principle. Consider its subset $f(A_x^{\mathfrak{D}}(y)) \cap A_y^{\mathfrak{R}}(x)$. If the cardinality of this subset is not constant with respect to $(x, y) \in A_x^{\mathfrak{R}} \times A_y^{\mathfrak{D}}$, then by the continuity of the intersections, there exists $(x_0, y_0) \in A_x^{\mathfrak{R}} \times A_y^{\mathfrak{D}}$ so that some point $p \in f(A_x^{\mathfrak{D}}(y_0)) \cap (\{x_0\} \times \mathbb{C})$ touches $\partial_h\mathcal{A}^{\mathfrak{R}}$. Then, it follows that $f^{-1}(p) \in \mathcal{A}^{\mathfrak{D}}$ and $p \in f^{-1}(\partial_h\mathcal{A}^{\mathfrak{R}})$, contradicting to the condition (ii) of the (BCC).

Moreover, if $f : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ satisfies the (BCC), then the distance between $\pi_x(\partial_v\mathcal{A}^{\mathfrak{D}})$ and $\pi_x(f^{-1}(\mathcal{A}^{\mathfrak{R}}) \cap \mathcal{A}^{\mathfrak{D}})$ is strictly positive. In particular, the inclusion $\iota : V \to D$ in the proof of Theorem 2.14 has the property that $[\pi_x^{\mathfrak{D}} \circ \iota(V)]_\varepsilon \subset \pi_x^{\mathfrak{D}}(D)$ for some $\varepsilon > 0$ which only depends on the distance above and does not depend on the choice of the disk D, where $[X]_\varepsilon$ is the ε-neighborhood of X. If f satisfies the (OCC), it then follows that there exists $\lambda > 1$ which does not depend on the vector v and the disk D so that $|v|_{\pi_x^{\mathfrak{D}}(V)} \geq \lambda|v|_{\pi_x^{\mathfrak{D}}(D)}$. Thus, there exists $\lambda > 1$ so that $\|Df(v)\|_h \geq \lambda\|v\|_h$ holds.

The argument above works for $f^{-1} : \mathcal{A}^{\mathfrak{R}} \to \mathcal{A}^{\mathfrak{D}}$ as well so that f^{-1} expands the pair of the vertical Poincaré cone fields. When $\mathcal{A}^{\mathfrak{D}} = \mathcal{A}^{\mathfrak{R}} = \mathcal{A}$, we may conclude that f is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A})$ by Lemma 2.2. Q.E.D.

The argument so far can be trivially extended to the setting

$$f : \bigsqcup_{1\leq j\leq M_{\mathfrak{D}}} \mathcal{A}_j^{\mathfrak{D}} \longrightarrow \bigsqcup_{1\leq k\leq M_{\mathfrak{R}}} \mathcal{A}_k^{\mathfrak{R}},$$

where each $\mathcal{A}_i^*$ is an open set in $\mathbb{C}^2$ biholomorphic to a Poincaré box of the form $A_x^* \times A_y^*$ (then, two natural projections for $\mathcal{A}_i^*$ corresponding to π_x^* and π_y^* and the notion of horizontal/vertical Poincaré cone fields in $\mathcal{A}_i^*$ can be defined), and the domain and the range are assumed to be the *disjoint* unions of $\{\mathcal{A}_i^*\}_{1\leq i\leq M_*}$. Then, we have the following

Corollary 2.18 (**Hyperbolicity Criterion II).** *If $f : \mathcal{A}_j^{\mathfrak{D}} \to \mathcal{A}_k^{\mathfrak{R}}$ satisfies the (BCC) and the (OCC) for each $1 \leq j \leq M_{\mathfrak{D}}$ and each $1 \leq k \leq M_{\mathfrak{R}}$, then f expands the pair of the horizontal Poincaré cone fields and contracts the pair of the vertical Poincaré cone fields on their unions. In particular, if $\mathcal{A}_i^{\mathfrak{D}} = \mathcal{A}_i^{\mathfrak{R}} = \mathcal{A}_i$ for all $1 \leq i \leq M \equiv M_{\mathfrak{D}} = M_{\mathfrak{R}}$ and $f : \mathcal{A}_j \to \mathcal{A}_k$ satisfies the (BCC) and the (OCC) for all $1 \leq j, k \leq M$, then f is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n(\bigsqcup_{1\leq i\leq M} \mathcal{A}_i)$.*

Confer Subsection 4.1, where we present a similar criterion for hyperbolicity when Poincaré boxes may have overlaps.

2.4. **Proof of Theorem C.** Thanks to Corollary 2.17, we can give explicit bounds on parameter regions of hyperbolic maps in the complex Hénon family. Notice that Hubbard and Oberste–Vorth [HO2], Fornæss and Sibony [FS] and Bedford and Smillie [BS$\mathbb{C}$3] did not give any specific bounds on the possible perturbation width which keeps the hyperbolicity.

Corollary 2.19. *The complex Hénon map:*

$$f_{c,b} : (x, y) \longmapsto (x^2 + c - by, x)$$

with $|c| > 2(1 + |b|)^2$ *is a hyperbolic horseshoe on* J.

Proof. Let $\mathcal{D}_R = \Delta_x(0; R) \times \Delta_y(0; R)$ where $R = (|b| + 1 + \sqrt{(|b| + 1)^2 + 4|c|})/2$. Then, as in Example above, $f : \mathcal{D}_R \to \mathcal{D}_R$ satisfies the (BCC). The (OCC) for $f : \mathcal{D}_R \to \mathcal{D}_R$ is written as $|c - by| > R$ for all $|y| \leq R$. A sufficient condition for this is given by $|c| - |b|R > R$. It is then not difficult to obtain the desired estimate from this inequality by Corollary 2.17. Q.E.D.

Remark 2.20. *Compare with* [DN] *where hyperbolic horseshoes in the real Hénon family on* $\mathbb{R}^2$ *are considered by using the Euclidean metric. We notice that our estimate is better than that in* [DN]. *This is an advantage of the use of the Poincaré metric. Confer also Theorem D on much sharper estimates for the real hyperbolic horseshoe locus of the Hénon family on* $\mathbb{R}^2$.

Corollary 2.21. *The complex Hénon map which satisfies either*

(i) $c = 0$ *and* $|b| < (\sqrt{2} - 1)/2$ *(an attractive fixed point case), or*
(ii) $c = -1$ *and* $|b| < 0.02$ *(an attractive cycle of period two case)*

is hyperbolic on J.

We start with a general remark which will be used in the rest of this paper. Let $R > 0$ as in the previous corollary and $\mathcal{A} = A_x \times A_y$ where $A_x = \Delta_x(0; R) \setminus \bigcup_{i=1}^N H_i$ and $A_y = \Delta_y(0, R)$. Suppose that $f : \mathcal{A} \to \mathcal{A}$ satisfies the (BCC), the (OCC) and $|b| < 1$. Then, either $\overline{f(H_i \times \Delta_y(0; R))} \subset \mathrm{int}(H_j \times \Delta_y(0; R))$ or $\overline{f(H_i \times \Delta_y(0; R))} \subset V^+$ holds for all $1 \leq i \leq N$. By the Kobayashi hyperbolicity of the bidisks $H_i \times \Delta_y(0; R)$, it then follows that every orbit which eventually mapped into some $H_i \times \Delta_y(0; R)$ converges to an attractive cycle or tends to infinity. Thus, we have $K^+ \subset V^- \cup \bigcap_{n=0}^{\infty} f^{-n}(\mathcal{A}) \cup \{\text{attractive cycles}\}$. Because $K \subset \mathcal{D}_R$ and K is invariant, one gets $K \subset \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A}) \cup \{\text{attractive cycles}\}$ and K is hyperbolic. In particular, $J \subset \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A})$ follows since we know that $K = J \cup \{\text{attractive cycles}\}$ when $|b| < 1$ and K is a hyperbolic set.

Proof of Corollary 2.21. We first prove (i). Define the constant $\alpha_c = 1 - \sqrt{1 - 4c}$. Note that the fixed points of the quadratic polynomial $p_c(x) = x^2 + c$ are given by $\alpha_c/2$ and their multipliers are α_c. A sufficient condition for $p_c(x)$ to have an attracting fixed point is thus

$$|\alpha_c| < 1. \tag{2.1}$$

Now, let us examine the (BCC). Our task is to find (c, b) so that there is $r > 0$ with the following property: if x satisfies $|x - \alpha_c/2| < r$, then

$$|\pi_x \circ f_{c,b}(x, y) - \alpha_c/2| < r$$

for all $|y| < R$. Writing $x = te^{i\theta} + \alpha_c/2$ $(0 \le t < r)$ the above condition becomes

$$|te^{i\theta}(te^{i\theta} + \alpha_c) - by| < r$$

for all $|y| < R$. A sufficient condition for this is given by $r(r + |\alpha_c|) + |b|R < r$. Let us put $g(s) = s(s + |\alpha_c|) + |b|R - s$. Then, $g(0) = |b|R > 0$. So, $g(s) = 0$ has a real positive root if and only if (2.1) and

$$(|\alpha_c| - 1)^2 - 4|b|R > 0 \tag{2.2}$$

are satisfied. Conversely, let $r > 0$ be the largest root of $s(s + |\alpha_c|) + |b|R = s$ and let $H = \{x \in \Delta_x(0; R) : |x - \alpha_c/2| \le r\}$. Define $\mathcal{A} = (\Delta_x(0; R) \setminus H) \times \Delta_y(0; R)$. Then, the above argument shows that $f_{c,b} : \mathcal{A} \to \mathcal{A}$ satisfies the (BCC).

We remark that the hole $H \times \Delta_y(0; R)$ contains the critical set $\mathcal{C}$ because $0 \in H$. So the (BCC) implies the (OCC). The Hénon map which satisfies the two conditions (2.1) and (2.2) above is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n_{c,b}(\mathcal{A})$. By putting $c = 0$, we obtain $|b| < (\sqrt{2} - 1)/2 \approx 0.2071$. This finishes the proof of (i).

Next we prove (ii). The polynomial $p_{-1}(x) = x^2 - 1$ has two super–attractive periodic points $\{0, -1\}$ of period 2. Let $r_1 > 0$ and $r_2 > 0$ be small (which we will determine later), and put $H_1 \equiv \{x \in \Delta_x(0; R) : |x - 0| \le r_1\}$, $H_2 \equiv \{x \in \Delta_x(0; R) : |x - (-1)| \le r_2\}$, $A_x = \Delta_x(0; R) \setminus (H_1 \cup H_2)$, $A_y = \Delta_y(0; R)$ and $\mathcal{A} = A_x \times A_y$. A sufficient condition for $f_{-1,b} : \mathcal{A} \to \mathcal{A}$ to satisfy the (BCC) is that $|\pi_x \circ f_{-1,b}(x, y) - (-1)| < r_2$ for all $(x, y) \in H_1 \times \Delta_y(0; R)$ and $|\pi_x \circ f_{-1,b}(x, y) - 0| < r_1$ for all $(x, y) \in H_2 \times \Delta_y(0; R)$. A sufficient condition for these can be witten as

$$r_1^2 + |b|R < r_2 \quad \text{and} \quad r_2(r_2 + 2) + |b|R < r_1.$$

To get a better bound for b, we want to find $r_1 > 0$ and $r_2 > 0$ so that both $r_2 - r_1^2$ and $r_1 - r_2(r_2 + 2)$ become as large as possible. Thus, it is necessary to estimate

$$r \equiv \sup_{r_1>0,\, r_2>0} \min\{r_2 - r_1^2,\, r_1 - r_2(r_2 + 2)\},$$

and an easy calculation shows that $r > 0.04$. By solving $|b|R < 0.04$, we obtain $|b| < 0.02$. This is a sufficient condition for the (BCC).

We again remark that $H_1 \times \Delta_y(0; R)$ contains the critical set $\mathcal{C}$ because $0 \in H_1$. So the (BCC) implies that $\overline{\pi_x \circ f_{-1,b}(\mathcal{C})} \subset \mathrm{int}H_2$, and thus the (OCC) is automatically satisfied when $|b| < 0.02$. This proves (ii). Q.E.D.

Remark 2.22. *According to numerical experiments for the complex Hénon maps with real parameters performed by Oliva (see Section 4.1 of* [Ol]*), the Hénon map with* $(c, b) = (-1, 0.13)$ *seems not conjugate on the Julia set to the projective limit of* p_{-1}.

By using Corollary 2.17, we can recover the following assertion which was originally obtained in [HO2, FS] for the quadratic polynomial case (see [BS$\mathbb{C}$3] for the general degree case).

Corollary 2.23. *For every expanding polynomial $p(x)$ of one variable, the generalized complex Hénon map $f_{p,b}$ is hyperbolic for b sufficiently close to zero.*

Proof. Recall that a polynomial map $p(x)$ in one variable is expanding on its Julia set J_p if and only if every critical point of $p(x)$ converges either to an attractive cycle or to infinity.

Define $B_x = \{x \in \mathbb{C} : G(x) < \min_{1 \le i \le N} G(c_i)\}$, where $G(x)$ is the Green function of $p(x)$ and $\{c_i\}_{i=1}^N$ are the critical points of $p(x)$ which tend to infinity. If there is no critical point of $p(x)$ which tend to infinity, we simply put $B_x = \{x \in \mathbb{C} : G(x) < 1\}$. Let H_x be the points in $\mathbb{C}$ whose Poincaré distance in the Fatou set of $p(x)$ to the set of attracting periodic points including infinity is equal to or less than one. Define $\mathcal{A} = (B_x \setminus H_x) \times \Delta_y(0; R)$, where $R > 0$ is sufficiently large. Then, $f_{p,b} : \mathcal{A} \to \mathcal{A}$ satisfies the (BCC) and the (OCC) when b sufficiently close to zero. In fact, $\mathrm{dist}(p(\partial B_x), B_x) \ge \delta$ and $\mathrm{dist}(p(H_x), \partial H_x) \ge \delta$ for some $\delta > 0$ with respect to the Euclidean distance in $\mathbb{C}$ by the construction of B_x and H_x. Thus, we have $\mathrm{dist}(p(\partial A_x), A_x) \ge \delta$ and the (BCC) follows when $|b|$ is sufficiently close to zero.

The critical set $\mathcal{C}$ of $f_{p,b}$ coincides with $\bigcup_{i=1}^N \{c_i\} \times \Delta_y(0; R)$. Thus, when b is close to zero, $\pi_x \circ f(\mathcal{C})$ is contained in the $(|b|R)$-neighborhood of $\bigcup_{i=1}^N \{p(c_i)\}$ which is contained in either H_x or the complement of B_x. So, the (OCC) follows. By applying Corollary 2.17, we get the conclusion. This finishes the proof. Q.E.D.

3. Fusion of One–Dimensional Polynomials

This section is dedicated to the proof of Theorem B. In Subsection 3.1, a detailed topological model of fusion is analyzed. This model will be realized as an actual generalized Hénon map by constructing a polynomial $p_0(x)$ in one variable whose Julia set has special geometric properties (see Corollary 3.4) in the second subsection. These geometric properties will be essential for proving hyperbolicity of the Hénon map in Subsection 3.3 and in 3.4 we analyze the topology of the Julia set of the generalized Hénon map to finish the proof.

3.1. **Model study of fusion.** In this subsection we only consider cubic polynomials for simplicity. Although the degree of the actual polynomial appeared in Theorem B may be higher than three, the most relevant point of our construction can be described in the cubic case.

Think of two cubics $p_1(x)$ and $p_2(x)$ so that $p_2(x) = p_1(x) + \delta$ for some $\delta > 0$, both have negative leading coefficients and have two real critical points $c_1 > c_2$. Let $\Delta_x(0; R) = \{|x| < R\}$ and $\Delta_y(0; R) = \{|y| < R\}$. Take $R > 0$ sufficiently large so that $\partial\Delta_x(0; R) \times \Delta_y(0; R) \subset \mathrm{int}V^+$ and $\Delta_x(0; R) \times \partial\Delta_y(0; R) \subset \mathrm{int}V^-$ hold. Assume that p_i satisfies $p_1(c_2) < -R$, $p_2(c_2) < -R$ and $p_2(c_1) > R$ so that the orbits $|p_1^k(c_2)|$, $|p_2^k(c_1)|$ and $|p_2^k(c_2)|$ go to infinity as $k \to \infty$. Assume also that c_1 is a super–attractive fixed point for p_1. Define $B_{y,1}$ to be the connected component of $p_1^{-1}(\Delta_y(0; R))$ containing c_1 and $B_{y,2}$ to be the other component. Let H be a closed neighborhood of c_1 which is contained in the attractive basin of c_1. Put $\mathcal{A}_1 = (\Delta_x(0; R) \setminus H) \times B_{y,1}$ and $\mathcal{A}_2 =$

$\Delta_x(0;R)\times B_{y,2}$. Now, we assume that there exists a generalized Hénon map f with

$$f|_{\mathcal{A}_i}(x,y)\approx(p_i(x),x) \tag{3.1}$$

for $i=1,2$.

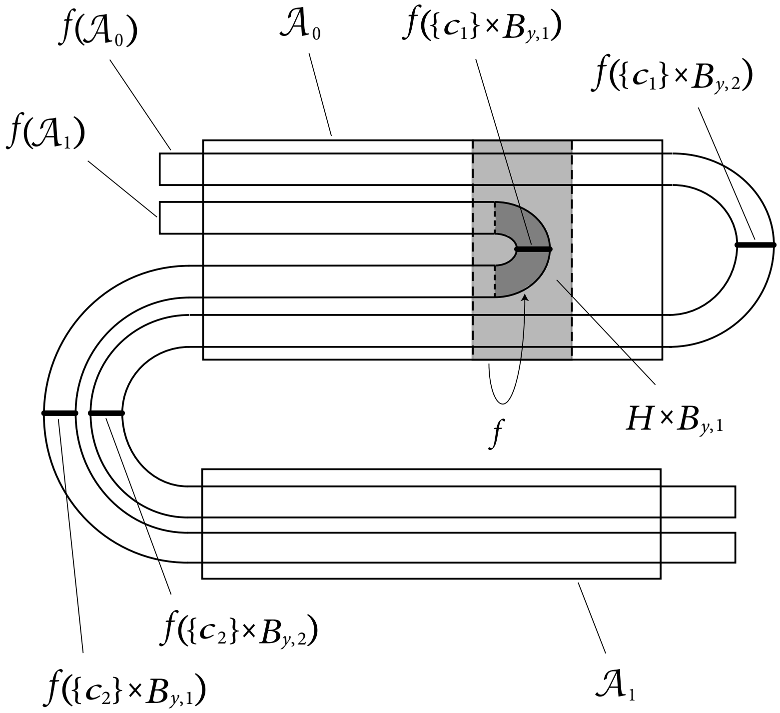

Figure 2. A model for fusion of two polynomials.

(a) Consider $f:\mathcal{A}_1\to\mathcal{A}_1\cup\mathcal{A}_2$. Then, the (BCC) would hold since

$$\overline{f(H\times B_{y,1})}\approx\overline{p_1(H)\times H}\subset\mathrm{int}(H\times B_{y,1})$$

by (3.1) and $R>0$ is large (see Figure 2). Also the (OCC) would hold since

$$\overline{f(\{c_1\}\times B_{y,1})}\approx\overline{\{p_1(c_1)\}\times\{c_1\}}\subset\mathrm{int}(H\times B_{y,1})$$

and

$$\overline{f(\{c_2\}\times B_{y,1})}\approx\overline{\{p_1(c_2)\}\times\{c_2\}}\subset\mathrm{int}V^+$$

again by (3.1). Thus we may conclude that $f:\mathcal{A}_1\to\mathcal{A}_1\cup\mathcal{A}_2$ satisfies the (OCC) and the (BCC) if the argument above is verified rigorously.

(b) Consider $f:\mathcal{A}_2\to\mathcal{A}_1\cup\mathcal{A}_2$. Since $\mathcal{A}_2$ does not have any holes like H and $R>0$ is large, the (BCC) would hold for f on $\mathcal{A}_2$. Also the (OCC) would hold since

$$\overline{f(\{c_1\}\times B_{y,2})}\approx\overline{\{p_2(c_1)\}\times\{c_1\}}\subset\mathrm{int}V^+$$

and

$$\overline{f(\{c_2\}\times B_{y,2})}\approx\overline{\{p_2(c_2)\}\times\{c_2\}}\subset\mathrm{int}V^+$$

(again, see Figure 2). Thus we may conclude that $f:\mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2$ satisfies the (OCC) and the (BCC) if the argument above is justified.

Combining these two considerations, we may expect that $f:\mathcal{A}_1\cup\mathcal{A}_2 \to \mathcal{A}_1\cup\mathcal{A}_2$ is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A}_1\cup\mathcal{A}_2)$ by Corollary 2.18.

In the successive subsections we will justify the argument above. The problem thus is to find a nice polynomial p in one variable (not necessarily of degree three), $b\in\mathbb{C}$ and domains $\mathcal{A}_i$ so that the argument above works. In fact, we will show the following more detailed version of Theorem B:

Theorem 3.1 (**Detailed Theorem B**). *For any $0<\delta<1/2$ there exists an expanding polynomial $p_0(x)$ with the following property: take any $b_0\in\mathbb{C}$ with $\delta<|b_0|<1-\delta$ and take any continuous one–parameter family $\{f_{p_0,b_\mu}\}_{\mu\in[0,1]}$ connecting a small perturbation f_{p_0,b_1} of $p_0(x)$ and f_{p_0,b_0}, then (i) f_{p_0,b_0} is hyperbolic, and (ii) f_{p_0,b_μ} is not hyperbolic at some $\mu_0\in(0,1)$.*

Apparently this implies Theorem B in the Introduction. The proof of the theorem above occupies the rest of this section.

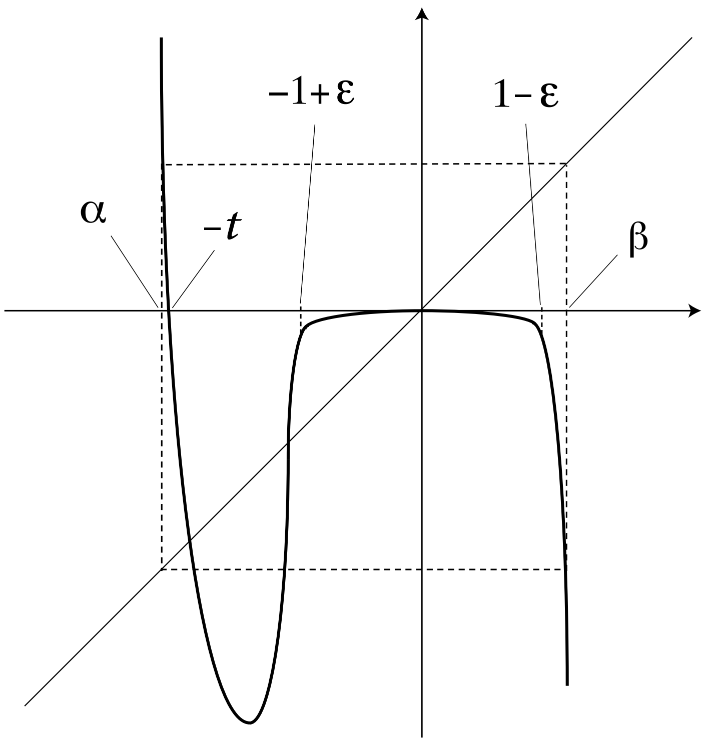

Figure 3. The graph of the polynomial $p_{c,t,l}(x)$.

3.2. **A one–dimensional map.** In this subsection we construct a polynomial in one variable p_0 with its Julia set of "good shape".

For $c < 0$, $t > 1$ and an even $l \in \mathbb{N}$, let us put

$$p(x) = p_{c,t,l}(x) \equiv c(x+t)x^l.$$

Evidently $p(x)$ is bimodal on $\mathbb{R}$ and 0 is its superattractive fixed point. We let γ be the other critical point. Also, $p(x)$ has a repelling cycle of period two $\{\alpha, \beta\}$ with $\alpha < -t < 1 < \beta$. Parameter dependence of the behavior of $p(x)$ as a real dynamics is described in the next lemma.

Lemma 3.2. *For $p = p_{c,t,l} : \mathbb{R} \to \mathbb{R}$, the following hold:*

(i) *for any $\varepsilon > 0$ small, we have $p'(x) \to 0$ uniformly on $F = (-1+\varepsilon, 1-\varepsilon)$ as $l \to +\infty$, i.e. $p(x)$ becomes flat on F when l goes to infinity,*

(ii) length$(F)/(\beta - \alpha) \to 0$ *as $t \to +\infty$, i.e. the flat part F becomes relatively small in $[\alpha, \beta]$ when t goes to infinity,*

(iii) $|p'(x)|$ *on $p^{-1}([\alpha, \beta]) \setminus F$ tends to $+\infty$ as $c \to -\infty$ (in particular, $\beta \downarrow 1$ and $\alpha \uparrow -t$).*

The proof of this lemma is easy and thus omitted (see Figure 3).

Define $R_0 \equiv t + 1/(|c|t^{l-1})$ and put $|p|(x) \equiv |p(x)|$. The behavior of p as a complex dynamics is described in the next lemma.

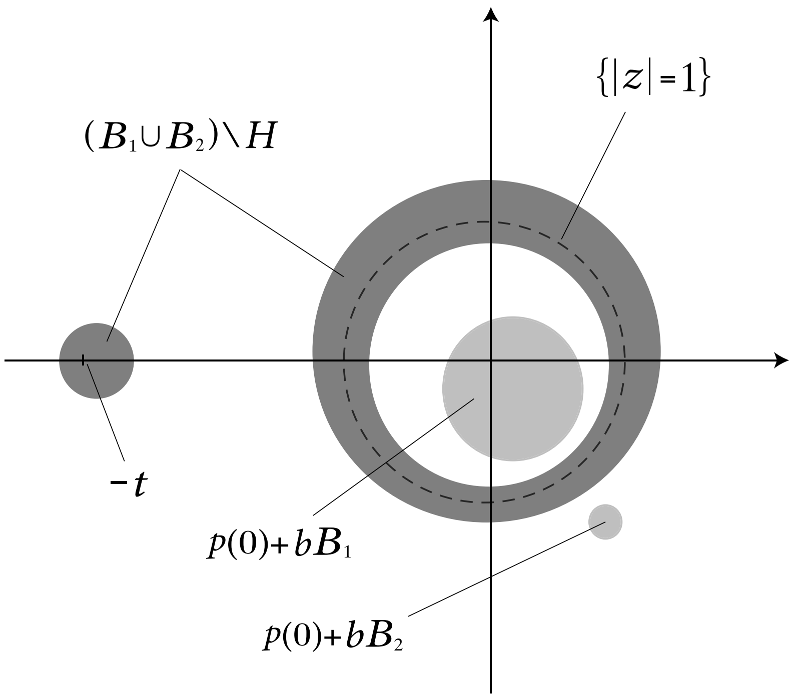

Figure 4. Geometry of the Julia set of $p_{c,t,l}(x)$.

Lemma 3.3. *For $p = p_{c,t,l} : \mathbb{C} \to \mathbb{C}$, the following hold:*

(i) *all critical points of $p(z)$ are 0 and γ (in particular, there are no critical points outside $\mathbb{R}$) and $p(z)$ is quadratic near $z = \gamma$,*

(ii) *$|p^k(z)| \leq |p|^k(|z|)$ for all $z \in \mathbb{C}$,*

(iii) *if $|z| > R_0$, then $|p^k(z)| \geq \lambda^k |z|$ for some $\lambda > 1$.*

Proof. The first statement (i) is trivial. The second claim (ii) follows from

$$|p(z)| = |cz^{l+1} + ctz^l| \leq |c||z|^{l+1} + |c||t||z|^l = |c|(|z| + t)|z|^l = |p|(|z|).$$

This inductively implies

$$|p^k(z)| = |p^{k-1}(p(z))| \leq |p|^{k-1}(|p(z)|) \leq |p|^{k-1}(|p|(|z|)) = |p|^k(|z|)$$

for $z \in \mathbb{C}$ because $|p|$ is monotone increasing on $\mathbb{R}_+$. If $|z| > R_0$, then we have $|c|t^{l-1}(|z| - t) > 1$. Since $|z| \geq \lambda |t|$ for some $\lambda > 1$, it follows that

$$|c(z + t)z^l| \geq |c||z|^{l+1} - |c||t||z|^l \geq \lambda^{l-1}|c|t^{l-1}(|z| - t)|z| \geq \lambda |z|.$$

This proves (iii) and thus we are done. Q.E.D.

We define $\Delta(0; R_0) = \{z \in \mathbb{C} : |z| \leq R_0\}$ with $R_0 > 0$ specified as above. The following corollary describes the shape of the Julia set of p.

Corollary 3.4 (Geometry of J_p). *Let $J_p \subset \mathbb{C}$ be the Julia set of $p = p_{c,t,l}$. Then, the following hold:*

(i) *for any $\varepsilon > 0$, we have $J_p \subset p^{-1}(\Delta(0; R_0)) \setminus H$ when l is large, where $H \equiv \overline{\Delta(0; 1 - \varepsilon)}$,*

(ii) *$p^{-1}(\Delta(0; R_0))$ has two connected components $B_1 \ni 0$ and $B_2 \ni -t$,*

(iii) *$\overline{B_1}$ is close to the unit disk $\{|z| \leq 1\}$ and $\overline{B_2}$ is close to the one-point set $\{-t\}$ in the Hausdorff topology when $c < 0$ is small and l is large,*

(iv) *the distance between B_1 and B_2 is controlled by t, i.e.* $\mathrm{dist}(B_1, B_2) \approx t - 1$ *when $c < 0$ is small (see Figure 4).*

Proof. (i) The flat part F extends to H in the complex plane as a part of the attractive basin of 0 thanks to the estimate in (ii) of Lemma 3.3. This and (i) of Lemma 3.2 imply the conclusion. (ii) p has only one quadratic critical value outside $\Delta(0; R)$. (iii) Take any κ with $t-1 > \kappa > 0$ and any z with $|z| = 1+\kappa$. Then, we see that $|p(z)| = |c||z+t|(1+\kappa)^l$ is large when $|c|$ is large. This implies that B_1 is contained in $\{|z| \leq 1 + \kappa\}$. But B_1 contains H, so $\overline{B_1}$ is close to $\{|z| \leq 1\}$. Since $|p'(z)| = |c||z|^{l-1}|(l+1)z+lt| \approx |c||z|^{l-1}|lz|$ is large near $z = -t$ when $|c|$ is large, $\overline{B_2}$ is close to the one point set $\{-t\}$. The claim (iv) then follows from (iii). Q.E.D.

3.3. **Intersection of Julia sets.** Now, we take any $0 < \delta < 1/2$. Our next task is to find $p_0 = p_{c,t,l}$ so that for any $b_0 \in \mathbb{C}$ with $\delta < |b_0| < 1 - \delta$, f_{p_0,b_0} satisfies the (BCC) and the (OCC) under a suitable choice of domains like $\mathcal{A}_1$ and $\mathcal{A}_2$ as in Subsection 3.1.

We put $H_x \equiv H$ and $B_{x,i} = B_{y,i} \equiv B_i$ for $i = 1, 2$, where B_i and H are given in Corollary 3.4. Let us define bidisks as follows: $\mathcal{B}_1 \equiv B_{x,1} \times B_{y,1}$, $\mathcal{B}_2 \equiv \Delta_x(0; R_0) \times B_{y,2}$ and $\mathcal{B}_3 \equiv B_{x,2} \times B_{y,1}$. In the place of $\mathcal{B}_i$, we will also consider $\mathcal{A}_1 \equiv (B_{x,1} \setminus H_x) \times B_{y,1}$,

$\mathcal{A}_2 \equiv \mathcal{B}_2$ and $\mathcal{A}_3 \equiv \mathcal{B}_3$. Given $X \subset \mathbb{C}$ and $\lambda \in \mathbb{C}$ we write $\lambda X = \{\lambda x : x \in X\}$ and $X + \lambda = \{x + \lambda : x \in X\}$. Let $[X]_\delta$ denote the δ-neighborhood of X.

Lemma 3.5. *Take $p = p_{c,t,l}$ where $c < 0$ is small enough and l is large enough. Then, for any $b \in \mathbb{C}$ with $0 < |b| < 1$, the map $f_{p,b} : \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \to \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ satisfies the (BCC).*

Proof. As we saw in the proof of the previous corollary, we know that $|p'(y)| \approx |c||y|^{l-1}|ly|$ becomes large near $B_{y,2}$. So, if $B_{y,2}$ is taken slightly larger, one gets $p(\partial B_{y,2}) \cap [|b|\Delta_y(0; R_0)]_{R_0} = \emptyset$. Similarly, we get $p(\partial B_{y,1}) \cap [|b|\Delta_y(0; R_0)]_{R_0} = \emptyset$ by taking $B_{y,1}$ slightly larger if necessarily. These imply that for any $x \in \Delta_x(0; R_0)$ we have $(p(\partial B_{y,i}) - x)/b \cap \Delta_y(0; R_0) = \emptyset$ for $i = 1, 2$ in the y-plane. Now let us recall the formula of the inverse generalized Hénon map $f^{-1} : (x, y) \mapsto (y, (p(y) - x)/b)$. It follows that $\mathrm{dist}(\pi_y \circ f^{-1}(\partial_h \mathcal{B}_i), \Delta_y(0; R)) > 0$ for $i = 1, 2, 3$.

By the same reasoning, we see that $p(B_{x,i}) \supset [\Delta_x(0; R_0)]_{|b|R_0}$ for $i = 1, 3$ in the x-plane. It then follows that $\mathrm{dist}(\pi_x \circ f(\partial_v \mathcal{B}_i), \Delta_x(0; R)) > 0$ for $i = 1, 3$. It is not difficult to see that the vertical boundary of $\mathcal{B}_2$ also has the same property. The conclusion follows. Q.E.D.

Corollary 3.6. *The sets K and J of $f_{p,b}$ as in the previous lemma are contained in $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ for all $0 < |b| < 1$.*

Proof. Apparently the proof of the previous Lemma 3.5 shows that K is contained in $(B_{x,1} \cup B_{x,2}) \times (B_{y,1} \cup B_{y,2})$. Q.E.D.

Now we check the conditions for hyperbolicity, i.e. the (BCC) and the (OCC) of the map $f_{p,b} : \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \to \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ for $\delta < |b| < 1 - \delta$ (notice that $\mathcal{B}_i$ is now replaced by $\mathcal{A}_i$). Let d_1 be the smallest r so that $\Delta(0; r) \supset B_1$, d_2 be the diameter of B_2 and h be the smallest r so that $\Delta(0; r) \supset p(H)$.

(a) Consider $f_{p,b} : \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. The (BCC) is confirmed by the proof of Lemma 3.5. Let γ be the unique critical point of p which is different from 0. Since we took $c < 0$ so small that $p(\gamma) < 0$ becomes very small, the only critical set which will concern with the (OCC) is $\{0\} \times B_{y,2} \subset \mathcal{A}_2$. Thus, a sufficient condition for $f_{p,b}$ to satisfy the (OCC) is given by

$$\mathrm{dist}(B_{x,1} \cup B_{2,x}, p(0) + bB_{y,2}) > 0 \tag{3.2}$$

in the x-plane. Note that if $|b|$ is too small, then $B_{x,1} \cap (bB_{y,2}) \neq \emptyset$ and $f_{p,b}$ is conjugate to the projective limit of p. A sufficient condition for $b \in \mathbb{C} \setminus \{0\}$ to satisfy (3.2) is

$$|\alpha||b| > \beta + 2d_2|b|. \tag{3.3}$$

We know by Corollary 2.18 that $f_{p,b} : \mathcal{A}_2 \to \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ is expanding with respect to the pair of the two horizontal Poincaré cone fields when (3.3) holds.

(b) Consider $f_{p,b} : \mathcal{A}_3 \to \mathcal{A}_2$. The (BCC) is confirmed by the proof of Lemma 3.5. Since $|p'(x)|$ does not vanish on $B_{x,2}$, we see that $\mathcal{C} = \emptyset$. Thus, the (OCC) is automatically satisfied. By Corollary 2.18 above, we may conclude that $f_{p,b} : \mathcal{A}_3 \to \mathcal{A}_2$ is expanding with respect to the pair of the two horizontal Poincaré cone fields.

(c) Consider $f_{p,b} : \mathcal{A}_1 \to \mathcal{A}_1 \cup \mathcal{A}_3$. By the proof of Lemma 3.5, we know that $f_{p,b} : \mathcal{B}_1 \to \mathcal{B}_1 \cup \mathcal{B}_3$ satisfies the (BCC). Thus, a sufficient condition for $f_{p,b} : \mathcal{A}_1 \to \mathcal{A}_1 \cup \mathcal{A}_3$ to satisfy the (BCC) is given by

$$\overline{[p(H_x)]_{d_1|b|}} \subset \mathrm{int} H_x. \tag{3.4}$$

To fulfill this condition (3.4), it is sufficient to choose $b \in \mathbb{C} \setminus \{0\}$ so that

$$h + d_1|b| < 1 - \varepsilon. \tag{3.5}$$

Since the only critical point of p in $B_{x,1}$ is 0, a sufficient condition for $f_{p,b}$ to satisfy the (OCC) is

$$\overline{p(0) + bB_{y,1}} \subset \mathrm{int} H_x \tag{3.6}$$

in the x-plane. A sufficient condition for (3.6) is given by

$$d_1|b| < 1 - \varepsilon. \tag{3.7}$$

By Corollary 2.18, we conclude that $f_{p,b} : \mathcal{A}_1 \to \mathcal{A}_1 \cup \mathcal{A}_3$ is expanding with respect to the pair of the two horizontal Poincaré cone fields when (3.5) and (3.7) are satisfied.

Combining these three cases, we know that $f_{p,b} : \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \to \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f_{p,b}^n(\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$ when (3.3), (3.5) and (3.7) hold. It is easy to see that $|\alpha| \downarrow t$, $\beta \downarrow 1$, $d_1 \downarrow 1$, $d_2 \downarrow 0$, $h \downarrow 0$ and $\varepsilon \downarrow 0$ when $l \to \infty$ and $|c|$ large. So, we conclude that for any $0 < \delta < 1/2$, there exists $p_0 = p_{c,t,l}$ with $|c|$ and l being large and $t > 1$ so that for any $b_0 \in \mathbb{C}$ with $\delta < |b_0| < 1 - \delta$, the generalized Hénon map f_{p_0,b_0} satisfies all the conditions (3.3), (3.5) and (3.7).

Recall that the (BCC) in (c) means $\overline{f_{p,b}(H_x \times B_{y,1})} \subset \mathrm{int}(H_x \times B_{y,1})$ so it follows that $J \cap (H_x \times B_{y,1}) = \emptyset$ by the Kobayashi hyperbolicity of $H_x \times B_{y,1}$. Thus, we have $J \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ and there is a unique attractive fixed point in $H_x \times B_{y,1}$. This shows the hyperbolicity of $f = f_{p_0,b_0}$ on J_f, where p_0 and b_0 chosen as above.

Remark 3.7. *A crucial point of the proof of hyperbolicity is to see how the two fattened Julia sets intersect with each other. More precisely, Let J_1 be the slice of $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ by $\{y = 0\}$ and J_2 be the slice of $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ by $\{x = 0\}$. Roughly speaking, we argued that, since there is no intersection between J_1 and $(-b)J_2$ (which followed from the special geometric properties described in Corollary 3.4), the (OCC) is satisfied. This consideration on "intersection geometry" of two fattened Julia sets will be also important in the proof of Theorem A in Section 4. Compare it to a work of Buzzard* [Bu1] *where he considered the stable intersection of two Julia sets without the notion of "thickness" of Cantor sets to discuss Newhouse phenomena in two complex variables* [Bu2].

3.4. **Proof of Theorem B.** Take any continuous one–parameter family $\{f_{p_0,b_\mu}\}_{\mu\in[0,1]}$ connecting a small perturbation f_{p_0,b_1} of $p_0(x)$ and the the hyperbolic generalized Hénon map f_{p_0,b_0} with $\delta < |b_0| < 1 - \delta$ we have constructed so far. To finish the proof of Theorem B, we prove that f_{p_0,b_μ} is not hyperbolic at some $\mu_0 \in (0, 1)$. To do this, the topology of the Julia sets of maps in this family will be analyzed.

Here we need the following terminology. A compact invariant set S of a homeomorphism g is called a *solenoid* of degree k if $g|_S$ is topologically conjugate to the projective limit of $\sigma : S^1 \to S^1$, $\sigma(\theta) = k\theta$. In this case, we say $g|_S$ a *solenoidal map* of degree k. Now, consider a map $f : A_x^{\mathfrak{D}} \times A_y^{\mathfrak{D}} \to A_x^{\mathfrak{R}} \times A_y^{\mathfrak{R}}$ between two Poincaré boxes.

Definition 3.8 (**Solenoidal and Horseshoe Type**). *Assume that $A_y^{\mathfrak{D}}$ and $A_y^{\mathfrak{R}}$ are topological disks. Assume moreover that $f : A_x^{\mathfrak{D}} \times A_y^{\mathfrak{D}} \to A_x^{\mathfrak{R}} \times A_y^{\mathfrak{R}}$ satisfies the (BCC) and the (OCC). Then, f is said to be of solenoidal type if $A_x^{\mathfrak{R}}$ is a topological annulus and $\pi_x^{\mathfrak{R}} \circ f(\mathcal{C})$ is contained in the unique bounded component of $\mathbb{C} \setminus A_x^{\mathfrak{R}}$. Similarly, f is said to be of horseshoe type if $A_x^{\mathfrak{R}}$ is a topological disk and $\pi_x^{\mathfrak{R}} \circ f(\mathcal{C})$ is contained in $\mathbb{C} \setminus A_x^{\mathfrak{R}}$. The degree of a such map can be defined since it satisfies the (CMC).*

An example of a map of solenoidal type is a small perturbation of $z \mapsto z^2 + c$ with $|c|$ small, and an example of a map of horseshoe type is a small perturbation of $z \mapsto z^2 + c$ with $|c|$ large. By following the argument in [HO2], it can be shown that, when $f : \mathcal{A} \to \mathcal{A}$ is a map of solenoidal type of degree k, then $f|_\Omega$ is topologically conjugate to a solenoidal map of the same degree, where $\Omega = \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A})$. Similarly, when f is a map of horseshoe type of degree k, then $f|_\Omega$ is topologically conjugate to the full shift with k symbols. (see also [IS] for a complete proof and more general treatment of these facts).

Consider now the continuous one–parameter family $\{f_{p_0,b_\mu}\}_{\mu\in[0,1]}$ and assume that f_{p_0,b_μ} is hyperbolic for all $\mu \in (0,1)$. We will conclude a contradiction from this. Let us write $f \equiv f_{p_0,b_0}$ and $g \equiv f_{p_0,b_1}$.

For each $* = f, g$, let us put $\mathcal{A}_1^* \equiv \mathcal{A}_1$, $\mathcal{A}_2^* \equiv \mathcal{B}_2$ and $\mathcal{A}_3^* \equiv \mathcal{B}_3$. Then, the following decomposition for f:

$$J_f = \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A}_1^f \sqcup \mathcal{A}_2^f \sqcup \mathcal{A}_3^f) = \bigsqcup_{\underline{\varepsilon}\in\{1,2,3\}^{\mathbb{Z}}} J_{\underline{\varepsilon}}^f,$$

is obtained, where

$$J_{\underline{\varepsilon}}^f \equiv \cdots \cap f^2(\mathcal{A}_{\varepsilon_{-2}}^f) \cap f(\mathcal{A}_{\varepsilon_{-1}}^f) \cap \mathcal{A}_{\varepsilon_0}^f \cap f^{-1}(\mathcal{A}_{\varepsilon_1}^f) \cap f^{-2}(\mathcal{A}_{\varepsilon_2}^f) \cap \cdots .$$

Note that $x \in J_{\underline{\varepsilon}}^f$ iff $f^n(x) \in \mathcal{A}_{\varepsilon_n}$ for all $n \in \mathbb{Z}$. Similarly, the decomposition for g:

$$J_g = \bigcap_{n\in\mathbb{Z}} g^n(\mathcal{A}_1^g \sqcup \mathcal{A}_2^g \sqcup \mathcal{A}_3^g) = \bigsqcup_{\underline{\varepsilon}\in\{1,2,3\}^{\mathbb{Z}}} J_{\underline{\varepsilon}}^g,$$

is obtained, where

$$J_{\underline{\varepsilon}}^g \equiv \cdots \cap g^2(\mathcal{A}_{\varepsilon_{-2}}^g) \cap g(\mathcal{A}_{\varepsilon_{-1}}^g) \cap \mathcal{A}_{\varepsilon_0}^g \cap g^{-1}(\mathcal{A}_{\varepsilon_1}^g) \cap g^{-2}(\mathcal{A}_{\varepsilon_2}^g) \cap \cdots .$$

Lemma 3.9. *For both $* = f, g$ we have the following:*

- (i) *if $\underline{\varepsilon} = \cdots 111.111 \cdots$, then $J_{\underline{\varepsilon}}^*$ becomes an invariant solenoid S_* of degree l,*
- (ii) *if $\varepsilon_i \neq 1$ for only finitely many and at least one $i \geq 0$, then each connected component of $J_{\underline{\varepsilon}}^*$ is either a topological circle or an empty set,*
- (iii) *if $\varepsilon_i \neq 1$ for infinitely many $i \geq 0$, then each connected component of $J_{\underline{\varepsilon}}^*$ is either a point or an empty set.*

Proof. First note that $g : \mathcal{A}_1^g \to \mathcal{A}_1^g$ is of solenoidal type of degree l (and same for f). Thus, the claim of (i) follows.

For (ii), we may assume that $\varepsilon_i = 1$ for all $i \geq 0$ and $\varepsilon_{-1} \neq 1$. It is then easy to see that $\cdots \cap g^{-2}(\mathcal{A}^g_{\varepsilon_2}) \cap g^{-1}(\mathcal{A}^g_{\varepsilon_1}) \cap \mathcal{A}^g_{\varepsilon_0}$ is homeomorphic to $S^1 \times B_{y,1}$, where each fiber is a holomorphic disk of degree one over $B_{y,1}$. Since $\varepsilon_{-1} \neq 1$, $g : \mathcal{A}_{\varepsilon_{-2}} \to \mathcal{A}_{\varepsilon_{-1}}$ is either of degree zero, one or of horseshoe type. Thus, we see that $\mathcal{A}^g_{\varepsilon_{-1}} \cap g(\mathcal{A}^g_{\varepsilon_{-2}}) \cap g^2(\mathcal{A}^g_{\varepsilon_{-3}}) \cap \cdots$ consists of holomorphic disks of degree one over $\pi_x(\mathcal{A}^g_{\varepsilon_{-1}})$ or an empty set. It then follows that each connected component of $J^g_{\underline{\varepsilon}} = \cdots \cap g^{-2}(\mathcal{A}^g_{\varepsilon_2}) \cap g^{-1}(\mathcal{A}^g_{\varepsilon_1}) \cap \mathcal{A}^g_{\varepsilon_0} \cap g(\mathcal{A}^g_{\varepsilon_{-1}} \cap g(\mathcal{A}^g_{\varepsilon_{-2}}) \cap g^2(\mathcal{A}^g_{\varepsilon_{-3}}) \cap \cdots)$ becomes either a topological circle or an empty set. The argument for the case $* = f$ is similar.

For (iii), we first see that each connected component of $\cdots \cap g^{-2}(\mathcal{A}^g_{\varepsilon_2}) \cap g^{-1}(\mathcal{A}^g_{\varepsilon_1}) \cap \mathcal{A}^g_{\varepsilon_0}$ is homeomorphic to either a holomorphic disk of degree one over $B_{y,1}$ or an empty set. Since $\mathcal{A}^g_{\varepsilon_{-1}} \cap g(\mathcal{A}^g_{\varepsilon_{-2}}) \cap g^2(\mathcal{A}^g_{\varepsilon_{-3}}) \cap \cdots$ consists of degree one disks over $\pi_x(\mathcal{A}^g_{\varepsilon_{-1}})$, the conclusion follows. The argument for the case $* = f$ is similar, and thus we are done. Q.E.D.

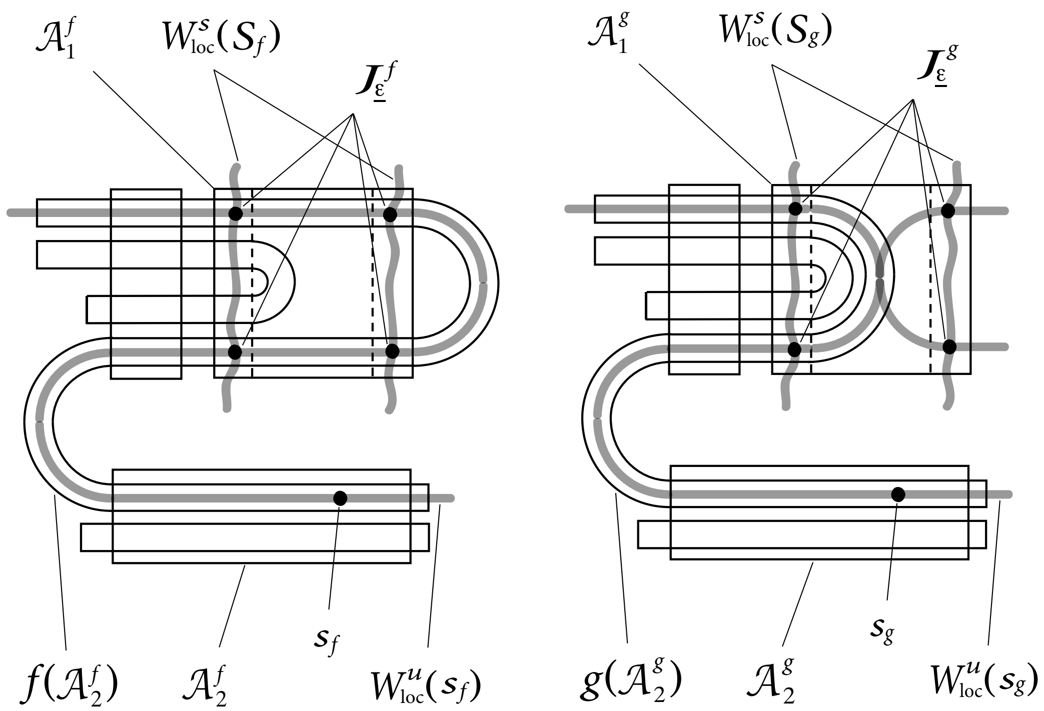

Figure 5. The Julia sets $J^f_{\underline{\varepsilon}}$ and $J^g_{\underline{\varepsilon}}$ for $\underline{\varepsilon} = \cdots 222.111 \cdots$.

For the proof of Theorem 3.1, we need the following more specific fact.

Corollary 3.10. *We have*

(i) *the set $J^f_{\underline{\varepsilon}}$ consists of exactly $l-1$ circles for $\underline{\varepsilon} = \cdots 222.111 \cdots$,*
(ii) *the set $J^g_{\underline{\varepsilon}}$ consists of exactly one circle for $\underline{\varepsilon} = \cdots 222.111 \cdots$.*

Proof. For (i), recall first that $W^s_{\mathrm{loc}}(S_f) = \cdots \cap f^{-2}(\mathcal{A}^f_1) \cap f^{-1}(\mathcal{A}^f_1) \cap \mathcal{A}^f_1$ is homeomorphic to $S^1 \times B_{y,1}$, where each fiber is a holomorphic disk of degree one over $B_{y,1}$. On the other hand, $W^u_{\mathrm{loc}}(s_f) = \mathcal{A}^f_2 \cap f(\mathcal{A}^f_2) \cap f^2(\mathcal{A}^f_2) \cap \cdots$ is a holomorphic disk of degree one over $\Delta_x(0;R)$. Since $f : \mathcal{A}^f_2 \to \mathcal{A}^f_1$ is a degree $l-1$ map of horseshoe type, we see that $J^f_{\underline{\varepsilon}} = W^s_{\mathrm{loc}}(S_f) \cap f(W^u_{\mathrm{loc}}(s_f))$ consists of exactly $l-1$ topological circles.

For (ii), we know that $W^s_{\mathrm{loc}}(S_g) = \cdots \cap g^{-2}(\mathcal{A}^g_1) \cap g^{-1}(\mathcal{A}^g_1) \cap \mathcal{A}^g_1$ is homeomorphic to $S^1 \times B_{y,1}$, where each fiber is a degree one disk over $B_{y,1}$. On the other hand, $W^u_{\mathrm{loc}}(s_g) = \mathcal{A}^g_2 \cap g(\mathcal{A}^g_2) \cap g^2(\mathcal{A}^g_2) \cap \cdots$ is a holomorphic disk of degree one over $\Delta_x(0;R)$. The only difference from (i) is that $g : \mathcal{A}^g_2 \to \mathcal{A}^g_1$ is a degree $l-1$ map of solenoidal type. Thus, we see that $J^g_{\underline{\varepsilon}} = W^s_{\mathrm{loc}}(S_g) \cap g(W^u_{\mathrm{loc}}(s_g))$ consists of one topological circle (see Figure 5). Q.E.D.

Remark 3.11. *We have $W^u(s_*) \cap S_* = \emptyset$.*

End of the proof of Theorem B. Since we are assuming that f_{p_0,b_μ} is hyperbolic for all $\mu \in [0,1]$, every point moves continuously with respect to μ. Corollary 3.6 says that the set of points in $\mathcal{B}_i \cap K$ for f_{p_0,b_μ} always stay in $\mathcal{B}_i \cap K$ when $\mu \in [0,1]$ moves from 0 to 1. Thus, the set of points in K with same itinerary for f should be homeomorphic to the same set for g. However, this is impossible by Corollary 3.10. It follows that f_{p_0,b_μ} cannot be hyperbolic for all $\mu \in [0,1]$. This completes the proof of Theorem 3.1, thus of Theorem B. Q.E.D.

4. Constructing Non–Perturbative Dynamics

In this section, we prove Theorem A. To achieve this, it is necessary to generalize Corollary 2.18 to the case where several Poincaré boxes have overlaps. Subsection 4.1 is dedicated to discuss a general treatment of this overlapping problem. In 4.2, we introduce a new coordinate system called the projective coordinates which will fit better than the Euclidean ones to our purpose. The next subsection explains the basic idea of interval arithmetic as well as how this technique is used to prove some results in complex analysis. In Subsection 4.4, we construct a topological model for the cubic Hénon map under consideration in the same spirit (the fusion) as Theorem B, and verify its hyperbolicity by integrating the tools explained in the previous subsections. In the last subsection, it is shown that the map has essentially two–dimensional dynamics to finish the proof of Theorem A. In the same subsection, a combinatorial description of the Julia set of the cubic Hénon map is given in Theorem 4.23.

4.1. **Gluing Poincaré boxes.** Let $\{\mathcal{A}_i\}_{i=0}^N$ be a family of Poincaré boxes in $\mathbb{C}^2$ each of which is biholomorphic to a product set of the form $A^i_x \times A^i_y$ with its horizontal

Poincaré cone field $\{C_p^{\mathcal{A}_i}\}_{p\in\mathcal{A}_i}$ in $\mathcal{A}_i$. Note, however, that here we are *not* assuming $\mathcal{A}_i$ are disjoint, so that at some point $p \in \mathcal{A} \equiv \bigcup_{i=0}^N \mathcal{A}_i$ there may be more than one horizontal Poincaré cones. Thus, a question is how to define a new cone on the overlaps of the Poincaré cones. Let us put $\Omega_{\mathcal{A}} \equiv \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A})$.

Definition 4.1 (**Gluing of Poincaré Boxes**). *For each point $p \in \mathcal{A}$, let us write $I(p) \equiv \{i : p \in \mathcal{A}_i\}$. We shall define a new cone field $\{C_p^{\cap}\}_{p\in\mathcal{A}}$ by*

$$C_p^{\cap} \equiv \bigcap_{i\in I(p)} C_p^{\mathcal{A}_i}$$

for $p \in \mathcal{A}$ and a metric $\|\cdot\|_{\cap}$ in it by

$$\|v\|_{\cap} \equiv \min\{\|v\|_{\mathcal{A}_i} : i \in I(p)\}$$

for $v \in C_p^{\cap}$.

Remark 4.2. *A priori we do not know if $C_p^{\cap}$ is a non–empty cone for $p \in \mathcal{A}$ with* $\mathrm{card}(I(p)) \geq 2$.

Given a subset $I \subset \{0, 1, \cdots, N\}$, let us write

$$\langle I\rangle \equiv \left(\bigcap_{i\in I}\mathcal{A}_i\right) \setminus \left(\bigcup_{j\in I^c}\mathcal{A}_j\right) = \{p \in \mathcal{A} : I(p) = I\}.$$

In what follows, we only consider the case $\mathrm{card}(I(p)) \leq 2$ for all $p \in \mathcal{A}$. One then sees, for example, $\langle i\rangle = \mathcal{A}_i \setminus \bigcup_{j\neq i}\mathcal{A}_j$ and $\langle i, j\rangle = \mathcal{A}_i \cap \mathcal{A}_j$. When there exists a point $p \in \langle I_1\rangle \cap \Omega_{\mathcal{A}}$ so that $f(p) \in \langle I_2\rangle$, we write $\langle I_1\rangle \to \langle I_2\rangle$ and call it an *allowed transition.* We also write $\mathcal{A}_i \Rightarrow \mathcal{A}_j$ if $f : \mathcal{A}_i \to \mathcal{A}_j$ satisfies the (BCC) and the (OCC).

A crucial step in the proof of Theorem A is to extend Corollary 2.18 as follows:

Proposition 4.3 (**Gluing Lemma**). *Let $p \in \mathcal{A} \cap f^{-1}(\mathcal{A})$. If for any $i \in I(f(p))$ there exists $j = j(i) \in I(p)$ such that $\mathcal{A}_j \Rightarrow \mathcal{A}_i$, then we have $Df(C_p^{\cap}) \subset C_{f(p)}^{\cap}$ and $\|Df(v)\|_{\cap} \geq \lambda\|v\|_{\cap}$.*

Proof. Since $\mathcal{A}_j \Rightarrow \mathcal{A}_i$, we have $Df(C_p^{\mathcal{A}_j}) \subset C_{f(p)}^{\mathcal{A}_i}$ and $\|Df(v)\|_{\mathcal{A}_i} \geq \lambda\|v\|_{\mathcal{A}_j}$ as in Corollary 2.18. By the very definitions of $C_p^{\cap}$ and $\|v\|_{\cap}$, it follows that

$$\begin{aligned} C_{f(p)}^{\cap} &= \bigcap_{i\in I(f(p))} C_{f(p)}^{\mathcal{A}_i} \\ &\supset \bigcap_{i\in I(f(p))} Df(C_p^{\mathcal{A}_{j(i)}}) \\ &\supset \bigcap_{j\in I(p)} Df(C_p^{\mathcal{A}_j}) \\ &= Df(C_p^{\cap}), \end{aligned}$$

and

$$\begin{aligned}\|Df(v)\|_\cap &= \min\{\|Df(v)\|_{\mathcal{A}_i} : i \in I(f(p))\}\\ &\geq \min\{\lambda\|v\|_{\mathcal{A}_{j(i)}} : i \in I(f(p))\}\\ &\geq \lambda \min\{\|v\|_{\mathcal{A}_j} : j \in I(p)\}\\ &= \lambda\|v\|_\cap.\end{aligned}$$

This proves the claim. Q.E.D.

The following fact has been already shown in Theorem C (iii). However, we here present another proof of it by using the gluing technique above.

Corollary 4.4. *The quadratic Hénon map $f_{c,b}(x,y) = (x^2 + c - by, x)$ is hyperbolic on K for $c = -1$ and $|b|$ sufficiently close to zero.*

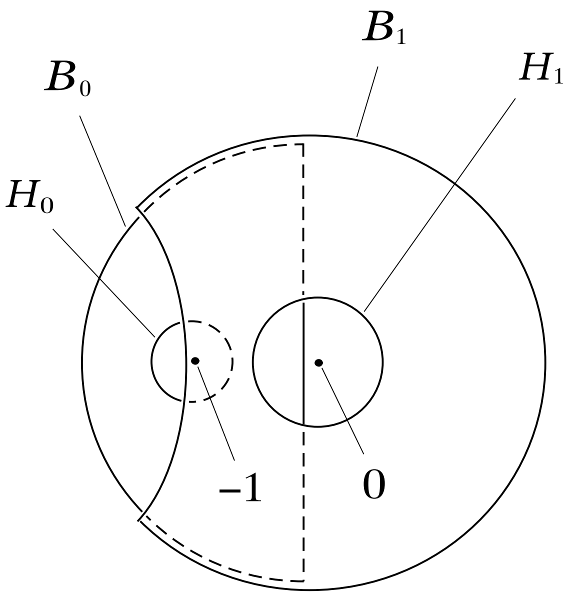

Figure 6. Disks for $p(x) = x^2 - 1$.

Proof. Let $p(x) = x^2 - 1$. Bedford and Smillie observed the following fact (private communication): there exist two topological disks $B_0 \ni -1 = p(0)$ and $B_1 \ni 0$ in $\mathbb{C}$ such that $\mathrm{dist}(p(\partial B_0), B_1) \geq \delta > 0$, $\mathrm{dist}(p(\partial B_1), B_0) \geq \delta > 0$, and $B_0 \cup B_1 \supset K_p$ (see Figure 6). Moreover, $p : p^{-1}(B_1) \cap B_0 \to B_1$ is proper of degree one and $p : p^{-1}(B_0) \cap B_1 \to B_0$ is proper of degree two.

Let $H_0 \ni p(0)$ and $H_1 \ni 0$ be the disjoint closed disks as in the proof of Corollary 2.21 and define $\mathcal{A}_i = (B_i \setminus H_i) \times \Delta_y(0; R)$. Then, $h \equiv f_{-1,b}$ satisfies $\mathcal{A}_0 \Rightarrow \mathcal{A}_1$, $\mathcal{A}_1 \Rightarrow \mathcal{A}_1$

(these are of degree one), and $\mathcal{A}_1 \Rightarrow \mathcal{A}_0$ (this is solenoidal type of degree two) when $|b|$ is sufficiently close to zero. All allowed transitions for h are $\langle 0\rangle \to \langle 1\rangle$, $\langle 0,1\rangle \to \langle 0\rangle$, $\langle 0,1\rangle \to \langle 0,1\rangle$, $\langle 0,1\rangle \to \langle 1\rangle$, $\langle 1\rangle \to \langle 0\rangle$, $\langle 1\rangle \to \langle 0,1\rangle$ and $\langle 1\rangle \to \langle 1\rangle$. It is then easy to see that the assumption of Proposition 4.3 is satisfied. Thus, the cone field $(\{C_p^{\cap}\}_{p\in\mathcal{A}\cap h^{-1}(\mathcal{A})}, \|\cdot\|_{\cap})$ is expanding. Since any horizontal Poincaré cone $C_p^{\mathcal{A}_i}$ contains the horizontal direction, we know that $C_p^{\cap}$ is non–empty everywhere. This proves the hyperbolicity of h. Q.E.D.

4.2. **Interval arithmetic.** A computer does not understand all real numbers. Let $\mathbb{F}^*$ be the set of real numbers which can be represented by binary floating point numbers no longer than a certain length of digits and put $\mathbb{F} \equiv \mathbb{F}^* \cup \{+\infty, -\infty\}$. Denote by $\mathfrak{I}$ the set of all closed intervals with their end points in $\mathbb{F}$. Given $x \in \mathbb{R}$, let $\downarrow x\downarrow$ be the largest number in $\mathbb{F}$ which is less than x and let $\uparrow x\uparrow$ be the smallest number in $\mathbb{F}$ which is greater than x (when such numbers do not exist in $\mathbb{F}^*$, we assign $-\infty$ and $+\infty$ in $\mathbb{F}^*$ respectively). It then follows that

$$x \in [\downarrow x\downarrow, \uparrow x\uparrow] \in \mathfrak{I}.$$

Interval arithmetic is a set of operations to output an interval in $\mathfrak{I}$ from given two intervals in $\mathfrak{I}$. It contains at least four basic operations: addition, differentiation, multiplication and division. Specifically, the addition of given two intervals $I_1 = [a, b]$, $I_2 = [c, d] \in \mathfrak{I}$ is defined by

$$I_1 + I_2 \equiv [\downarrow a + c\downarrow, \uparrow b + d\uparrow].$$

It then follows that $\{x + y \in \mathbb{R} : x \in I_1,\ y \in I_2\} \subset I_1 + I_2$ rigorously. The other three operations can be defined similarly. A point $x \in \mathbb{R}$ is represented as the small interval $[\downarrow x\downarrow, \uparrow x\uparrow] \in \mathfrak{I}$. We also write $[a, b] < [c, d]$ when $b < c$.

In this article interval arithmetic will be employed to prove rigorously the (BCC) and the (OCC) for a given polynomial diffeomorphism of $\mathbb{C}^2$. It should be easy to imagine how this technique is used for checking the (BCC); we simply cover the vertical boundary of $\mathcal{A}^{\mathfrak{D}}$ by small real four–dimensional cubes (i.e. product sets of four small intervals) in $\mathbb{C}^2$ and see how they are mapped by $\pi_x \circ f$. Thus, below we explain how interval arithmetic will be applied to check the (OCC).

The problem of checking the (OCC) in the Euclidean coordinates for a given generalized Hénon map $f_{p,b}$ reduces to finding the zeros of the derivative $\frac{d}{dx}(p(x) - by_0)$ for each fixed y_0. In the rest of this paper, for some reasons, we have to find a desired number of zeros of the derivative above in a specified region not only for $f_{p,b}$ itself but also for its twice iterate $f_{p,b}^2 : \mathcal{A}^{\mathfrak{D}} \to \mathcal{A}^{\mathfrak{R}}$ with respect to certain projective coordinates. In this case, the problem is to find the critical points of $\pi_u \circ f^2(x, y_0)$ in a specified region for each fixed $y_0 \in A_y^{\mathfrak{R}}$. Essentially, this means that one has to find the zeros for a family of polynomials $q_y(x)$ in x parameterized by $y \in A \subset \mathbb{C}$. To do this, we first apply Newton's method to know approximate locations of its zeros. However, this method can not tell how many zeros we found in the region since it does not detect the multiplicity of zeros.

In order to count the multiplicity we employ the idea of winding number. That is, we first fix $y \in A$ and write a small circle in the x–plane centered at the approximate location of a zero (which we had already found by Newton's method). We map the circle by q_y and count how it rounds around the image of the approximate zero, which gives both the existence and the number of zeros inside the small circle. Our method to count the winding number on computer is the following. We may assume that the image of the approximate zero is the origin of the complex plane. Cover the small circle by many tiny squares and map them by q_y. We then verify the following two points: (i) check that the images of the squares have certain distance from the origin which is much larger than the size of the image squares, and (ii) count the number of changes of the signs in the real and the imaginary parts of the sequence of image squares. These data tell how the image squares move one quadrant to another (note that the transition between the first and the third quadrants and between the second and the fourth are prohibited by (i)), and if the signs change properly, we are able to know the winding number of the image of the small circle.

An advantage of this method is that, since the winding number is integer–valued, its mathematical rigorous justification becomes easier (there is almost no room for round–off errors to be involved). Another advantage of this winding number method is its stability; once we check that the image of the circle by q_y rounds a point desired number of times for a fixed parameter y, then this is often true for any nearby parameters. So, by dividing the parameter set A into small squares and verifying the above points for each squares, we can rigorously trace the zeros of q_y for all $y \in A$.

4.3. **Projective coordinates.** Let $u = (u_x, u_y) \in \mathbb{C}^2$ and let L_u be a complex line in $\mathbb{C}^2$ so that $u \notin L_u$. Define $\mathbb{C}^2_u = \mathbb{C}^2 \setminus L'_u$, where L'_u is the unique complex line through u parallel to L_u. Let $\pi_u : \mathbb{C}^2_u \to L_u$ be the projection with respect to the focus $u = (u_x, u_y)$, i.e. for $z \in \mathbb{C}^2_u$ we let L be the unique complex line containing both u and z, then $\pi_u(z)$ is defined as the unique point $L \cap L_u$. We call u the *focus* of π_u.

Let u and v be two focuses and let L_u and L_v be two complex lines in general position in $\mathbb{C}^2$ such that $u \notin L_u$ and $v \notin L_v$. Consider the pair of corresponding projections (π_u, π_v).

Definition 4.5 (Projective Coordinates). *We call the pair of projections (π_u, π_v) the projective coordinates with respect to u, v, L_u and L_v.*

Evidently, the Euclidean coordinates correspond to the case $u = (0, \infty)$, $v = (\infty, 0)$, $L_u = \{y = 0\}$, $L_v = \{x = 0\}$, $L'_u = \emptyset$ and $L'_v = \emptyset$, .

Take two bounded topological disks $U_u \subset L_u$ and $U_v \subset L_v$ so that the following condition holds: $\pi_u^{-1}(z) \cap \pi_v^{-1}(U_v)$ is a bounded topological disk for any $z \in U_u$ and $\pi_u^{-1}(U_u) \cap \pi_v^{-1}(z)$ is a bounded topological disk for any $z \in U_v$.

Proposition 4.6. *Under this assumption, $\pi_u^{-1}(U_u) \cap \pi_v^{-1}(U_v)$ is biholomorphic to a rigid bidisk in $\mathbb{C}^2$.*

Proof. We will first show that the map:

$$F : (z, w) \longmapsto \pi_u^{-1}(z) \cap \pi_v^{-1}(w)$$

gives a biholomorphism between $U_u \times U_v$ and $\pi_u^{-1}(U_u) \cap \pi_v^{-1}(U_v)$. Evidently, it is surjective. By the assumption, the focus v is not contained in the cone $\pi_u^{-1}(U_u)$. Thus, for each fixed $z \in U_u$, the map above is an injective holomorphic map. Similarly, we know that for each fixed $w \in U_v$, the map above is an injective holomorphic map. Moreover, it is clear that $\pi^{-1}(z) \cap \pi_v^{-1}(U_v)$ and $\pi^{-1}(z') \cap \pi_v^{-1}(U_v)$ are disjoint when $z \neq z'$. Hence the map F is injective on the entire $\pi_u^{-1}(U_u) \cap \pi_v^{-1}(U_v)$. Since F is holomorphic in each variable and continuous, a standard argument shows that F is in fact holomorphic as a function of two variables. Thus, it follows that F is biholomorphic.

Now, the conclusion follows by applying Riemann mapping theorem to the bounded topological disks U_u and U_v. Q.E.D.

Definition 4.7 (Projective Bidisks). *We call $\pi_u^{-1}(U_u) \cap \pi_v^{-1}(U_v)$ a projective bidisk and write $U_u \times_P U_v$.*

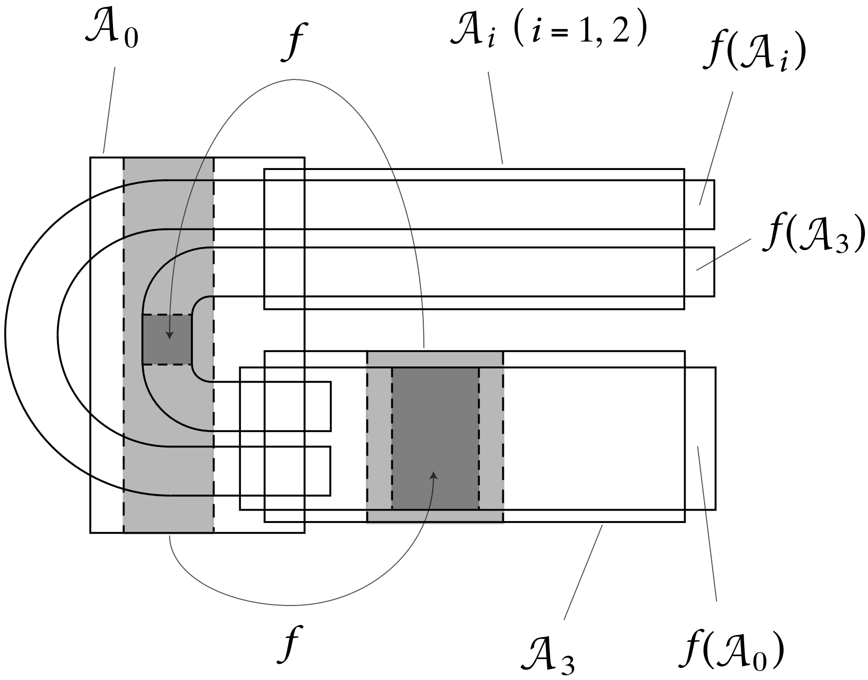

Figure 7. Four Poincaré boxes for the cubic Hénon map $f_{a,b}$.

Thus, Proposition 4.3 is valid in this projective bidisk setting as well. In what follows, the focuses we will use are enough separated with each other and are relatively far away from the place where the dynamics is interesting, so we may assume that the projective coordinates we will employ always satisfy the assumption of Proposition 4.6.

4.4. **Checking hyperbolicity.** Now let us suppose that $f = f_{a,b}$ is the cubic Hénon map with $(a, b) = (-1.35, 0.2)$ as in Theorem A. Our first step is to construct four Poincaré boxes $\{\mathcal{A}_i\}_{i=0}^3$ whose transitions are described in Figure 7 (note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are overlapped in the figure though they are disjoint). For more precise statements, see Propositions 4.12 and 4.16.

We first define $\mathcal{D}_i$ $(i = 0, 1, 2)$ as follows. Let D_x be an open hexagon inspired by an equi–potential curve of $p(z) = -z^3 - 1.35$. More precisely, we take 15 points in $\mathbb{C}$: $p_0 = -1.420 + 0.288i$, $p_1 = \overline{p_0}$, $p_2 = -0.800 - 0.656i$, $p_3 = -0.420 + 0.000i$, $p_4 = \overline{p_2}$, $p_5 = 0.963 + 1.075i$, $p_6 = 0.963 + 0.000i$, $p_7 = 0.000 + 0.000i$, $p_8 = -0.482 + 0.825i$, $p_9 = 0.428 + 1.375i$, $p_{10} = \overline{p_5}$, $p_{11} = p_6$, $p_{12} = p_7$, $p_{13} = \overline{p_8}$, $p_{14} = \overline{p_9}$, where $\overline{p_i}$ is the complex conjugate of p_i. Define D_x to be the hexagon $p_0p_1p_{14}p_{10}p_5p_9$. Note that, in particular, we then have

$$D_x \subset \{x \in \mathbb{C} : \operatorname{Re} x \geq -1.42\}.$$

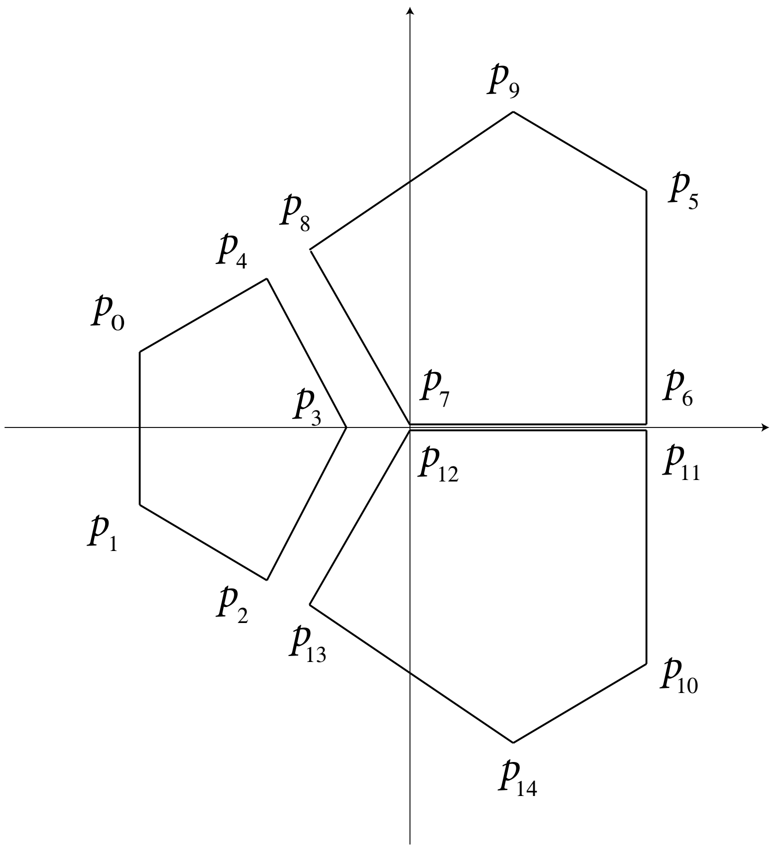

Figure 8. Three pentagons D_i.

Let $D_0 \subset D_x$ be the open pentagon $p_0p_1p_2p_3p_4$ (see Figure 8). Set $D'_y = \Delta(0; 1.05)$ and $\mathcal{D}_0 \equiv D_0 \times_P D'_y$. The projective coordinates to define this projective bidisk are given by the focuses $u = (u_x, u_y) = (-1.763356785556, 13.753270977536)$ and $v = (v_x, v_y) = (\infty, 0)$, and the complex lines $L_u = \{y = 0\}$ and $L_v = \{x = 0\}$. Let $D_1 \subset D_x$ be the open pentagon given by $p_5p_6p_7p_8p_9$ and $D_2 \subset D_x$ be the open pentagon defined by $p_{10}p_{11}p_{12}p_{13}p_{14}$. For $i = 1, 2$, we set $D_y = \Delta(0; 1.5)$ and define $\mathcal{D}_i \equiv D_i \times_P D_y$, where the projective coordinates to define this projective bidisk are the Euclidean ones, that is, given by the focuses $u = (u_x, u_y) = (0, \infty)$ and $v = (v_x, v_y) = (\infty, 0)$, and the complex lines $L_u = \{y = 0\}$ and $L_v = \{x = 0\}$. Finally, we put $\mathcal{D} \equiv \bigcup_{i=0}^{2} \mathcal{D}_i$.

Now we set $\mathcal{B}_i \equiv f(\mathcal{D}_i)$ for $i = 1, 2$. The coordinate system for $\mathcal{D}_i$ naturally induces a coordinate system for $\mathcal{B}_i$. Next we put $\mathcal{B}_0 \equiv \mathcal{D}_0$ and $\mathcal{B}_3 \equiv f(\mathcal{D}_0)$. Here, when $\mathcal{B}_3$ is defined, we fatten $\mathcal{D}_0$ slightly to the D_0-direction and shrink slightly to the D'_y-direction. The coordinate system for $\mathcal{B}_0$ is the same for $\mathcal{D}_0$. The coordinate system for $\mathcal{B}_3$ is the one induced by f from $\mathcal{D}_0$. Remark that, since we slightly modified $\mathcal{D}_0$ when we define $\mathcal{B}_3 = f(\mathcal{D}_0)$, the map $f : \mathcal{B}_0 \to \mathcal{B}_3$ automatically satisfies the (BCC).

To start the proof of Theorem A, we first check

Proposition 4.8. *$K \subset \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$.*

Proof. By the invariance of K, the conclusion is equivalent to $K \subset f^{-1}(\mathcal{D}_0) \cup \mathcal{D}$. Let $\mathcal{D}_R \equiv \Delta_x(0; R) \times \Delta_y(0; R)$ so that every point outside this bidisk tends to infinity either by forward or backward iterations. The claim of Proposition 4.8 immediately follows from the next fact which can be verified by the C++ program `filled.cc`. Q.E.D.

Numerical Check 1. *For any $x \in \mathcal{D}_R$ with $R = 1.5$, we have either $x \in \mathcal{D}$, $f(x) \in \mathcal{D}_0$, $f^2(x) \notin \mathcal{D}_R$ or $f^{-1}(x) \notin \mathcal{D}_R$.*

Next we will see how $\mathcal{B}_i$ are sitting in $\mathbb{C}^2$ and how they are mapped by f.

Lemma 4.9. *$\mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{B}_1 \cap \mathcal{B}_3 = \mathcal{B}_2 \cap \mathcal{B}_3 = \emptyset$.*

Proof. It is easy to see that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ since $D_1 \cap D_2 = \emptyset$ and the focuses for $\mathcal{D}_1$ and $\mathcal{D}_2$ are the same. It then follows that $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. Similarly, a simple computation of projective coordinates shows that $\mathcal{D}_0 \cap \mathcal{D}_i = \emptyset$ for $i = 1, 2$. It then follows that $\mathcal{B}_1 \cap \mathcal{B}_3 = \mathcal{B}_2 \cap \mathcal{B}_3 = \emptyset$ since $\mathcal{B}_3 = f(\mathcal{D}_0)$. Q.E.D.

By the lemma above, $\mathrm{card}(I(p)) \leq 2$ holds for all $p \in \bigcup_{i=0}^{3} \mathcal{B}_i$. Note that, moreover, the sets $\langle 1, 2\rangle$, $\langle 1, 3\rangle$, $\langle 2, 3\rangle$ are empty.

Lemma 4.10. *(a) $f(\mathcal{B}_i \setminus \mathcal{B}_0) \cap \mathcal{B}_3 = \emptyset$ and (b) $f(\mathcal{B}_0) \cap \mathcal{B}_i = \emptyset$ for $i = 1, 2$.*

Proof. (a) follows from the fact that $(\mathcal{B}_i \setminus \mathcal{B}_0) \cap \mathcal{B}_0 = \emptyset$. (b) is exactly what the previous lemma says. Q.E.D.

The next claim is a key to list up all the allowed transitions of points in $\Omega_{\mathcal{B}} \equiv \bigcap_{n \in \mathbb{Z}} f^n(\mathcal{B})$, where $\mathcal{B} \equiv \bigcup_{i=0}^{3} \mathcal{B}_i$.

Lemma 4.11. *$f(\langle 0\rangle \cap \Omega_{\mathcal{B}}) \cap \mathcal{B}_0 = \emptyset$.*

Proof. Let us assume that $x \in \langle 0\rangle \cap \Omega_{\mathcal{B}}$. Then, first $x \in \langle 0\rangle$ implies $x \in \mathcal{B}_0$ and $x \notin \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. On the other hand, by the invariance of $\Omega_{\mathcal{B}}$, $x \in \Omega_{\mathcal{B}}$ implies $x \in f(\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$. Since $\mathcal{B}_3 = f(\mathcal{B}_0)$, it follows that $x \in \mathcal{B}_0$, $x \notin \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ and $x \in f(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$. These conditions can be rewritten as $x \in \mathcal{D}_0$, $f^{-1}(x) \notin \mathcal{D}$ and $f^{-2}(x) \in \mathcal{D}$. We must show $f(x) \notin \mathcal{B}_0$ for such x. Hence the conclusion follows from the next Numerical Check 2 which is verified by the program `allowed.cc`. Q.E.D.

Numerical Check 2. *For any $x \notin \mathcal{D}$ such that $f(x) \in \mathcal{D}_0$ and $f^{-1}(x) \in \mathcal{D}$, we have $f^2(x) \notin \mathcal{D}_0$.*

Proposition 4.12. *Any allowed transition for a point in $\Omega_{\mathcal{B}}$ is one of the following: $\langle 0\rangle \to \langle 3\rangle$, $\langle 0,1\rangle \to \langle 0\rangle$, $\langle 0,1\rangle \to \langle 0,3\rangle$, $\langle 0,1\rangle \to \langle 3\rangle$, $\langle 1\rangle \to \langle 0\rangle$, $\langle 1\rangle \to \langle 0,1\rangle$, $\langle 1\rangle \to \langle 1\rangle$, $\langle 1\rangle \to \langle 0,2\rangle$, $\langle 1\rangle \to \langle 2\rangle$, $\langle 0,2\rangle \to \langle 0\rangle$, $\langle 0,2\rangle \to \langle 0,3\rangle$, $\langle 0,2\rangle \to \langle 3\rangle$, $\langle 2\rangle \to \langle 0\rangle$, $\langle 2\rangle \to \langle 0,1\rangle$, $\langle 2\rangle \to \langle 1\rangle$, $\langle 2\rangle \to \langle 0,2\rangle$, $\langle 2\rangle \to \langle 2\rangle$, $\langle 0,3\rangle \to \langle 0\rangle$, $\langle 0,3\rangle \to \langle 0,3\rangle$, $\langle 0,3\rangle \to \langle 3\rangle$, $\langle 3\rangle \to \langle 0\rangle$, $\langle 3\rangle \to \langle 0,1\rangle$, $\langle 3\rangle \to \langle 1\rangle$, $\langle 3\rangle \to \langle 0,2\rangle$ and $\langle 3\rangle \to \langle 2\rangle$ (thus there are 25 transitions).*

Proof. By the previous lemmas, the following transitions do not occur: $\langle 1\rangle \to \langle 3\rangle$, $\langle 1\rangle \to \langle 0,3\rangle$, $\langle 2\rangle \to \langle 3\rangle$, $\langle 2\rangle \to \langle 0,3\rangle$, $\langle 3\rangle \to \langle 3\rangle$, $\langle 3\rangle \to \langle 0,3\rangle$ (these follow from (a) of Lemma 4.10), $\langle 0\rangle \to \langle 0,1\rangle$, $\langle 0\rangle \to \langle 1\rangle$, $\langle 0\rangle \to \langle 0,2\rangle$, $\langle 0\rangle \to \langle 2\rangle$, $\langle 0,1\rangle \to \langle 0,1\rangle$, $\langle 0,1\rangle \to \langle 1\rangle$, $\langle 0,1\rangle \to \langle 0,2\rangle$, $\langle 0,1\rangle \to \langle 2\rangle$, $\langle 0,2\rangle \to \langle 0,1\rangle$, $\langle 0,2\rangle \to \langle 1\rangle$, $\langle 0,2\rangle \to \langle 0,2\rangle$, $\langle 0,2\rangle \to \langle 2\rangle$, $\langle 0,3\rangle \to \langle 0,1\rangle$, $\langle 0,3\rangle \to \langle 1\rangle$, $\langle 0,3\rangle \to \langle 0,2\rangle$, $\langle 0,3\rangle \to \langle 2\rangle$ (these follow from (b) of Lemma 4.10).

By the definition of $\mathcal{B}_3$ and $\langle 0\rangle$, the transition $\langle 0\rangle \to \langle 0\rangle$ does not occur. This is because $\langle 0\rangle \subset \mathcal{B}_0 \setminus \mathcal{B}_3$ and $f(\langle 0\rangle) \subset \mathcal{B}_3 \setminus f(\mathcal{B}_3) \subset \mathcal{B}_3$. By Lemma 4.11, the transition $\langle 0\rangle \to \langle 0,3\rangle$ does not occur. Thus, we finally get the list of all allowed transitions as above. Q.E.D.

Next we claim that $f : \mathcal{B}_i \to \mathcal{B}_j$ satisfies the (BCC) for some pairs of i and j. To do this, we need the next four Numerical Checks which are done by two programs `crossed-f.cc` and `crossed-i.cc`.

Numerical Check 3A. $f : \mathcal{D}_i \to \mathcal{D}_j$ $(1 \le i, j \le 2)$ *satisfies the (BCC) of degree one.*

Numerical Check 3B. $f^2 : \mathcal{D}_i \to \mathcal{D}_0$ $(i = 1, 2)$ *satisfies the (BCC) of degree three.*

Numerical Check 3C. $f : \mathcal{D}_0 \to \mathcal{D}_i$ $(i = 1, 2)$ *satisfies the (BCC) of degree one.*

Numerical Check 3D. $f^2 : \mathcal{D}_0 \to \mathcal{D}_0$ *satisfies the (BCC) of degree three.*

Lemma 4.13. *The following transitions: $\mathcal{B}_0 \to \mathcal{B}_3$, $\mathcal{B}_1 \to \mathcal{B}_0$, $\mathcal{B}_1 \to \mathcal{B}_1$, $\mathcal{B}_1 \to \mathcal{B}_2$, $\mathcal{B}_2 \to \mathcal{B}_0$, $\mathcal{B}_2 \to \mathcal{B}_1$, $\mathcal{B}_2 \to \mathcal{B}_2$, $\mathcal{B}_3 \to \mathcal{B}_0$, $\mathcal{B}_3 \to \mathcal{B}_1$ and $\mathcal{B}_3 \to \mathcal{B}_2$ satisfy the (BCC).*

Proof. We will analyze each transition in order.

1. $f : \mathcal{B}_0 \to \mathcal{B}_3$.
Recall that, when we defined $\mathcal{B}_3$, we fatten $\mathcal{D}_0$ slightly to the vertical direction and shrink slightly to the horizontal direction. Thus, the map $f : \mathcal{B}_0 \to \mathcal{B}_3$ automatically satisfies the (BCC) of degree one.

2. $f : \mathcal{B}_i \to \mathcal{B}_j$ *(*$1 \le i, j \le 2$*).*
By Numerical Check 3A, we see that $f : f(\mathcal{D}_i) \to f(\mathcal{D}_j)$ $(1 \le i, j \le 2)$ satisfies the (BCC). This means that $f : \mathcal{B}_i \to \mathcal{B}_j$ satisfies the (BCC) of degree one.

3. $f : \mathcal{B}_i \to \mathcal{B}_0$ *(*$i = 1, 2$*).*
By Numerical Check 3B, we see that $f : f(\mathcal{D}_i) \to \mathcal{D}_0$ $(i = 1, 2)$ satisfies the (BCC). This means that $f : \mathcal{B}_i \to \mathcal{B}_0$ $(i = 1, 2)$ satisfies the (BCC) of degree three.

4. $f : \mathcal{B}_3 \to \mathcal{B}_i$ *(*$i = 1, 2$*).*
By Numerical Check 3C, we see that $f : f(\mathcal{D}_0) \to f(\mathcal{D}_i)$ $(i = 1, 2)$ satisfies the (BCC). This means that $f : \mathcal{B}_3 \to \mathcal{B}_i$ satisfies the (BCC) of degree one.

5. $f : \mathcal{B}_3 \to \mathcal{B}_0$.
By Numerical Check 3D, we see that $f : f(\mathcal{D}_0) \to \mathcal{D}_0$ satisfies the (BCC). This means that $f : \mathcal{B}_3 \to \mathcal{B}_0$ satisfies the (BCC) of degree three. Q.E.D.

Next we define four Poincaré boxes $\{\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$. Let $H_0 \equiv \Delta_x(-1.11275; 0.105)$ and let $\mathcal{A}_0 \equiv (D_0 \setminus H_0) \times_P D'_y$, where the product is with respect to the projective coordinates for $\mathcal{B}_0 = \mathcal{D}_0$. We also define $\mathcal{A}_3 \equiv f(\mathcal{A}_0)$, i.e. the hole of $\mathcal{A}_3$ is the image of $H_0 \times_P D'_y$ by f.

Lemma 4.14. *The following transitions:* $\mathcal{A}_0 \to \mathcal{A}_3$*,* $\mathcal{A}_1 \to \mathcal{A}_0$*,* $\mathcal{A}_1 \to \mathcal{A}_1$*,* $\mathcal{A}_1 \to \mathcal{A}_2$*,* $\mathcal{A}_2 \to \mathcal{A}_0$*,* $\mathcal{A}_2 \to \mathcal{A}_1$*,* $\mathcal{A}_2 \to \mathcal{A}_2$*,* $\mathcal{A}_3 \to \mathcal{A}_0$*,* $\mathcal{A}_3 \to \mathcal{A}_1$ *and* $\mathcal{A}_3 \to \mathcal{A}_2$ *satisfy the (BCC).*

Proof. Thanks to Lemma 4.13, the transitions we have to care are $\mathcal{A}_0 \to \mathcal{A}_3$ and $\mathcal{A}_3 \to \mathcal{A}_0$. Since we defined the hole of $\mathcal{A}_3$ to be $f(H_0 \times_P D'_y)$, we only need to see that the hole of $\mathcal{A}_3$ is mapped into the hole of $\mathcal{A}_0$. This means that we have to check that $H_0 \times_P D'_y$ is mapped into itself by f^2. For this, we employ computer assistance again. Before stating the rigorous result, let us show below some analytic pre–estimate.

There are attractive periodic points of period two: one is $p_1 = (0.0622, -1.1252) \in \mathcal{B}_3$ and the other is $p_2 = (-1.1252, 0.0622) \in \mathcal{B}_0$. The diameter of $\mathcal{B}_3$ in the u-coordinate direction is approximately

$$\Delta y \approx \mathrm{diam}(D'_y)|b|/|D^v f(p_2)| = 2 \times 1.05 \times 0.2/(3 \times (-1.1252)^2) < 0.12,$$

where $D^v f$ means the derivative in the v-direction. Let r be the radius in the v-direction of the hole in $\mathcal{B}_0$ containing p_2. Then the (BCC) is satisfied if

$$r|D^v f(p_2)||D^v f(p_1)| + |b|\Delta y < r.$$

This inequality is transfered to $r \times 3.80 \times 0.01 + 0.024 < r$ and this is satisfied when $r \ge 0.03$. In fact, by taking $H_0 \equiv \Delta_x(-1.11275; 0.105)$, we rigorously obtain the following claim by using a computer program called `hole.cc`.

Numerical Check 4. $\pi_u \circ f^2(\partial H_0 \times_P D'_y) \subset \mathrm{int} H_0$.

Here, recall that π_u means the projection to the u-coordinate direction in $\mathcal{B}_0$. By combining this fact and Numerical Check 3C, we know that $\mathcal{A}_3 \to \mathcal{A}_0$ satisfies the (BCC). Thus, we are done. Q.E.D.

Lemma 4.15. *The transitions $\mathcal{A}_1 \to \mathcal{A}_0$, $\mathcal{A}_2 \to \mathcal{A}_0$ and $\mathcal{A}_3 \to \mathcal{A}_0$ satisfy the (OCC).*

Proof. We remark that the other transitions except for the three above are of degree one, so we do not need to check the (OCC).

Let us first show that $f : \mathcal{A}_3 \to \mathcal{A}_0$ satisfies the (OCC). Since we have defined as $\mathcal{A}_3 = f(\mathcal{A}_0)$, it is sufficent to see that $f^2 : \mathcal{A}_0 \to \mathcal{A}_0$ satisfies the (OCC). To do this, first take the intersection γ_{y_0} of the vertical boundary $\partial_v(H_0 \times_P D'_y)$ and $\{y = y_0\}$ for each $y_0 \in D'_y$. Note that γ_{y_0} is homeomorphic to a circle. If we can check that $\pi_u(f^2(\gamma_{y_0}))$ rounds around an appropriate point three times for all $y_0 \in D'_y$, then we know that $f^2 : \mathcal{A}_0 \to \mathcal{A}_0$ satisfies the (OCC).

In fact, the next Numerical Check 5A can be verified by employing the program `sign.cc`. Given a closed curve γ in $\mathbb{C}$ and a point $\alpha \in \mathbb{C}$, we let $\mathrm{Wind}(\alpha, \gamma)$ be the winding number of γ with respect to α.

Numerical Check 5A. *Take any $y_0 \in D'_y$. Then, for $\gamma = \gamma_{y_0} \equiv \partial H_0 \times_P \{y_0\}$ we have*

(i) $\pi_u \circ f^2(\gamma) \subset \mathrm{int} H_0$, *and*
(ii) $\mathrm{Wind}(-1.154, \pi_u \circ f^2(\gamma)) = 3$.

By Numerical Check 3D, the number of critical points of $\pi_u \circ f|_{D_0 \times_P \{y_0\}}$ for each $y_0 \in D'_y$ is $3 - 1 = 2$. The condition (ii) of Numerical Check 5A says that there are two critical points inside γ, so there is no more critical points outside. The condition (i) says that the two critical values are in $\mathrm{int} H_0$. In particular, this implies that

$$\pi_u \circ f(\mathrm{Crit}(\pi_x \circ f|_{\mathcal{B}_3})) \subset \mathrm{int} H_0$$

is satisfied. Note that the condition (i) above follows from Numerical Check 4.

Next, we show that $f : \mathcal{A}_i \to \mathcal{A}_0$ $(i = 1, 2)$ satisfies the (OCC). For this, let us first present the following rough estimate. Assume that, for simplicity, the horizontal coordinate of $\mathcal{B}_i$ $(i = 1, 2, 3)$ induced by f from $\mathcal{D}_i$ $(i = 1, 2)$ or from $\mathcal{B}_0$ is close to the Euclidean horizontal coordinate. Then, the dynamical critical set in $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ is close to the slice of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ by the y-axis $\{x = 0\}$. This slice consists of three disks. Their centers form the vertices of a triangle and have distance approximately $|a|^{1/3} = (1.35)^{1/3} \approx 1.1$ from the origin. The radius of each disk is approximately $R \times |b|/(3 \times 1.1^2) \approx 1.5 \times 0.2/(3 \times 1.1^2) < 0.1$. In particular, the distance from the real part of the slice of $\mathcal{B}_1$ and $\mathcal{B}_2$ to the origin is approximately $1.1/2 = 0.55$. When we map this by $\pi_u \circ f$, the real part of the images of the slice is close to $a - by = -1.35 - 0.2 \times 0.55 = -1.46$, since the radius of the disk in the image by f is about $0.1 \times |b| = 0.02$ which is small. If the images of the three disks do not have intersection with D_x, then the (OCC) for $\mathcal{A}_i \to \mathcal{A}_0$ $(i = 1, 2)$ follows.

In fact, by using the program `newton.cc` which combines Newton's method with the winding number argument as in Subsection 4.2, we get the following rigorous claim, and the estimate above turns out to be quite accurate. The part of Newton's method computes an approximate position of $\alpha_{y_0,i}$ in the next Numerical Check 5B, which is a zero of $\frac{\partial}{\partial x}(\pi_u \circ f^2|_{D_i \times_P \{y_0\}})$ where $y_0 \in D_y$.

Numerical Check 5B. *Take any $i = 1, 2$ and $y_0 \in D_y$. Then, we can find $\alpha = \alpha_{y_0,i} \in D_i$ so that $\gamma = \gamma_{y_0,i} \equiv \partial\Delta_x(\alpha; 0.04) \times_P \{y_0\} \subset D_i \times_P \{y_0\}$ and*

(i) $\pi_u \circ f^2(\gamma) \subset \{x \in \mathbb{C} : \mathrm{Re}\, x < -1.425\}$, *and*
(ii) $\mathrm{Wind}(\pi_u \circ f^2(\alpha, y_0), \pi_u \circ f^2(\gamma)) = 3$.

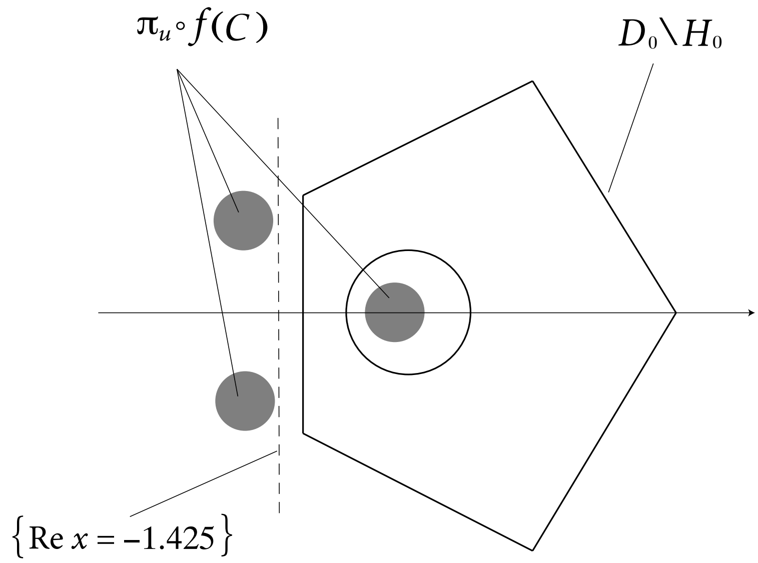

Figure 9. $\pi_u \circ f(\mathcal{C})$ and $D_0 \setminus H_0$.

By Numerical Check 3B, we know that the number of critical points of $\pi_u \circ f^2|_{D_i \times_P \{y_0\}}$ for each $y_0 \in D_y$ is $3 - 1 = 2$. The condition (ii) of Numerical Check 5B says that there are two critical points inside γ, and there is no more critical points outside. The condition (i) of Numerical Check 5B says that the two critical values are contained in $\{x \in \mathbb{C} : \mathrm{Re}\, x < -1.425\}$. Recall that $D_x \cap \{x \in \mathbb{C} : \mathrm{Re}\, x < -1.425\} = \emptyset$. In particular, we have

$$\pi_u \circ f(\mathrm{Crit}(\pi_u \circ f|_{\mathcal{B}_1 \cup \mathcal{B}_2})) \cap \overline{D_x} = \emptyset.$$

Figure 9 above describes the relative position of $\pi_u \circ f(\mathcal{C})$ with respect to $D_0 \setminus H_0$, where $\mathcal{C} = \mathrm{Crit}(\pi_u \circ f|_{\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3})$.

Here, the formulae of the Newton's method for $\pi_u \circ f^2|_{D_i \times_P \{y_0\}}$ are given as follows. Fix $y \in \Delta'_y$ and consider

$$\begin{aligned}\begin{pmatrix} p(x,y) \\ q(x,y) \end{pmatrix} &\equiv f^2 \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x^9 + 3(by-a)x^6 + 3(by-a)^2x^3 - bx + b^3y^3 - 3b^2ay^2 + 3ba^2y - a^3 + a \\ -x^3 + a - by \end{pmatrix}.\end{aligned}$$

By differentiating each coordinate successively , we get

$$\begin{aligned}\begin{pmatrix} dp(x,y) \\ dq(x,y) \end{pmatrix} &\equiv \frac{\partial}{\partial x} f^2 \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 9x^8 + 18(by-a)x^5 + 9(by-a)^2x^2 - b \\ -3x^2 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} ddp(x,y) \\ ddq(x,y) \end{pmatrix} &\equiv \frac{\partial^2}{\partial x^2} f^2 \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 72x^7 + 90(by-a)x^4 + 18(by-a)^2x \\ -6x \end{pmatrix}.\end{aligned}$$

Let $u = (u_x, u_y) = (-1.763356785556, 13.753270977536)$ be one of the two focuses of $\mathcal{D}_0$. For each fixed $y_0 \in D_y$ we try to find the critical points of $\pi_u \circ f^2|_{D_i \times_P \{y_0\}}$, i.e.

$$\mathbb{C} \ni x \longmapsto \frac{p(x,y_0) - u_x}{q(x,y_0) - u_y}(-u_y) + u_x.$$

That is, we search for the zeros of its derivative:

$$N(x) \equiv (-u_y)\frac{dp(q-u_y) - dq(p-u_x)}{(q-u_y)^2}$$

of the map above by Newton's method. To do this, we differentiate it once more to get

$$N'(x) = (-u_y)\frac{ddp(q-u_y)^2 - ddq(p-u_x)(q-u_y) - 2dpdq(q-u_y) + 2dqdq(p-u_x)}{(q-u_y)^3}.$$

Thus, the map to iterate is given by

$$\begin{aligned} x \longmapsto\ & x - \frac{N(x)}{N'(x)} \\ =\ & x - \frac{dp(q-u_y)^2 - dq(p-u_x)(q-u_y)}{ddp(q-u_y)^2 - ddq(p-u_x)(q-u_y) - 2dpdq(q-u_y) + 2dqdq(p-u_x)}\end{aligned}$$

By using this formula, we get an approximate position of $\alpha = \alpha_{y_0,i}$.

Next we draw a circle of radius 0.04 centered at α in $D_i \times_P \{y_0\}$, and see how many times its image rounds around $\pi_u \circ f^2(\alpha, y_0)$. In this way we can verify Numerical Check 5B. This proves Lemma 4.15. Q.E.D.

By Lemma 4.14 and Lemma 4.15, we obtain the

Proposition 4.16. *$\mathcal{A}_0 \Rightarrow \mathcal{A}_3$, $\mathcal{A}_1 \Rightarrow \mathcal{A}_0$, $\mathcal{A}_1 \Rightarrow \mathcal{A}_1$, $\mathcal{A}_1 \Rightarrow \mathcal{A}_2$, $\mathcal{A}_2 \Rightarrow \mathcal{A}_0$, $\mathcal{A}_2 \Rightarrow \mathcal{A}_1$, $\mathcal{A}_2 \Rightarrow \mathcal{A}_2$, $\mathcal{A}_3 \Rightarrow \mathcal{A}_0$, $\mathcal{A}_3 \Rightarrow \mathcal{A}_1$, and $\mathcal{A}_3 \Rightarrow \mathcal{A}_2$.*

It is easy to see that any allowed transition listed in Proposition 4.12 satisfies the assumption of Proposition 4.3 by Proposition 4.16, so we have the

Corollary 4.17. *For the cubic Hénon map f, the pair $(\{C_p^{\cap}\}_{p \in \mathcal{A} \cap f^{-1}(\mathcal{A})}, \|\cdot\|_{\cap})$ forms an expanding cone field.*

To deduce hyperbolicity of f from this claim, our next task is to show $C_p^{\cap} \neq \emptyset$ for $p \in \Omega_{\mathcal{A}}$. For this, we use Proposition 4.16 and the invariance of the cone fields $\{C_p^{\mathcal{A}_i}\}_{p\in\mathcal{A}_i}$ and show $C_p^{\cap} \neq \emptyset$ only on certain proper subset of $\Omega_{\mathcal{A}}$ inductively.

Lemma 4.18. *With $i = 1, 2$, we have $C_p^{\mathcal{A}_0} \cap C_p^{\mathcal{A}_i} \neq \emptyset$ for all $p \in \mathcal{E}_i^1 \equiv \langle 0, i\rangle \cap f(\mathcal{A})$.*

Proof. By Proposition 4.12, either $f^{-1}(p) \in \mathcal{A}_1$, $f^{-1}(p) \in \mathcal{A}_2$ or $f^{-1}(p) \in \mathcal{A}_3$ holds for any $p \in \mathcal{E}_i^1 = \langle 0, i\rangle \cap f(\mathcal{A})$. Then the conclusion follows because either $Df(C_{f^{-1}(p)}^{\mathcal{A}_1}) \subset C_p^{\mathcal{A}_0} \cap C_p^{\mathcal{A}_i}$ (since $\mathcal{A}_1 \Rightarrow \mathcal{A}_0$ and $\mathcal{A}_1 \Rightarrow \mathcal{A}_i$), $Df(C_{f^{-1}(p)}^{\mathcal{A}_2}) \subset C_p^{\mathcal{A}_0} \cap C_p^{\mathcal{A}_i}$ (since $\mathcal{A}_2 \Rightarrow \mathcal{A}_0$ and $\mathcal{A}_2 \Rightarrow \mathcal{A}_i$) or $Df(C_{f^{-1}(p)}^{\mathcal{A}_3}) \subset C_p^{\mathcal{A}_0} \cap C_p^{\mathcal{A}_i}$ (since $\mathcal{A}_3 \Rightarrow \mathcal{A}_0$ and $\mathcal{A}_3 \Rightarrow \mathcal{A}_i$) holds depending on whether $f^{-1}(p) \in \mathcal{A}_1$, $f^{-1}(p) \in \mathcal{A}_2$ or $f^{-1}(p) \in \mathcal{A}_3$. Q.E.D.

From this lemma and the invariance of the cone fields, we obtain that $C_p^{\mathcal{A}_0} \cap C_p^{\mathcal{A}_3} \neq \emptyset$ for $p \in \mathcal{E}_i^2 \equiv \langle 0, 3\rangle \cap f(\mathcal{E}_i^1)$. Inductively,

Lemma 4.19. *We have $C_p^{\mathcal{A}_0} \cap C_p^{\mathcal{A}_3} \neq \emptyset$ for $p \in \mathcal{E}_i^{n+1} \equiv \langle 0, 3\rangle \cap f(\mathcal{E}_i^n)$, where $i = 1, 2$ and $n = 1, 2, \cdots$.*

Proof. For $p \in \mathcal{E}_i^2$, we have $q = f^{-1}(p) \in \mathcal{E}_i^1$. Since $\mathcal{A}_0 \Rightarrow \mathcal{A}_3$ and $\mathcal{A}_i \Rightarrow \mathcal{A}_0$, we have $Df(C_q^{\mathcal{A}_0} \cap C_q^{\mathcal{A}_i}) \subset C_p^{\mathcal{A}_0} \cap C_p^{\mathcal{A}_3}$. By Lemma 4.18, we see $C_q^{\mathcal{A}_0} \cap C_q^{\mathcal{A}_3} \neq \emptyset$ for $q \in \mathcal{E}_i^1$. This proves the claim for the case $n = 1$. The proof for general case is similar. Q.E.D.

Now, the next task is to define a non–empty cone field on

$$(\Omega_{\mathcal{A}} \cap \langle 0, 3\rangle) \setminus \bigcup_{i=1,2} \bigcup_{n\geq 1} \mathcal{E}_i^n.$$

This set consists of the points in $\Omega_{\mathcal{A}} \cap \langle 0, 3\rangle$ whose backward orbits remain in $\langle 0, 3\rangle$. Take a point p from the set above such that there exists the smallest $N > 0$ with $f^N(p) \notin \langle 0, 3\rangle$. Then, we construct a new cone $C_p^{\cap}$ by "pulling–back and shrinking" the cone $C_{f^N(p)}^{\cap}$. That is, we define $C_p^{\cap}$ as a subcone of $Df^{-N}(C_{f^N(p)}^{\cap})$ such that $C_p^{\cap}$ does not converge to the entire $T_q\mathbb{C}^2$ when p converges to $\bigcap_{n\in\mathbb{Z}} f^n(\langle 0, 3\rangle)$. We also define the norm $\|v_p\|_{\cap}$ to be smaller than $\lambda^{-N}\|Df^N(v_p)\|_{\cap}$ for $v_p \in C_p^{\cap}$ so that $\|v_p\|_{\cap}$ does not diverge when p converges to $\bigcap_{n\in\mathbb{Z}} f^n(\langle 0, 3\rangle))$. By construction, this defines an expanding cone field.

So far, the remaining question is how to define expanding/contracting cone fields on $\bigcap_{n\in\mathbb{Z}} f^n(\langle 0, 3\rangle)$. Notice that $\bigcap_{n\in\mathbb{Z}} f^n(\langle 0, 3\rangle)$ is completely invariant. We again abandon the cone $C_p^{\cap} = C_p^{\mathcal{A}_0} \cap C_p^{\mathcal{A}_3}$ on this set and define a new cone field. More precisely, in certain coordinate, we try to find a bidisk $\mathcal{V} \supset \bigcap_{n\in\mathbb{Z}} f^n(\langle 0, 3\rangle)$ and prove that $f : \mathcal{V} \to \mathcal{V}$ is a crossed mapping of degree one. In fact, we let $\mathcal{V} = V_x \times_P V_y$ be a projective bidisk given by $v = (0.84901\infty, -0.52838\infty)$ (the unstable direction of the unique saddle fixed point in the third quadrant), $u = (0.40838\infty, 0.84901\infty)$, $L_v = \{x = -0.71\}$, $L_u = \{y = -0.76\}$, $V_x = \Delta(-0.71; 0.55)$ and $V_y = \Delta(-0.76; 0.3)$. When we write $u = (\alpha\infty, \beta\infty)$, the complex lines parallel to the v–axis in the projective coordinate are defined to be $\{\alpha y - \beta x = \gamma\}$ for $\gamma \in \mathbb{C}$.

Numerical Check 6. *With this definition, it can be checked that*

(i) $\mathcal{V} \supset \bigcap_{|n|\le 2} f^n(\mathcal{D}_0)$*, and*

(ii) $f : \mathcal{V} \to \mathcal{V}$ *is a crossed mapping of degree one.*

The assertion (i) above is verified by the program `bidisk.cc` and the second one (ii) is verified by `saddle-f.cc` and `saddle-i.cc`.

Thus, we redefine $C_p^{\cap} \equiv C_p^{\mathcal{V}}$ and $\|v\|_{\cap} \equiv \|v\|_{\mathcal{V}}$ on $\bigcap_{n\in\mathbb{Z}} f^n(\langle 0,3\rangle)$.

Remark 4.20. *From Numerical Check 6 above, it then turns out that* $\bigcap_{n\in\mathbb{Z}} f^n(\langle 0,3\rangle)$ *consists of a single saddle fixed point.*

By Lemma 4.18, Lemma 4.19 and Numerical Check 6, we conclude

Corollary 4.21. $C_p^{\cap}$ *is non–empty for all* $p \in \Omega_{\mathcal{A}}$*.*

Since $f : \mathcal{A}_i \to \mathcal{A}_0$ is a map of horseshoe type for $i = 1, 2$ by Numerical Check 5B, we have the following consequence.

Corollary 4.22. $\Omega_{\mathcal{A}}$ *is not connected.*

We know that f has an attractive cycle of period two by Kobayashi hyperbolicity of the holes, i.e. $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{B}) \setminus \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A}) = \{\text{the attractive cycles}\}$. By Proposition 4.8, it follows that $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{B}) = \Omega_{\mathcal{B}} = K$. Thus, we have $K = \Omega_{\mathcal{A}} \cup \{\text{the attractive cycles}\}$. Since we know that f is hyperbolic on $\Omega_{\mathcal{A}}$ by Corollary 4.17 and Corollary 4.21, we conclude that f is hyperbolic on K, and that $\Omega_{\mathcal{A}} = J$.

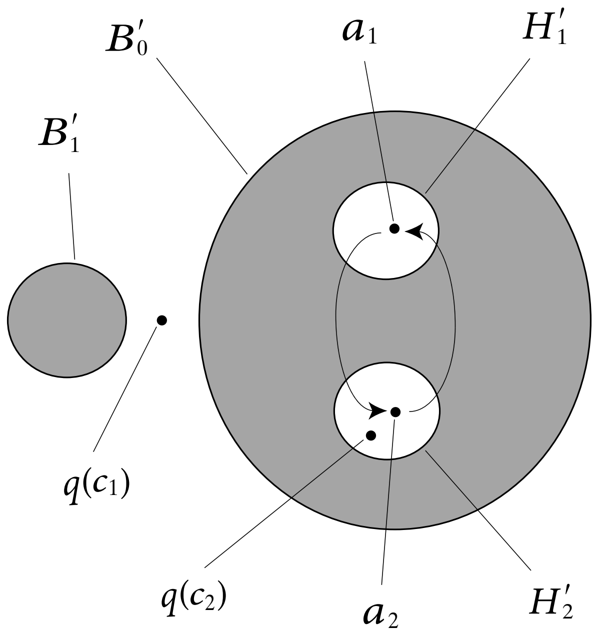

Figure 10. Disks for $q(x)$ with its attractive cycle $\{a_1, a_2\}$.

4.5. **Proof of Theorem A.** Our task here is to show that the cubic Hénon map f under consideration is not topologically conjugate on J to a small perturbation of any expanding polynomial in one variable in order to finish the proof of Theorem A. Moreover, the tools to show the hyperbolicity gives combinatorial description of the topology of its Julia set as in Theorem 4.23. To do this, we decompose the Julia set of f combinatorially as in Subsection 3.4 and analyze how they are glued together.

Proof of Theorem A. Assume that f is topologically conjugate on its Julia set J to a small perturbation g of some expanding polynomial $q(x)$ in one variable, i.e. $g = f_{q,b}$ where b is sufficiently close to zero. Then $q(x)$ would be cubic by comparing their entropies. Since f has an attractive cycle of period two, q ought to have a unique attractive cycle $\{a_1, a_2\}$ of period two as well by comparing the number of periodic points.

Let c_1 and c_2 be the critical points of q. If both $q^n(c_1)$ and $q^n(c_2)$ diverge to infinity, then q can not have attractive cycles. Thus, this is not the case. If both $q^n(c_1)$ and $q^n(c_2)$ converge to the attractive cycle, it then follows that the Julia set of of g, thus of f is connected, which contradicts to Corollary 4.22. So, exactly one of these two orbits has to converges to the attractive cycle.

Assume that $q^n(c_1)$ converges to the attractive cycle $\{a_1, a_2\}$ of period two. Then, there exist two disjoint topological open disks B_0^g and B_1^g such that (i) B_1^g contains the attractive cycle, (ii) $q(c_1) \in B_1^g$ and $q(c_2) \notin B_0^g \cup B_1^g$, (iii) $\mathrm{dist}(q(\partial B_0^g \cup \partial B_1^g), B_i^g) \geq \delta$ for some $\delta > 0$ for $i = 1, 2$. Moreover, one can find mutually disjoint two closed topological disks H_1^g and H_2^g in B_1^g such that (a) $a_1 \in \mathrm{int} H_1^g$ and $a_2 \in \mathrm{int} H_2^g$, (b) $q(c_1) \in \mathrm{int}(H_1^g \cup H_2^g)$, (c) $q(H_i^g) \subset \mathrm{int} H_j^g$ for $i \neq j$ (see Figure 10). Let us put $\mathcal{A}_0^g \equiv B_0^g \times \Delta_y(0; R)$ and $\mathcal{A}_1^g \equiv (B_1^g \setminus (H_1^g \cup H_2^g)) \times \Delta_y(0; R)$ for a sufficiently large $R > 0$. Then, by (iii) and (c), it follows that $g : \mathcal{A}_0^g \cup \mathcal{A}_1^g \to \mathcal{A}_0^g \cup \mathcal{A}_1^g$ satisfies the (BCC). Also by (ii) and (b), it follows that $g : \mathcal{A}_0^g \cup \mathcal{A}_1^g \to \mathcal{A}_0^g \cup \mathcal{A}_1^g$ satisfies the (OCC). Since $g : \mathcal{A}_0^g \to \mathcal{A}_0^g$ satisfies the (BCC) of degree one by (iii), we know that there is only one saddle fixed point s_2^g in $\mathcal{A}_0^g$. Thus, the other two fixed points s_0^g and s_1^g of g should belong to $\bigcap_{n \in \mathbb{Z}} g^n(\mathcal{A}_1^g)$. Note that $\bigcap_{n \in \mathbb{Z}} g^n(\mathcal{A}_1^g)$ is connected. In fact, $\bigcap_{n \in \mathbb{Z}} g^n(\mathcal{A}_1^g)$ is homeomorphic to the Julia set of a small perturbation of $p_{-1}(z) = z^2 - 1$ (see the proof of Theorem 6.1 at the end of Section 6 as well as [IS]). It follows that the connected component of J_g containing s_0^g and s_1^g consists of uncountably many points.

Next let us consider f. Let $\mathcal{A}^f = \bigcup_{i=0}^3 \mathcal{A}_i^f$, where $\mathcal{A}_i^f \equiv \mathcal{A}_i$ are the Poincaré boxes appeared in the proof of Theorem A. We know that $f : \mathcal{A}_i^f \to \mathcal{A}_i^f$ ($i = 1, 2$) satisfies the (BCC) of degree one, thus each $\mathcal{A}_i^f$ contains exactly one saddle fixed point s_i^f. Since $f^{-1} : \mathcal{A}_0^f \to \mathcal{A}_i^f$ is of horseshoe type, the connected component of $f^{-1}(\mathcal{A}^f) \cap \mathcal{A}^f$ containing s_i^f is equal to the connected component of $f^{-1}(\mathcal{A}_0^f \cup \mathcal{A}_1^f \cup \mathcal{A}_2^f) \cap \mathcal{A}^f$ containing s_i^f (see Figure 7). More generally, the connected component of $f^{-n}(\mathcal{A}^f) \cap \mathcal{A}^f$ containing s_i^f is equal to the connected component of $f^{-n}(\mathcal{A}_0^f \cup \mathcal{A}_1^f \cup \mathcal{A}_2^f) \cap \mathcal{A}^f$ containing s_i^f for $n \geq 0$. It then follows that the connected component of $\bigcap_{n \geq 0} f^{-n}(\mathcal{A}^f)$ containing s_i^f is equal to the connected component of $\bigcap_{n \geq 0} f^{-n}(\mathcal{A}_0^f \cup \mathcal{A}_1^f \cup \mathcal{A}_2^f) \cap \mathcal{A}^f$ containing s_i^f. This means

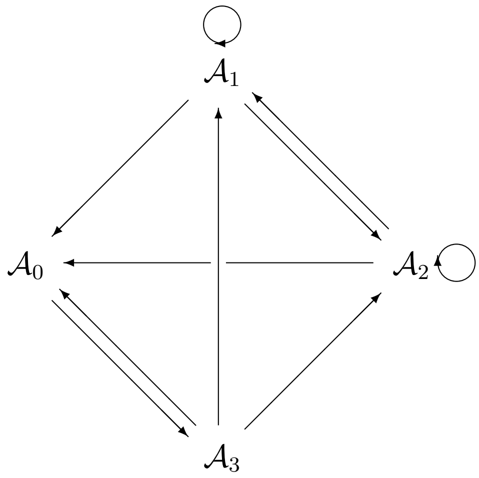

Diagram 1. The transition diagram for Σ_f.

that the connected component of $\bigcap_{n\geq 0} f^{-n}(\mathcal{A}^f)$ containing s_i^f forms a vertical disk of degree one in $\mathcal{A}_i^f$ for $i = 1, 2$. Similarly, since $f : \mathcal{A}_i^f \to \mathcal{A}_0^f$ is of horseshoe type, the connected component of $f(\mathcal{A}^f) \cap \mathcal{A}^f$ containing s_i^f is equal to the connected component of $f(\mathcal{A}_i^f) \cap \mathcal{A}^f$ containing s_i^f. More generally, the connected component of $f^n(\mathcal{A}^f) \cap \mathcal{A}^f$ containing s_i^f is equal to the connected component of $f^n(\mathcal{A}_i^f) \cap \mathcal{A}^f$ containing s_i^f. It then follows that the connected component of $\bigcap_{n\geq 0} f^n(\mathcal{A}^f)$ containing s_i^f is equal to the connected component of $\bigcap_{n\geq 0} f^n(\mathcal{A}_i^f) \cap \mathcal{A}^f$ containing s_i^f. This means that the connected component of $\bigcap_{n\geq 0} f^n(\mathcal{A}^f)$ containing s_i^f forms a horizontal disk of degree one in $\mathcal{A}_i^f$. Thus, the connected component of $J_f = \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A}^f)$ containing s_i^f is the intersection of a horizontal disk of degree one and the vertical disk of degree one in $\mathcal{A}_i^f$ which turns out to be exactly one point $\{s_i^f\}$ for $i = 1, 2$.

Since any conjugacy between f and g maps their fixed points as well as the connected components of the Julia sets containing them homeomorphically, we arrive at a contradiction. This finishes the proof of Theorem A. Q.E.D.

In the rest of this subsection, we investigate more carefully the topology of the Julia set J_f. As before, we decompose the Julia set J_f of the cubic Hénon map f by using the family of the Poincaré boxes $\{\mathcal{A}_0^f, \mathcal{A}_1^f, \mathcal{A}_2^f, \mathcal{A}_3^f\}$. Let $\Sigma_f \subset \{0, 1, 2, 3\}^{\mathbb{Z}}$ be the subshift of finite type defined by the allowed words $\{03, 10, 11, 12, 20, 21, 22, 30, 31, 32\}$ of length two. In other words, a symbol sequence belongs to Σ_f iff it contains the words 00, 01, 02, 13, 23, 33 nowhere. Compare with the transitions $\mathcal{A}_i^f \Rightarrow \mathcal{A}_j^f$ appeared in Proposition 4.16 and see the Diagram 1. As before, given $\underline{\varepsilon} \in \Sigma_f$, we set

$$J_{\underline{\varepsilon}}^f \equiv \cdots \cap f^2(\mathcal{A}_{\varepsilon_{-2}}^f) \cap f(\mathcal{A}_{\varepsilon_{-1}}^f) \cap \mathcal{A}_{\varepsilon_0}^f \cap f^{-1}(\mathcal{A}_{\varepsilon_1}^f) \cap f^{-2}(\mathcal{A}_{\varepsilon_2}^f) \cap \cdots .$$

Note that $\{J^f_{\underline{\varepsilon}}\}_{\underline{\varepsilon}\in\Sigma_f}$ are not mutually disjoint any more. In stead, we will consider a formal disjoint union $\bigsqcup_{\underline{\varepsilon}\in\Sigma_f} J^f_{\underline{\varepsilon}}$. Then, the inclusion map $\iota : J^f_{\underline{\varepsilon}} \to J_f$ naturally induces $\iota_f : \bigsqcup_{\underline{\varepsilon}\in\Sigma_f} J^f_{\underline{\varepsilon}} \to J_f$. Let $\underline{\alpha} \equiv \cdots 0303.0303\cdots$ and $\underline{\alpha}' \equiv \cdots 3030.3030\cdots$. By Numerical Check 6, we know that there is a unique saddle fixed point $s^f \in \mathcal{A}_0\cap\mathcal{A}_3$. The following describes topology of the Julia set combinatorially by gluing several pieces $J^f_{\underline{\varepsilon}}$ in terms of the map ι_f.

Theorem 4.23 (**Combinatorics of** J_f). *Consider the map:*

$$\iota_f : \bigsqcup_{\underline{\varepsilon}\in\Sigma_f} J^f_{\underline{\varepsilon}} \longrightarrow J_f.$$

(i) *ι_f is a continuous surjection and its restriction $\iota_f|_{J^f_{\underline{\varepsilon}}} : J^f_{\underline{\varepsilon}} \to J_f$ is injective for each $\underline{\varepsilon}\in\Sigma_f$.*

(ii) *We have $\mathrm{card}(\iota_f^{-1}(x)) \le 2$ for all $x\in J_f$. If $\mathrm{card}(\iota_f^{-1}(x)) = 2$, then $x\in W^s(s^f)$.*

(iii) (a) *$J^f_{\underline{\varepsilon}}$ is a solenoid if and only if $\underline{\varepsilon} = \underline{\alpha}$ or $\underline{\alpha}'$.*

(b) *$J^f_{\underline{\varepsilon}}$ consists of topological circles if and only if there exists $N\in\mathbb{Z}$ such that $\varepsilon_N\varepsilon_{N+1}\varepsilon_{N+2}\cdots = 0303\cdots$ and $\underline{\varepsilon}\neq\underline{\alpha}$, $\underline{\alpha}'$.*

(c) *For the other $\underline{\varepsilon}$, $J^f_{\underline{\varepsilon}}$ is either a Cantor set or finitely many points.*

(iv) *The identifications by ι_f occur only between the solenoids and the topological circles. That is, if $\iota_f(J^f_{\underline{\varepsilon}})\cap\iota_f(J^f_{\underline{\varepsilon}'})\neq\emptyset$ and $\underline{\varepsilon}\neq\underline{\varepsilon}'$, then $J^f_{\underline{\varepsilon}}$ and $J^f_{\underline{\varepsilon}'}$ are either*

(a) *two solenoids,*

(b) *a solenoid and topological circles, or*

(c) *topological circles.*

In the case (a), we have $\underline{\varepsilon} = \underline{\alpha}$ and $\underline{\varepsilon}' = \underline{\alpha}'$. In the cases (b) and (c), we have

$$\underline{\varepsilon} = \cdots(i^{(2)}_{k_2}\cdots i^{(2)}_1)(03)^{l_2}(i^{(1)}_{k_1}\cdots i^{(1)}_1)(03)^{l_1}\mathbf{0}(30)^\infty$$

and

$$\underline{\varepsilon}' = \cdots(i^{(2)}_{k_2}\cdots i^{(2)}_1)(03)^{l_2}(i^{(1)}_{k_1}\cdots i^{(1)}_1)(03)^{l_1}\mathbf{i}(03)^\infty$$

for some $0\le k_j\le\infty$ and $0\le l_j\le\infty$ (note that, if $l_1=\infty$ and $k_1=l_2=k_2=\cdots=0$, then $J^f_{\underline{\varepsilon}} = J^f_{\underline{\alpha}}$ becomes a solenoid and corresponds to the case (b)). Here, the decimal points can be placed anywhere in $\underline{\varepsilon}$ and $\underline{\varepsilon}'$ in such a way that the 0 in bold and the i in bold are in the same digit.

(v) *In the case (a) above, we have $\iota_f(J^f_{\underline{\alpha}})\cap\iota_f(J^f_{\underline{\alpha}'}) = \{s^f\}$. In particular,*

$$\iota_f|_{J^f_{\underline{\alpha}}\sqcup J^f_{\underline{\alpha}'}} : (J^f_{\underline{\alpha}}\sqcup J^f_{\underline{\alpha}'})\setminus\iota_f^{-1}(s^f) \longrightarrow \iota_f(J^f_{\underline{\alpha}}\sqcup J^f_{\underline{\alpha}'})\setminus\{s^f\}$$

is injective.

Proof. (iii) The proof of this claim is same as Lemma 3.9, thus omitted.

(i) We can easily check that for any allowed transition $\langle I_1\rangle\to\langle I_2\rangle$ in Proposition 4.16 there exist $i_1\in I_1$ and $i_2\in I_2$ such that the word i_1i_2 of length two is allowed in Σ_f. This shows the surjectivity of ι_f. The injectivity is trivial from the definition of ι_f.

(v) Since $\underline{\alpha} = \cdots 0303.0303 \cdots$ and $\underline{\alpha}' = \cdots 3030.3030 \cdots$, we see that $f(\iota_f(J^f_{\underline{\alpha}})) = \iota_f(J^f_{\underline{\alpha}'})$ and $f(\iota_f(J^f_{\underline{\alpha}'})) = \iota_f(J^f_{\underline{\alpha}})$. Take $x \in \iota_f(J^f_{\underline{\alpha}}) \cap \iota_f(J^f_{\underline{\alpha}'})$. Then, it follows that $f^n(x) \in \iota_f(J^f_{\underline{\alpha}}) \cap \iota_f(J^f_{\underline{\alpha}'}) \subset \langle 0, 3\rangle \equiv \mathcal{A}_0 \cap \mathcal{A}_3$ for all $n \in \mathbb{Z}$ by the invariance of the two solenoids. By the Numerical Check 6, we know that the only point which stays in $\langle 0, 3\rangle$ for all forward and backward iterates is s^f. Thus, $\iota_f(J^f_{\underline{\alpha}}) \cap \iota_f(J^f_{\underline{\alpha}'}) = \{s^f\}$

(ii) We first prove that, if $\mathrm{card}(\iota_f^{-1}(x)) \geq 2$, then $x \in W^s(s^f)$. Assume that two distinct points $\tilde{x}_1 \in J^f_{\underline{\varepsilon}}$ and $\tilde{x}_2 \in J^f_{\underline{\varepsilon}'}$ satisfy $x \equiv \iota_f(\tilde{x}_1) = \iota_f(\tilde{x}_2)$. Since the restriction $\iota_f|_{J^f_{\underline{\varepsilon}}}$ is injective, it follows that $\underline{\varepsilon} = \cdots \varepsilon_{-1}.\varepsilon_0\varepsilon_1 \cdots \neq \underline{\varepsilon}' = \cdots \varepsilon'_{-1}.\varepsilon'_0\varepsilon'_1 \cdots$. Evidently, $f^n(\iota_f(\tilde{x}_1))$ and $f^n(\iota_f(\tilde{x}_2)$ belong to the same Poincaré box for each $n \in \mathbb{Z}$. Since $\mathcal{A}_0 \cap \mathcal{A}_3 \neq \emptyset$ and $\mathcal{A}_0 \cap \mathcal{A}_i \neq \emptyset$ for $i = 1, 2$, and $\mathcal{A}_j \cap \mathcal{A}_k = \emptyset$ for the other choices of j and k, it follows that either (a) $\varepsilon_n = \varepsilon'_n$, (b) $\varepsilon_n = 0$ and $\varepsilon'_n = 3$, or (c) $\varepsilon_n = i$ and $\varepsilon'_n = 3$ $(i = 1, 2)$ for all $n \in \mathbb{Z}$. Then, either (b) or (c) holds for some n_0 because $\underline{\varepsilon} \neq \underline{\varepsilon}'$. Consider first the case (b) and assume that $\varepsilon_{n_0} = 0$ and $\varepsilon'_{n_0} = 3$. This means that $f^{n_0}(\iota_f(\tilde{x}_1)) = f^{n_0}(\iota_f(\tilde{x}_2)) \in \langle 0, 3\rangle = \mathcal{A}_0 \cap \mathcal{A}_3$. Here, recall Proposition 4.12 which says that the only allowed transitions from $\langle 0, 3\rangle$ are $\langle 0, 3\rangle \to \langle 0\rangle$, $\langle 0, 3\rangle \to \langle 0, 3\rangle$ and $\langle 0, 3\rangle \to \langle 3\rangle$. This means that neither ε_{n_0+1} nor ε'_{n_0+1} can be i $(i = 1, 2)$. By the Diagram 1, we conclude that the only possibility is $\varepsilon_{n_0+1} = 3$ and $\varepsilon'_{n_0+1} = 0$. This again means that $f^{n_0+1}(\iota_f(\tilde{x}_1)) = f^{n_0+1}(\iota_f(\tilde{x}_2)) \in \langle 0, 3\rangle$. By repeating this argument, we see that $f^m(x) = f^m(\iota_f(\tilde{x}_1)) = f^m(\iota_f(\tilde{x}_2)) \in \langle 0, 3\rangle$ for all $m \geq n_0$. This implies $x \in W^s(s^f)$ by Numerical Check 6. The argument for the case (c) is similar.

Next we prove $\mathrm{card}(\iota_f^{-1}(x)) \leq 2$ for all $x \in J_f$. Assume $\mathrm{card}(\iota_f^{-1}(x)) \geq 3$ for some $x \in J_f$. Put $\iota_f^{-1}(x) = \{\tilde{x}, \tilde{x}', \tilde{x}''\}$ and $\tilde{x} \in J^f_{\underline{\varepsilon}}$, $\tilde{x}' \in J^f_{\underline{\varepsilon}'}$ and $\tilde{x}'' \in J^f_{\underline{\varepsilon}''}$. The above argument shows that there exist N, N', $N'' \in \mathbb{Z}$ such that $\varepsilon_N\varepsilon_{N+1}\varepsilon_{N+2} \cdots = 0303 \cdots$, $\varepsilon'_{N'}\varepsilon'_{N'+1}\varepsilon'_{N'+2} \cdots = 0303 \cdots$ and $\varepsilon''_{N''}\varepsilon''_{N''+1}\varepsilon'_{N''+2} \cdots = 0303 \cdots$. Since $0303 \cdots$ is periodic of period 2 by the shift map, two out of these three symbol sequences have the same future itinerary. We may assume that there exists $M \in \mathbb{Z}$ such that $\varepsilon_M\varepsilon_{M+1}\varepsilon_{M+2} \cdots = \varepsilon'_M\varepsilon'_{M+1}\varepsilon'_{M+2} \cdots = 0303 \cdots$. Now, we look at Diagram 1. The only transition which comes to $\mathcal{A}_3$ is from $\mathcal{A}_0$ and the only transitions which comes to $\mathcal{A}_0$ are from $\mathcal{A}_3$ and from $\mathcal{A}_i$ $(i = 1, 2)$. Since $\underline{\varepsilon} \neq \underline{\varepsilon}'$, it follows that there exists $L \in \mathbb{Z}$ such that $\varepsilon_L\varepsilon_{L+1}\varepsilon_{L+2} = i0303 \cdots$ and $\varepsilon'_L\varepsilon'_{L+1}\varepsilon'_{L+2} = 30303 \cdots$. Since $\mathcal{A}_i \cap \mathcal{A}_3 = \emptyset$ for $i = 1, 2$, we see that $f^L(\iota_f(J^f_{\underline{\varepsilon}})) \cap f^L(\iota_f(J^f_{\underline{\varepsilon}'})) = \emptyset$. This is a contradiction, thus we are done.

(iv) The argument in the proof of (ii) shows that if $\iota_f(J^f_{\underline{\varepsilon}}) \cap \iota_f(J^f_{\underline{\varepsilon}'}) \neq \emptyset$ and $\underline{\varepsilon} \neq \underline{\varepsilon}'$, then there exist N, $N' \in \mathbb{Z}$ such that $\varepsilon_N\varepsilon_{N+1}\varepsilon_{N+2} \cdots = 0303 \cdots$ and $\varepsilon'_{N'}\varepsilon'_{N'+1}\varepsilon'_{N'+2} \cdots = 0303 \cdots$. By (iii), $J^f_{\underline{\varepsilon}}$ and $J^f_{\underline{\varepsilon}'}$ are either two solenoids, a solenoid and topological circles, or topological circles. In the case (a), we have $\underline{\varepsilon} = \underline{\alpha}$ and $\underline{\varepsilon}' = \underline{\alpha}'$ again by (iii). In the cases (b) and (c), we may assume that $\underline{\varepsilon} \neq \underline{\alpha}$, $\underline{\alpha}'$. Since the only transition to either $\mathcal{A}_0$ or to $\mathcal{A}_3$ (but not from $\mathcal{A}_0$ or from $\mathcal{A}_3$) is from $\mathcal{A}_i$ $(i = 1, 2)$, there exists $M \in \mathbb{Z}$ such that $\varepsilon'_M\varepsilon'_{M+1}\varepsilon'_{M+2} \cdots = i0303 \cdots$. Then, either $\varepsilon_M\varepsilon_{M+1}\varepsilon_{M+2} \cdots = 30303 \cdots$ or $\varepsilon_M\varepsilon_{M+1}\varepsilon_{M+2} \cdots = 03030 \cdots$. If $\varepsilon_M = 3$, then $f^M(\iota_f(J^f_{\underline{\varepsilon}})) \cap f^M(\iota_f(J^f_{\underline{\varepsilon}'})) = \emptyset$ because $\mathcal{A}_i \cap \mathcal{A}_3 = \emptyset$ for $i = 1, 2$, which is a contradiction. Thus, the former case does not happen and $\varepsilon_M\varepsilon_{M+1}\varepsilon_{M+2}\cdots = 03030\cdots$. In the same way, we see that $(\varepsilon_{M-1}, \varepsilon'_{M-1})$ is either $(3,3)$ or (i,i) $(i = 1,2)$. If $(\varepsilon_{M-1}, \varepsilon'_{M-1}) = (3,3)$, then $(\varepsilon_{M-2}, \varepsilon'_{M-2})$ should be $(0,0)$. If $(\varepsilon_{M-1}, \varepsilon'_{M-1}) = (i,i)$, then $(\varepsilon_{M-2}, \varepsilon'_{M-2})$ should be either $(3,3)$ or (j,j) $(j = 1,2)$. By continuing this argument, we obtain the desired expression for $\underline{\varepsilon}$ and $\underline{\varepsilon}'$. This proves Theorem 4.23. Q.E.D.

5. Applications to the Hénon Family in $\mathbb{R}^2$

This section is dedicated to study the dynamics of the Hénon family:

$$f = f_{c,b} : (x,y) \longmapsto (x^2 + c - by, x)$$

with real coefficients as a self–map of $\mathbb{R}^2$. In the real parameter space of such Hénon family, we define the *maximal entropy locus* $\mathcal{M}$ and the *hyperbolic horseshoe locus* $\mathcal{H}$ as in Introduction. The next subsection describes an algorithm to check if a given Hénon map is a hyperbolic horseshoe in $\mathbb{R}^2$. In Subsection 5.2, an algorithm which verifies if a given Hénon map defined on $\mathbb{R}^2$ has entropy strictly less than $\log 2$ is presented. Combining these, we give an outline of the proof of Theorem D in Subsection 5.3.

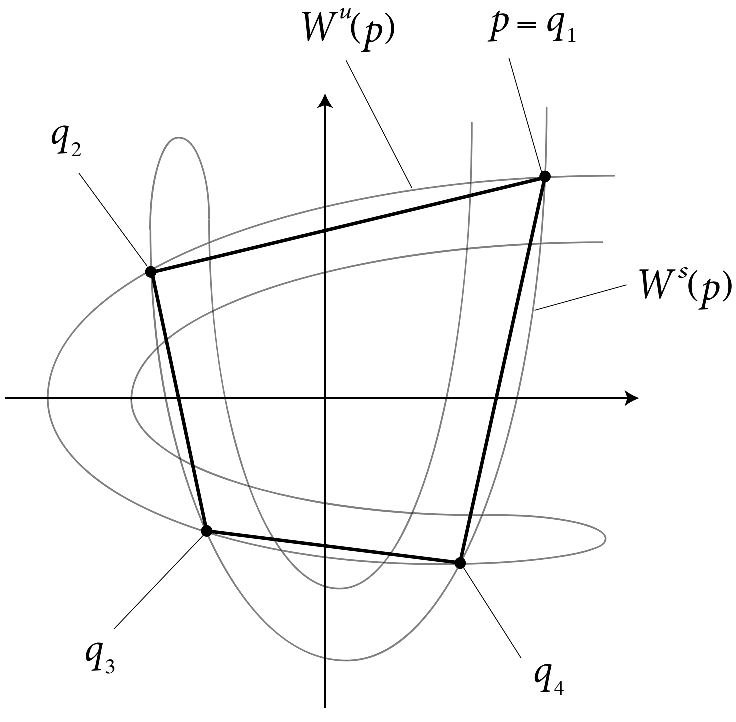

Figure 11. A trellis and a projective bidisk in $\mathbb{R}^2$.

5.1. **Hyperbolic horseshoes.** The algorithm to check a given real Hénon map to be a hyperbolic horseshoe in $\mathbb{R}^2$ is simply a combination of our hyperbolicity criteria combined with the idea of projective bidisks which are determined from the trellis formed by stable and unstable manifolds of the Hénon map in $\mathbb{R}^2$.

For simplicity, let us consider the map $f_{c,b}$ where $(c, b) = (-5.90, 1.00)$. Note that this parameter choice has appeared as c_1 in the list of Theorem D for $b = 1$. Then, there is a unique saddle fixed point p in the first quadrant. Consider the stable and the unstable manifolds $W^s(p)$ and $W^u(p)$ of p. One can find the four "most outer intersections" of these manifolds which we will denote by $p = q_1$, q_2, q_3 and q_4 (see Figure 11). Let u be the intersection of the two lines which contain the segments $\overline{q_2q_3}$ and $\overline{q_1q_4}$, and let L_u be the x–axis of $\mathbb{C}^2$. Let v be the intersection of the two lines which contain the segments $\overline{q_1q_2}$ and $\overline{q_3q_4}$, and let L_v be the y–axis of $\mathbb{C}^2$. These data define two projections π_u and π_v. We take a topological disk U_u in L_u which contains the segment $\pi_u(\overline{q_1q_2})$ and another topological disk U_v in L_v which contains the segment $\pi_v(\overline{q_2q_3})$. With these data, we define a projective bidisk by $\mathcal{B} \equiv U_u \times_P U_v$. By choosing appropriate U_u and U_v, one can prove with computer assistance that $f : \mathcal{B} \to \mathcal{B}$ satisfies the (BCC) and the (OCC) with respect to the projective coordinates. For other parameter choices $c < -5.90$, we find the four points as the "most outer intersections" of the invariant manifolds for the new map and continue the same procedure as before. This proves that $f_{c,b}|_{\mathbb{R}^2}$ is a hyperbolic horseshoe for $b = 1$ and all $c \leq -5.90$.

5.2. **Non–maximal entropy.** The algorithm to verify non–maximality of entropy of a given Hénon map defined on $\mathbb{R}^2$ relies on some ideas from the pluripotential theory (comparison of two pluricomplex Green functions, etc.) and a result in [BLS] combined with rigorous numerics. Below we may assume that $|b| \leq 1$ since any $f_{c,b}^{-1}$ is affinely conjugate to $f_{c',1/b}$ for some c'.

Let us put $R \equiv (1 + |b| + \sqrt{(1 + |b|)^2 + 4|c|})/2$, $V^+ = V_R^+ \equiv \{(x, y) \in \mathbb{C}^2 : |x| \geq R, |x| \geq |y|\}$ and $V^- = V_R^- \equiv \{(x, y) \in \mathbb{C}^2 : |y| \geq R, |y| \geq |x|\}$. We employ the norm $\|(x, y)\| \equiv \max\{|x|, |y|\}$ for $(x, y) \in \mathbb{C}^2$. Let $\mathcal{D}_R \equiv \Delta_x(0; R) \times \Delta_y(0; R) = \{(x, y) \in \mathbb{C}^2 : \|(x, y)\| < R\}$ be a bidisk.

In what follows we shall estimate

$$G(x, y) \equiv \max\{G^+(x, y), G^-(x, y)\}$$

from above and from below, where

$$G^{\pm}(x, y) \equiv \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|f^{\pm n}(x, y)\|$$

are the *Green functions* for $K^{\pm}$ and $\log^+ x = \max\{0, \log x\}$.

Lemma 5.1. *Let $|b| \leq 1$. For any $(x, y) \in \mathbb{C}^2$, we have*

$$G(x, y) \leq \log(2/|b|) + \max\{\log R, \log \|(x, y)\|\}.$$

Proof. By the definition of R, we see that $f(V_R^+) \subset V_R^+$. This and the estimate $\|f(x, y)\| = |x^2 + c - by| \leq |x|^2 + (1 + |b|)|x| + |c| \leq 2|x|^2 = 2\|(x, y)\|^2$ imply $\|f^n(x, y)\| \leq 2^{2^n-1}\|(x, y)\|^{2^n}$ for $(x, y) \in V_R^+$, so $G^+(x, y) \leq \log 2 + \log^+ \|(x, y)\|$ for all $(x, y) \in V_R^+$.

Because G^+ is plurisubharmonic and continuous on $\mathbb{C}^2$, we have $G^+(x,y) \leq \log 2 + \log R$ for all $(x,y) \in \mathcal{D}_R$ and $G^+(x,y) \leq \log 2 + \log|y|$ for all $(x,y) \in V_R^-$ by the Maximum Modulus Principle (to see this, we restrict G^+ to the complex line $\mathbb{C}_{y_0} = \{y = y_0\}$ to get a subharmonic function). This gives the desired bound for G^+.

For $G^-(x,y)$, we see $\|f^{-1}(x,y)\| \leq (2/|b|)\|(x,y)\|^2$ for $(x,y) \in V_R^-$ to obtain the conclusion. Q.E.D.

We next define

$$g_R(z) \equiv \log\left|\left(\sqrt{\frac{z-R}{z+R}}+1\right)\Big/\left(\sqrt{\frac{z-R}{z+R}}-1\right)\right|$$

for $z \in \mathbb{C}$, where the branch of the square root above is chosen so that $|\arg\sqrt{\cdot}\,| \leq \pi/2$. Notice that $g_R(z)$ is the Green function of the interval $[-R,R] \subset \mathbb{R} \subset \mathbb{C}$ in the potential theoretic sense. We hereafter use the notation $g(z) \equiv g_R(z)$.

Lemma 5.2. *If $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) = \log 2$, then we have*

$$G(x,y) \geq \max\{g(x), g(y)\}$$

for any $(x,y) \in \mathbb{C}^2$.

Proof. It is shown in [BS$\mathbb{C}$1] that $G(x,y)$ is the *pluricomplex Green function* for $K \equiv K^+ \cap K^-$, that is,

$$G(x,y) = \sup\{h(x,y) : h \text{ is p.s.h., } h|_K = 0 \text{ and } h(x,y) \leq \log^+\|(x,y)\| + C\}$$

(see the book [Kl] for more on this). First note that $\max\{g(x), g(y)\}$ is plurisubharmonic and satisfies $\max\{g(x), g(y)\} \leq \log^+\|(x,y)\| + C$ for some $C > 0$. If $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) = \log 2$, then $K \subset [-R,R] \times [-R,R] \subset \mathbb{R}^2$ by [BLS]. This implies that $\max\{g(x), g(y)\} = 0$ on K. Thus, we are done. Q.E.D.

Let $f^n(x,y) = (x_n, y_n)$. The next two lemmas generalize the previous estimates in terms of iterated points (x_n, y_n), which often gives sharper bounds for $G(x,y)$.

Lemma 5.3. *For any $(x,y) \in \mathbb{C}^2$ and $n \in \mathbb{Z}$, we have*

$$\frac{\log(2/|b|) + \max\{\log R, \log\|(x_n,y_n)\|, \log\|(x_{-n},y_{-n})\|\}}{2^{|n|}} \geq G(x,y).$$

Proof. Remark first that $G^\pm(f^n(x,y)) = 2^{\pm n}G^\pm(x,y)$. By applying Lemma 5.1 to (x_n,y_n) and (x_{-n},y_{-n}), we see

$$\log(2/|b|) + \max\{\log R, \log\|(x_n,y_n)\|\} \geq \max\{2^n G^+(x,y), 2^{-n}G^-(x,y)\}$$

and

$$\log(2/|b|) + \max\{\log R, \log\|(x_{-n},y_{-n})\|\} \geq \max\{2^{-n}G^+(x,y), 2^n G^-(x,y)\}$$

for any $n \in \mathbb{Z}$. When $n \geq 0$, these inequalities imply

$$\frac{\log(2/|b|) + \max\{\log R, \log\|(x_{-n},y_{-n})\|, \log\|(x_{-n},y_{-n})\|\}}{2^n} \geq \max\{G^+(x,y), G^-(x,y)\}$$

because $2^n \geq 2^{-n}$. The case $n \leq 0$ is similar. Q.E.D.

Lemma 5.4. *If* $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) = \log 2$*, then we have*

$$G(x, y) \geq \frac{\max\{g(x_m), g(y_m)\}}{2^{|m|}}$$

for any $(x, y) \in \mathbb{C}^2$ *and any* $m \in \mathbb{Z}$.

Proof. Suppose that $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) = \log 2$. By Lemma 5.2 we obtain

$$\begin{aligned} G(z) &= \max\{G^+(x, y), G^-(x, y)\} \\ &\geq \frac{\max\{2^m G^+(x, y), 2^{-m} G^-(x, y)\}}{2^m} \\ &= \frac{\max\{G^+(x_m, y_m), G^-(x_m, y_m)\}}{2^m} \\ &\geq \frac{\max\{g(x_m), g(y_m)\}}{2^m} \end{aligned}$$

if $m \geq 0$, because $2^m \geq 2^{-m}$. The case $m \leq 0$ is similar. Q.E.D.

The next criterion will be useful to confirm us that certain Hénon maps on $\mathbb{R}^2$ do not have maximal entropy.

Corollary 5.5 (Non–Maximal Entropy Criterion). *Let* $|b| \leq 1$. *The following two conditions are equivalent:*

(i) $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) < \log 2$,

(ii) *there exist* n, $m \in \mathbb{Z}$ *and* $(x, y) \in \mathbb{C}^2$ *such that*

$$\frac{\log(2/|b|) + \max\{\log R, \log \|(x_n, y_n)\|, \log \|(x_{-n}, y_{-n})\|\}}{2^{|n|}} < \frac{\max\{g(x_m), g(y_m)\}}{2^{|m|}}.$$

Proof. Suppose first that $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) = \log 2$. By combining the two previous lemmas, we see that

$$\frac{\log(2/|b|) + \max\{\log R, \log \|(x_n, y_n)\|, \log \|(x_{-n}, y_{-n})\|\}}{2^{|n|}} \geq \frac{\max\{g(x_m), g(y_m)\}}{2^{|m|}}.$$

for any n, $m \in \mathbb{Z}$ and $(x, y) \in \mathbb{C}^2$. Thus, (ii) implies (i).

Conversely, if $h_{\mathrm{top}}(f|_{\mathbb{R}^2}) < \log 2$, then there exists a point $(x, y) \in K \setminus \mathbb{R}^2$ (see the last section of [BLS]). This point (x, y) satisfies $f^i(x, y) \in \mathcal{D}_R$ for any $i \in \mathbb{Z}$, so the left–hand side of the inequality in (ii) tends to zero as $n \to \infty$. However, the right–hand side is strictly positive for $m = 0$ because (x, y) is not contained in $[-R, R] \times [-R, R] \subset \mathbb{R}^2$. This proves that (i) implies (ii). Q.E.D.

5.3. **Proof of Theorem D.** In this subsection, we explain how the previous two criteria will be implemented specifically. As we will see in the proof of Proposition 5.7, the verification of non–maximality of entropy involves many (typically more than 10) times iterate of a map. Moreover, the parameter of the map we will iterate will not be a single value but move in a small subset in the parameter space. Thus, we here need a serious use of interval arithmetic technique essentially. Since the implementation of the hyperbolic horseshoe algorithm is obvious, only the algorithm for the non–maximality of entropy is discussed here. An example of our result is

Theorem 5.6. *For $b = 1$ and $c \geq -5.699310678222$, we have $h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2}) < \log 2$.* [2]

To show this, we proceed as follows.

Proposition 5.7. *Let $U \equiv [-5.69 - 10^{-6}, -5.69 + 10^{-6}] \times [1 - 10^{-6}, 1 + 10^{-6}]$. Then, for all $(c, b) \in U$, we have $h_{\mathrm{top}}(f_{c,b}|_{\mathbb{R}^2}) < \log 2$.*

Proof. Let us set $f_0 \equiv f_{c_0,b_0}$, where $(c_0, b_0) = (-5.69, 1)$. We first compute (without using interval arithmetic) the position of the saddle fixed point of f_0 in the first quadrant of $\mathbb{R}^2$ and its unstable direction.

Next, take a disk of radius ≈ 0.0001 centered at the saddle and tangent to the unstable direction, and distribute 10000000 points in it. For each such point, we check (without interval arithmetic) if it satisfies the condition (ii) of Corollary 5.5 for f_0. As an example of such points, we find the point:

$$\begin{aligned}(x_0, y_0) = (&2.50725794266830481760 \cdots + i0.00240486218696596074 \cdots, \\ &3.42948343899702701165 \cdots + i0.00035808913230478455 \cdots).\end{aligned}$$

Now let $f \equiv f_{c,b}$, where the parameter (c, b) is no more a single point but taken to be the product set U. One can compute $(x_k, y_k) = f^k(x_0, y_0)$ for such map by using interval arithmetic. Note that since U is a product of intervals, the outputs for x_k and y_k also become intervals. Then, as inequalities between intervals (see Subsection 4.2), one sees for $n = 13$ and $m = 2$,

$$\begin{aligned}\frac{\log 2 + \max\{\log R, \log \|(x_n, y_n)\|, \log \|(x_{-n}, y_{-n})\|\}}{2^{|n|}} &\leq 0.00270958478835030574 \\ &< 0.00479440418560278959 \\ &\leq \frac{\max\{g(x_m), g(y_m)\}}{2^{|m|}}\end{aligned}$$

for all $(c, b) \in U$. This proves the claim. Q.E.D.

Proof of Theorem 5.6. Let $f_0 \equiv f_{c_0,b_0}$, where $(c_0, b_0) = (-5.69 + 10^{-6}, 1)$. Near the point (x_0, y_0) in the previous proposition, we distribute several points and try to find a new point (x_0, y_0) which satisfies (ii) of Corollary 5.5 for f_0 (but without using interval arithmetic here).

Now, let us set $f \equiv f_{c,b}$, where the parameter (c, b) is no more a single point but taken to be the interval:

$$[-5.69 + 10^{-6}, -5.69 + 2 \cdot 10^{-6}] \times \{1\}.$$

For the new (x_0, y_0) and this f, we verify the condition (ii) of Corollary 5.5 by using interval arithmetic.

[2] Zin Arai has announced that the Hénon map $f_{c,b}|_{\mathbb{R}^2}$ is a hyperbolic horseshoe for all $c < -5.69995$ when $b = 1.00$.

We continue this process until the value of c falls in a region where we already know that the entropy is less than $\log 2$ (see [BS$\mathbb{R}$1]). It is also possible to do this procedure to the other direction, i.e. the direction where c decreases. We observed that this procedure stops at $c = -5.699310678222$. This completes the proof. Q.E.D.

The values of c_2 for other choices of b as shown in Theorem D are obtained in the same way.

Remark 5.8. *The denominator $\sqrt{(z-R)/(z+R)} - 1$ in the formula of $g_R(z) = g(z)$ becomes very close to zero when $|z|$ is large, which may be a cause of round–off errors. To avoid this, during the above computation we should check if this value is not too close to zero.*

Remark 5.9. *The reason why our algorithm for c_2 does not work when $-0.4 < b < 0.3$ as in Theorem D is the following. First, the Jacobian determinant of f^{-1} is large when $|b|$ is small. This implies that the computation of a long backward orbit may increase the round–off error quite rapidly. Another reason is that, when $|b|$ becomes smaller, we observe that the "size" of $J \setminus \mathbb{R}^2$ becomes smaller in some sense so that it becomes more difficult to find a point (x_0, y_0) in $\mathbb{C}^2$ which satisfies (ii) of Corollary 5.5.*

6. Conjectures, Problems and Final Remarks

In this final section we collect several conjectures, open problems and remarks concerning the results discussed in the previous sections to conclude this paper. Recall that at the end of the Introduction, one conjecture (Conjecture 1) concerning the boundary of the hyperbolic horseshoe locus and the maximal entropy locus for the Hénon family on $\mathbb{R}^2$ has been already presented.

Another interesting question is to analyze several structures in the complex parameter space of the complex Hénon family (more generally, polynomial diffeomorphisms of a fixed degree). A first step towards this may be to study the "Mandelbrot set" i.e. the connectedness locus in the parameter space. However, the Julia sets constructed in Theorems A and B are disjoint. It is thus natural to ask the following

Problem 1. *Find a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$ with essentially two–dimensional dynamics whose Julia set is connected.*

Such a polynomial diffeomorphism will give us the first non-trivial example to which the theory of Bedford and Smillie [BS$\mathbb{C}$7] can apply, where they have defined the external rays and studied some combinatorial properties of connected hyperbolic Julia sets *à la* Douady and Hubbard.

The reason why we chose a cubic map in Theorem A is that we must require the vertical hight of the hole of $\mathcal{A}_3$ not to be too large. Recall the formula of the Hénon map. In its second coordinate, we have x. This means that the vertical hight of the hole in $\mathcal{A}_3$ is the same as the horizontal width of the hole in $\mathcal{A}_0$. In the first coordinate of the formula, we have the term by. Thus, in order to satisfy the "a hole into a hole" condition (i.e. the hole in $\mathcal{A}_3$ should be mapped into the hole of $\mathcal{A}_0$), the vertical hight of $\mathcal{A}_3$ times $|b|$ should be smaller than the horizontal width of the hole in $\mathcal{A}_0$. Note that, since the hole in $\mathcal{A}_0$ is contained in the Fatou–Bieberbach attractive basin of the attractive two–cycle, our chance to verify the "a hole into a hole" condition heavily depends on the shape of the Fatou–Bieberbach domain. We observe that, if the degree becomes larger, the vertical hight of the hole in the Poincaré box corresponding to $\mathcal{A}_3$ gets smaller and we have more chance to satisfy the "a hole into a hole" condition. In fact, for the quadratic case we failed to check the "a hole into a hole" condition because of the reason above. On the other hand, when the degree is large, the critical set roughly becomes the union of many small disks which form a circle–shape in the y–axis (note that, in the case of the map in Theorem A, the critical set approximately consists of three disks near the y-axis) and we have less chance for the (OCC) to hold (*cf.* Figure 9). Thus, we are led to ask the following

Problem 2. *Find a hyperbolic complex Hénon map of degree two with essentially two–dimensional dynamics. Can its Julia set be connected?*

Beside the examples of the maps presented by Oliva [Ol] which are conjectured to have hyperbolic and connected Julia sets, we find another good candidate of such complex Hénon map as follows.

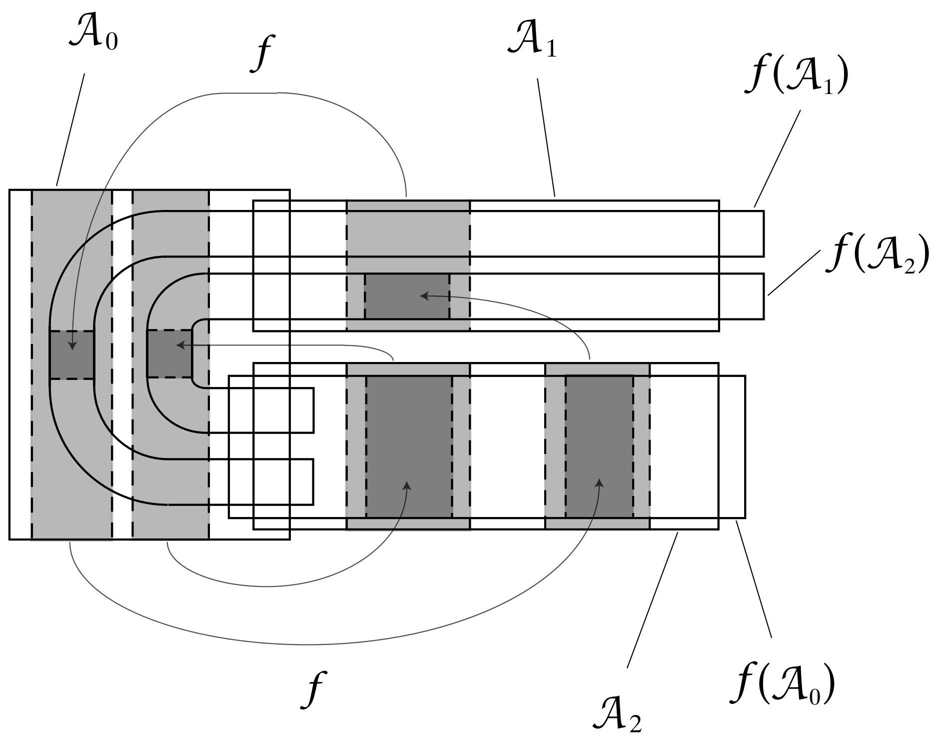

Figure 12. Poincaré boxes for the map in Conjecture 2.

Conjecture 2. *There exists a parameter value near* $(b, c) = (0.2, -1.325)$ *for which the corresponding quadratic Hénon map has a hyperbolic and connected Julia set. Moreover, it has two attractive cycles of period two and three.*

In fact we can construct a topological model for the Hénon map in Conjecture 2 in terms of three Poincaré boxes as in Figure 12. However, since the map in Conjecture 2 has a saddle periodic point of period three which is very close to the attractive 3–cycle, we guess that the verification of the (BCC) and the (OCC) would be hard.

We do not still know if the hyperbolic generalized Hénon map f_{p_0,b_0} constructed in Theorem B is not topologically conjugate to the projective limit of any expanding polynomial in one variable. However, here is one fact we can prove concerning this conjugacy problem.

Theorem 6.1. *If* f_{p_0,b_0} *in Theorem B is conjugate to the projective limit of some expanding polynomial* $q(x)$ *of one variable, then* q *is topologically conjugate to* p_0*.*

Thus, if the following conjecture holds, then f_{p_0,b_0} becomes the first example of essentially two–dimensional map which is shown to be hyperbolic without computer assistance.

Conjecture 3. *The map* f_{p_0,b_0} *in Theorem B is not topologically conjugate on its Julia set to a small perturbation* f_{p_0,b_1} *of the expanding polynomial* $p_0(x)$*.*

Even in the case that f_{p_0,b_0} in Theorem B is topologically conjugate to a small perturbation f_{p_0,b_1} of the expanding polynomial $p_0(x)$, it would be still interesting to consider the following Problem 3. Notice that by taking affine conjugacy, we may assume that any polynomial p of degree d has of the form $p(x) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$ which will be identified with $(a_{d-2}, \cdots, a_0) \in \mathbb{C}^{d-1}$. Let

$$\mathcal{H}_d \equiv \big\{(p, b) \in \mathbb{C}^{d-1} \times \mathbb{C}^{\times} : f_{p,b} \text{ is hyperbolic}\big\}.$$

Problem 3. *Suppose that* f_{p_0,b_0} *is topologically conjugate to* f_{p_0,b_1} *on their Julia sets. Let* d *be the degree of* p_0*. Then, are these maps in the same connected component of* $\mathcal{H}_d$ *or in different connected components of* $\mathcal{H}_d$*?*

Proof of Theorem 6.1. Assume that $f = f_{p_0,b_0}$ in Theorem B is conjugate on J_f to a small perturbation $g = f_{q,b}$ of q, where $q(x)$ is some expanding polynomial of one variable.

First, by comparing the entropy on $\mathbb{C}^2$, we see $d \equiv \deg g = \deg q = \deg f = l + 1$. Since f is hyperbolic, the multiplicity of every periodic point is one. It then follows that f has exactly d^n periodic points of period n counted without multiplicity. Moreover, f has only one attractive fixed point which is outside J_f, and the other periodic points are saddles in J_f. By the existence of conjugacy between J_f and J_g, we know that a small perturbation of q must have the same number of saddle periodic points. Since each attractive cycle of q persists under a small perturbation, it follows that g has a unique attractive fixed point and does not have other attractive cycles. Since q is hyperbolic, each critical point of q goes either to an attractive cycle or to infinity. So, suppose that

$d_a - 1$ critical points of q (counted with multiplicity) are attracted to the attractive fixed point. We define $d_a^f = d-1$ and $d_a^g = d_a$.

Given q as above, we take a large $R > 0$ so that $J_q \subset \Delta \equiv \{|z| < R\}$. Then, there is the smallest $M \geq 0$ so that $q^{-M}(\Delta)$ does not contain the critical points of q whose orbits go to infinity. Suppose that $q^{-M}(\Delta)$ has L components. Let H^0 be a closed neighborhood of the attractive fixed point so that $q(H^0) \subset \mathrm{int}H^0$. Define inductively H^k to be the connected component of $q^{-1}(H^{k-1})$ containing H^{k-1}. Since any critical point of q with bounded orbit tends to the attractive fixed point, there exists the smallest $N > 0$ such that H^N contains all critical points of q whose orbits go to the attractive fixed point. Note that H^N is connected, simply connected and $q(H^N) \subset \mathrm{int}H^N$. Let $\mathcal{A}_1^g$, $\mathcal{A}_2^g$, $\cdots$, $\mathcal{A}_L^g$ be the components of $(q^{-M}(\Delta) \setminus H^N) \times \Delta$, where $H_1(\mathcal{A}_1^g;\mathbb{Z}) = \mathbb{Z}$ and $H_1(\mathcal{A}_i^g;\mathbb{Z}) = \{0\}$ for $i \neq 1$. We then have the following decomposition:

$$J_g = \bigcap_{n\in\mathbb{Z}} g^n(\mathcal{A}_1^g \sqcup \cdots \sqcup \mathcal{A}_L^g) = \bigsqcup_{\underline{\varepsilon}\in\{1,\cdots,L\}^{\mathbb{Z}}} J_{\underline{\varepsilon}}^g,$$

where $J_{\underline{\varepsilon}}^g \equiv \cdots \cap g^2(\mathcal{A}_{\varepsilon_{-2}}^g) \cap g(\mathcal{A}_{\varepsilon_{-1}}^g) \cap \mathcal{A}_{\varepsilon_0}^g \cap g^{-1}(\mathcal{A}_{\varepsilon_1}^g) \cap g^{-2}(\mathcal{A}_{\varepsilon_2}^g) \cap \cdots$. Recall that a similar decomposition of J_f for f can be obtained:

$$J_f = \bigcap_{n\in\mathbb{Z}} f^n(\mathcal{A}_1^f \sqcup \mathcal{A}_2^f \sqcup \mathcal{A}_3^f) = \bigsqcup_{\underline{\varepsilon}\in\{1,2,3\}^{\mathbb{Z}}} J_{\underline{\varepsilon}}^f,$$

where $J_{\underline{\varepsilon}}^f \equiv \cdots \cap f^2(\mathcal{A}_{\varepsilon_{-2}}^f) \cap f(\mathcal{A}_{\varepsilon_{-1}}^f) \cap \mathcal{A}_{\varepsilon_0}^f \cap f^{-1}(\mathcal{A}_{\varepsilon_1}^f) \cap f^{-2}(\mathcal{A}_{\varepsilon_2}^f) \cap \cdots$.

Lemma 6.2. *For both $* = f, g$ we have the following:*

(i) *if $\underline{\varepsilon} = \cdots 111.111 \cdots$, then $J_{\underline{\varepsilon}}^*$ becomes an invariant solenoid S^* of degree d_i^*,*
(ii) *if $\varepsilon_i \neq 1$ for only finitely many and at least one $i \geq 0$, then each connected component of $J_{\underline{\varepsilon}}^*$ is either a topological circle or an empty set,*
(iii) *if $\varepsilon_i \neq 1$ for infinitely many $i \geq 0$, then each connected component of $J_{\underline{\varepsilon}}^*$ is either a point or an empty set.*

Proof. Same as Lemma 3.9, thus omitted. Q.E.D.

Note that each pathwise connected component of the solenoid S^* is non-compact but the ones in the cases (ii) and (iii) are compact. Moreover, each pathwise connected component of J_* is contained in $J_{\underline{\varepsilon}}^*$ for some $\underline{\varepsilon}$. It follows that the conjugacy map sends S^f homeomorphically to S^g. The entropy of a solenoidal map is given by the logarithm of its degree, so from (i) of the previous lemma one gets the following

Corollary 6.3. *We have $d_a^g = d_a^f = d-1$.*

With the previous corollary, the conclusion of Theorem 6.1 immediately follows from a result on homotopy conjugacy between two expanding systems which is due to J. Smillie. However, for the self–consistency of this paper, we here quote its simplified statement for our purpose. Consult [IS] for a statement in full generality as well as a similar line of argument for hyperbolic polynomial diffeomorphisms of $\mathbb{C}^2$ to model their dynamics by the projective limits of one–dimensional maps.

$$\begin{tikzcd}
& & X^{\phi}_{n+2} \arrow[r, dashed, "h_{n+2}"] \arrow[dll, "\phi"'] \arrow[d, "\iota"'] & X^{\psi}_{n+2} \arrow[dll, "\psi"] \arrow[d, "\iota"] \\
X^{\phi}_{n+1} \arrow[r, "h_{n+1}"'] \arrow[d, "\iota"'] & X^{\psi}_{n+1} \arrow[d, "\iota"] & X^{\phi}_{n+1} \arrow[r, "h_{n+1}"] \arrow[dll, "\phi"] & X^{\psi}_{n+1} \arrow[dll, "\psi"] \\
X^{\phi}_{n} \arrow[r, "h_n"'] & X^{\psi}_{n} & &
\end{tikzcd}$$

Diagram 2. Commutative diagrams up to homotopy.

Definition 6.4. *We call $\Phi = (\phi, \iota, X_0, X_1)$ an expanding system if*

(i) *X_0 and X_1 are bounded regions in $\mathbb{C}$,*
(ii) *$X_1 = \phi^{-1}(X_0)$ and $\overline{X_1} \subset X_0$,*
(iii) *$\iota : X_1 \to X_0$ is an inclusion map, and*
(iv) *$\phi : X_1 \to X_0$ is a holomorphic covering.*

Since $\overline{X_1} \subset X_0$, it follows that $\phi : X_1 \to X_0$ is an expanding map in the following sense: there exist $\delta > 1$ and $\varepsilon > 0$ so that

$$d_{X_0}(\phi(x), \phi(y)) \geq \delta \cdot d_{X_0}(\iota(x), \iota(y))$$

whenever $d_{X_1}(x, y) \leq \varepsilon$. Here, d_Y denotes the Poincaré metric in $Y \subset \mathbb{C}$.

Given an expanding system $\Phi = (\phi, \iota, X_0, X_1)$ we will consider a sequence of spaces $X^{\phi}_n = \phi^{-1}(X^{\phi}_{n-1})$. Note that X^{ϕ}_n has the *universality*, that is, for any space Y and a pair of maps $\alpha : Y \to X^{\phi}_{n+1}$ and $\beta : Y \to X^{\phi}_{n+1}$ so that $\iota \circ \alpha = \phi \circ \beta$, there exists $h : Y \to X^{\phi}_{n+2}$ with $\iota \circ h_{n+2} = \beta$ and $\phi \circ h_{n+2} = \alpha$ (in fact, h is given by $\iota^{-1} \circ \beta$).

Definition 6.5. *Let $\Phi = (\phi, \iota, X^{\phi}_0, X^{\phi}_1)$ and $\Psi = (\psi, \iota, X^{\psi}_0, X^{\psi}_1)$ be two expanding systems. We say that Φ is homotopy conjugate to Ψ if there are two maps $h_0 : X^{\phi}_0 \to X^{\psi}_0$ and $h_1 : X^{\phi}_1 \to X^{\psi}_1$ so that (a) $h_0 \circ \phi = \psi \circ h_1$, and (b) $h_0 \circ \iota$ is homotopic to $\iota \circ h_1$. We call the pair (h_0, h_1) a homotopy conjugacy from Φ to Ψ.*

Given two expanding systems $\Phi = (\phi, \iota, X^{\phi}_0, X^{\phi}_1)$ and $\Psi = (\psi, \iota, X^{\psi}_0, X^{\psi}_1)$ and a homotopy conjugacy (h_0, h_1) from Φ to Ψ, below we will inductively construct a sequence of maps $h_n : X^{\phi}_n \to X^{\psi}_n$ and show that h_n uniformly converges to some $h_\infty : X^{\phi}_\infty \to X^{\psi}_\infty$, where $X^{\phi}_\infty = \bigcap_{n=0}^{\infty} X^{\phi}_n$ and $X^{\psi}_\infty = \bigcap_{n=0}^{\infty} X^{\psi}_n$.

$$\begin{tikzcd}
 & X^{\psi}_{n+1} \arrow[dl, "\iota"'] & & \\
X^{\psi}_{n} & & X^{\psi}_{n+2} \arrow[ul, "\psi"] \arrow[dl, "\iota"] & X^{\phi}_{n+2} \arrow[l, dashed, "h_{n+2}"'] \arrow[ull, "\alpha"'] \arrow[dll, "\beta"] \\
 & X^{\psi}_{n+1} \arrow[ul, "\psi"] & &
\end{tikzcd}$$

Diagram 3. Existence of h_{n+2}.

We will use the following simplified version of a more general result due to Smillie in [IS] to complete Theorem 6.1.

Theorem 6.6 (**Homotopy Conjugacy**)**.** *Let Φ and Ψ be two expanding systems. Assume that (h_0, h_1) is a homotopy conjugacy from Φ to Ψ. Then, it induces a semi–conjugacy $h_\infty : X^\phi_\infty \to X^\psi_\infty$ from ϕ to ψ, i.e. $h_\infty \circ \phi = \psi \circ h_\infty$ on X^ϕ_∞.*

The proof of this theorem proceeds as follows. We first construct a sequence of maps $h_n : X^\phi_n \to X^\psi_n$ inductively.

Lemma 6.7. *Assume that we are given $h_n : X^\phi_n \to X^\psi_n$, $h_{n+1} : X^\phi_{n+1} \to X^\psi_{n+1}$ and a homotopy $H_n(\cdot, t) : X^\phi_{n+1} \to X^\psi_n$ such that (i) $h_n \circ \phi = \psi \circ h_{n+1}$, (ii) $H_n(\cdot, 0) = h_n \circ \iota$ and (iii) $H_n(\cdot, 1) = \iota \circ h_{n+1}$. Then, there exist $h_{n+2} : X^\phi_{n+2} \to X^\psi_{n+2}$ and a homotopy $H_{n+1}(\cdot, t) : X^\phi_{n+2} \to X^\psi_{n+1}$ such that (i) $h_{n+1} \circ \phi = \psi \circ h_{n+2}$, (ii) $H_{n+1}(\cdot, 0) = h_{n+1} \circ \iota$ and (iii) $H_{n+1}(\cdot, 1) = \iota \circ h_{n+2}$ (see Diagram 2).*

Proof. Since $\psi : X^\psi_{n+1} \to X^\psi_n$ is a covering, the homotopy $\phi \circ H_n(\cdot, t) : X^\phi_{n+2} \to X^\psi_n$ lifts to a homotopy $H_{n+1}(\cdot, t) : X^\phi_{n+2} \to X^\psi_{n+1}$ so that $H_{n+1}(\cdot, 0) = h_{n+1} \circ \iota$. We then have $H_n(\cdot, 1) \circ \phi = \psi \circ H_{n+1}(\cdot, 1)$, which implies $\psi \circ H_{n+1}(\cdot, 1) = \iota \circ h_{n+1} \circ \phi$. By putting $\alpha \equiv h_{n+1} \circ \phi$ and $\beta \equiv H_{n+1}(\cdot, 1)$, this condition can be rewritten as $\iota \circ \alpha = \beta \circ \iota$. The universality of the pull–back X^ψ_{n+2} implies that there exists a map $h_{n+2} : X^\phi_{n+2} \to X^\psi_{n+2}$ so that $\iota \circ h_{n+2} = \beta = H_{n+1}(\cdot, 1)$ and $\psi \circ h_{n+2} = \alpha = h_{n+1} \circ \phi$ (see Diagram 3). Thus, we are done. Q.E.D.

Proof of Theorem 6.6. Let $l_Y(\gamma(\cdot))$ be the length of the image of a curve $\gamma : [0, 1] \to Y$ with respect to the Poincaré metric in Y. To finish the proof of Theorem 6.6, we prove the uniform convergence of h_n.

First recall that $\psi : X^\psi_{n+1} \to X^\psi_n$ is an isometry with respect to the Poincaré metrics of X^ψ_{n+1} and X^ψ_n since it is a covering, and $\iota : X^\psi_{n+1} \to X^\psi_n$ is a contraction since $\overline{X^\psi_{n+1}} \subset X^\psi_n$. We thus see that there exists $0 < \lambda < 1$ with

$$\begin{aligned} l_{X^\psi_n}(\iota \circ H_{n+1}(x, \cdot)) &\leq \lambda \cdot l_{X^\psi_{n+1}}(H_{n+1}(x, \cdot)) \\ &= \lambda \cdot l_{X^\psi_n}(\psi \circ H_{n+1}(x, \cdot)) \\ &= \lambda \cdot l_{X^\psi_n}(H_{n+1}(\phi(x), \cdot)) \end{aligned}$$

for all $x \in X^\phi_{n+2}$. It then follows that

$$l_{X^\psi_0}(\iota^n \circ H_n(x, \cdot)) \leq \lambda^n \sup_{x \in X^\phi_1} l_{X^\psi_0}(H_0(x, \cdot)) = C\lambda^n.$$

Thus, the limit function

$$\lim_{n\to\infty} h_n = h_\infty : X^\phi_\infty \to X^\psi_\infty$$

can be defined and continuous. It is then not difficult to see that h_∞ gives a semi–conjugacy from ϕ to ψ. This completes the proof of Theorem 6.6. Q.E.D.

End of the proof of Theorem 6.1. Let us write $p = p_0$. As before, we let $\Delta = \{|x| < R\}$ for a sufficiently large $R > 0$. There is the smallest number M' such that $D^p \equiv p^{-M'}(\Delta)$ does not contain the unique critical value of p which goes to infinity. We let H_0 be a closed neighborhood of the attractive fixed point of p so that $p(H_0) \subset \mathrm{int} H_0$, and H_k be the connected component of $p^{-1}(H_{k-1})$ containing the attractive fixed point of p. There is the smallest number N' such that $H^p \equiv H_{N'}$ contains all the critical values which converge to the attractive fixed point. From the previous corollary, the number of critical points which diverge to infinity for p is the same for q, and it is one. Thus, $p^{-1}(D^p \setminus H^p)$ has two connected components A^p_1 and A^p_2 both of which are topological annulus. We assume that A^p_1 is the one which is surrounding the attractive fixed point. Similarly we can define A^q_1 and A^q_2 for the other polynomial q. Again from the previous corollary, the number of critical points which are attracted to the unique attractive fixed point for p is the same for q, and it is $d-2$. Thus, $p : A^p_1 \to D^p \setminus H^p$ is a non–branched covering of degree $d-1$ and $p : A^p_2 \to D^p \setminus H^p$ is injective. These claims are valid for the other polynomial q as well.

Now, we let $X^p_0 \equiv D^p \setminus H^p$, $X^p_1 \equiv p^{-1}(X^p_0)$, and similarly $X^q_0 \equiv D^q \setminus H^q$, $X^q_1 \equiv q^{-1}(X^q_0)$. These define two expanding systems $P = (p, \iota, X^p_0, X^p_1)$ and $Q = (q, \iota, X^q_0, X^q_1)$. Due to the observation in the previous paragraph, we can choose two homeomorphisms $h_0 : X^p_0 \to X^q_0$ and $h_1 : X^p_1 \to X^q_1$ so that the pair (h_0, h_1) gives a homotopy conjugacy from P to Q. By Theorem 6.6, this induces a semi–conjugacy $h_\infty : X^p_\infty \to X^q_\infty$ from p to q.

Similarly the homotopy conjugacy $(g_0, g_1) = (h_0^{-1}, h_1^{-1})$ from Q to P induces a semi–conjugacy $g_\infty : X^q_\infty \to X^p_0$ from q to p. It is not difficult to see that $h_\infty = g_\infty^{-1}$ and it then gives a topological conjugacy between $p = p_0$ on $X^p_\infty = J_p$ and q on $X^q_\infty = J_q$. This finishes the proof of Theorem 6.1. Q.E.D.

We conclude this article with the following

Problem 4. *Find more examples of a hyperbolic polynomial diffeomorphism of $\mathbb{C}^2$ possibly of degree two and/or with connected Julia set. Find an algorithm which automatically constructs Poincaré boxes for a given polynomial diffeomorphism and detects hyperbolicity as well as combinatorics of the Julia set.*

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