UNIQUE ERGODICITY OF HARMONIC CURRENTS ON SINGULAR FOLIATIONS OF $\mathbb{P}^2$

Abstract. Let $\mathcal{F}$ be a holomorphic foliation of $\mathbb{P}^2$ by Riemann surfaces. Assume all the singular points of $\mathcal{F}$ are hyperbolic. If $\mathcal{F}$ has no algebraic leaf, then there is a unique positive harmonic $(1,1)$ current T of mass one, directed by $\mathcal{F}$. This implies strong ergodic properties for the foliation $\mathcal{F}$. We also study the harmonic flow associated to the current T.

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1. Introduction

Let $\mathcal{F}$ be a holomorphic foliation of the complex projective space $\mathbb{P}^2$. Our purpose is to study the ergodic properties of $\mathcal{F}$, using the theory of harmonic currents as developed by the authors in [8].

We first recall a few facts. Let $\pi : \mathbb{C}^3 \to \mathbb{P}^2$ denote the canonical projection. The foliation $\pi^*\mathcal{F}$ can be defined in $\mathbb{C}^3$ by a global $1-$form $\omega_0 = a_1(x)dx_1 + a_2(x)dx_2 + a_3(x)dx_3$ where the $a_j(x)$ are homogeneous polynomials of the same degree $\delta \geq 1$ without common factors. Moreover since every line through the origin is in the kernel of ω_0, they satisfy the condition $\sum x_i a_i(x) = 0$

The degree of $\mathcal{F}$ is by definition deg $\mathcal{F} = d :=$deg $\delta - 1$. It represents the number of tangencies of a generic line L, with $\mathcal{F}$. Let $Fol(d)$ denote the space of foliations of degree d. The space of coefficients of 1 forms of degree δ is a projective space. The subspace given by $\sum x_i a_i = 0$ is a linear subspace, so also a projective space. The subspace of 1 forms of degree δ of the form $H\lambda$ where H is a homogenous polynomial of degree $0 < \delta' < \delta$ and λ is a $1-$ form of degree $\delta - \delta'$ is an algebraic subvariety. So together this gives that $Fol(d)$ is the complement of an algebraic subvariety of some $\mathbb{P}^N$. It follows from the Bézout theorem that the foliation $\mathcal{F}$ has a finite number of singularities bounded uniformly by some function of the degree. If in a coordinate chart $U, \mathcal{F}$ is defined by $\omega_1 = \alpha(z,w)dz + \beta(z,w)dw$, then $\text{sing}(\mathcal{F}) \cap U = \{\alpha = \beta = 0\}$. We can assume that all the singular points are in the same $\mathbb{C}^2$, $\{p_j = (\alpha_j, \beta_j)\}_{j \leq N}$.

Definition 1. Suppose there is a change of coordinates around p_j sending p_j to 0 and such that $\omega_0(z,w) = zdw - \lambda wdz + \mathcal{O}(z,w)^2$ where $\lambda = a + ib$ and b is a nonzero number. We say in this case that the singularity is hyperbolic and that we are in the Poincaré domain. If λ is real we say that the singularity is in the Siegel domain.

The following is a classical fact due to Poincaré, see [5].

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Theorem 1. *Suppose that the singular point is hyperbolic. Then there is a local biholomorphic change of coordinates so that the form ω_0 in these coordinates can be written $\omega_0 = zdw - \lambda wdz$ (with the same λ).*

We remark that the form ω_0 is invariant under scaling except for multiplication by a constant which of course does not affect the zero set. Hence we can assume that the linearization is valid in a fixed large ball, in particular in a neighborhood of the unit bidisc.

The following result is due to Lins Neto, Soares [11] (we give only the two dimensional version, their result is also valid in $\mathbb{P}^k$).

Theorem 2. *There exists a real Zariski open subset $\mathcal{H}(d) \subset Fol(d)$ such that any $\mathcal{F} \subset \mathcal{H}$ satisfies:*
i) $\mathcal{F}$ has exactly $\frac{(d+2)!}{2!d!}$ hyperbolic singularities and no other singular points.
ii) $\mathcal{F}$ has no invariant algebraic curve.

The global behavior of foliations is not well understood. It is unknown whether every leaf of a given foliation $\mathcal{F}$, clusters at a singular point. This problem, known as the problem of existence of a minimal exceptional set is discussed in [6] and [2] for example. It is conjectured in [10] that a generic holomorphic foliation by Riemann surfaces in $\mathbb{P}^k$ has dense leaves. Recently Loray and Rebelo [12] have constructed non empty open sets of holomorphic foliations by Riemann surfaces in $\mathbb{P}^k$ such that every leaf is dense.

L. Garnett [9] has introduced the notion of harmonic measure for smooth foliations (without singularities) of a compact Riemannian manifold. She studied their ergodic properties. The article by Candel [4] contains a recent approach to that theory. In [8] the authors have shown that a $\mathcal{C}^1$ laminated set without singularities carry a unique harmonic current of mass 1 directed by the lamination. Very recently Deroin and Klepsyn [7] developed the theory of diffusion on transversally conformal foliations and they showed that there are only finitely many harmonic measures.

For holomorphic foliations (with singularities) of $\mathbb{P}^2$ the following analogue was proved in [1]. It is valid for laminations by Riemann surfaces with a small set of singularities, see [1] and [8].

Theorem 3. *Let $\mathcal{F}$ be a holomorphic foliation of $\mathbb{P}^2$. There exists a positive current T on $\mathbb{P}^2$, of bidimension $(1,1)$ and mass 1 which is harmonic, i.e. $i\partial\overline{\partial}T = 0$. Moreover in any flow box B, (without singular points) the current can be expressed as*

$$T = \int h_\alpha [V_\alpha] d\mu(\alpha).$$

The functions h_α are positive harmonic on the local leaves V_α and μ is a Borel measure on the transversal. The function $H : B \to \mathbb{R}^+$, $H_{|V_\alpha} = h_\alpha$ is Borel measurable.

Observe that if $\mathcal{F}$ is defined in B by a smooth form ω_0, then $T \wedge \omega_0 = 0$. We will say that the current is directed by $\mathcal{F}$.

A theory of intersection of positive harmonic currents of bidegree $(1,1)$ is developed in [8]. The main purpose of the present article is, using that intersection theory, to prove:

Theorem 4. *Let $\mathcal{F}$ be a holomorphic foliation in $\mathbb{P}^2$ without algebraic leaves. Assume that all singular points of $\mathcal{F}$ are hyperbolic. Then there is a unique positive harmonic current T of mass one, directed by $\mathcal{F}$.*

A consequence of Theorem 4 and of results from [8] is that the foliations $\mathcal{F}$ with only hyperbolic singular points are uniquely ergodic in a very strong sense, see Corollary 1. We will show a similar uniqueness result for some classes of foliations with non hyperbolic singularities, see Remark 2, page 56.

Observe that under the assumption of Theorem 4 there is no non zero positive closed current directed by $\mathcal{F}$, see [8] and Brunella [3] for a general discussion of closed cycles on foliations by Riemann surfaces.

The intersection theory of positive harmonic currents in [8] is valid on compact Kähler manifolds. We just recall a few facts restricting to $\mathbb{P}^2$.

Let T be a positive harmonic current of bidegree $(1,1)$ in $\mathbb{P}^2$, i.e. $i\partial\overline{\partial}T = 0$. Let ω denote the standard Kähler form on $\mathbb{P}^2$. Then T can be written as

$$T = c\omega + \partial S + \overline{\partial S} + i\partial\overline{\partial}u$$

with $c \geq 0$ and S is a $(0,1)$ form such that $S, \partial S, \overline{\partial}S$ are in L^2 and $u \in L^1$. The current $\overline{\partial}S$ depends only on T and is zero only if T is closed. So the quantity $\int \overline{\partial}S \wedge \partial\overline{S}$ which we called energy measures how far T is from being closed. The expression

$$\int T \wedge T := \int (c\omega + \partial S + \overline{\partial S}) \wedge (c\omega + \partial S + \overline{\partial S})$$

makes sense and is finite. It is independent on the choice of S. Moreover if T_1 and T_2 are 2 positive harmonic currents such that $\int T_1 \wedge T_2 = 0$, then T_1 and T_2 are proportional. On the other hand the currents directed by holomorphic foliations can be expressed in a flow box B as

$$T = \int h_\alpha [V_\alpha] d\mu(\alpha)$$

as described in Theorem 3. It is hence possible to consider the geometric self intersection of such currents. More precisely consider suitable automorphisms Φ_ϵ of $\mathbb{P}^2$ which are close to the identity. For a current T directed by a foliation $\mathcal{F}$, it is possible to define the geometric intersection $T \wedge_g \Phi_{\epsilon *}(T)$ as the measure on the complement of the singular points given locally by the expression

$$\sum_{p \in J^\epsilon_{\alpha,\beta}} h_\alpha(p) h^\epsilon_\beta(p) d\mu(\alpha) d\mu(\beta) \qquad (1)$$

where $J^\epsilon_{\alpha,\beta}$ denotes the points of intersection of the plaque L_α and the plaque $(\Phi_\epsilon)_* L_\beta$. Since $\int T_1 \wedge T_2 = \lim_{\epsilon\to 0} \int T_1 \wedge_g T_{2,\epsilon}$ ([8], Lemma 19), to show that $\int T_1 \wedge T_2 = 0$ it is enough to count the number of points of intersection of a given plaque with perturbed plaques and estimate the harmonic functions. This is done in [8] (Theorem 6.2) when we assume that the currents T_1, T_2 are supported on a minimal laminated compact set, which is transversally of class $\mathcal{C}^1$.

Indeed the minimality hypothesis is not used and the argument there gives the following stronger result.

Theorem 5. *Let $\mathcal{F}$ be a $\mathcal{C}^1$ lamination with singularities by Riemann surfaces in $\mathbb{P}^2$. Assume that there is a laminated compact set X without singularities. Then there is a unique positive harmonic current T, of mass 1, directed by $\mathcal{F}$.*

Proof. We know there is a harmonic current T_1 of mass 1, supported on X. Let T_2 be another such current directed by $\mathcal{F}$, but not necessarily supported by X. The argument in [8] Theorem 6.2 shows that $\lim_{\epsilon\to 0} T_1 \wedge_g T_{2,\epsilon} = 0$. Hence $\int T_1 \wedge T_2 = 0$. Therefore T_1 and T_2 are proportional. □

We now deal with the case where the foliation is holomorphic and the current T contains in its support singular points (which are all hyperbolic).

We will prove the following more general result than Theorem 4.

Theorem 6. *(MAIN THEOREM) Let $\mathcal{F}$ be a holomorphic foliation of $\mathbb{P}^2$ without algebraic leaves. Let X be a closed invariant set for $\mathcal{F}$. Assume that all singular points of X are hyperbolic. Then there is a unique positive harmonic current T of mass 1, directed by X.*

The result is valid for a laminated set $(X, \mathcal{L}, E)$ where $X \setminus E$ is a $\mathcal{C}^1$ lamination by Riemann surfaces. The set $E = \{p_1, \dots, p_\ell\}$ is a finite set and in a neighborhood U_j of every singular point p_j we assume that $X \cap U_j$ is holomorphically equivalent to a lamination contained in $z = Cw^{\lambda_j}, \lambda_j = a_j + ib_j, b_j \neq 0$. One of the consequences of the main theorem is Corollary 1 (p 55) which says that appropriate weighted averages of the leaves always converge to the current T. This is a strong ergodic theorem. The uniqueness of T also permits to show that $\lambda \to T_\lambda$ is continuous when λ varies in a holomorphic family of foliations as considered in the main theorem.

It is easy to see that $\overline{\partial} T = \overline{\tau} \wedge T$, τ is a $(0,1)$ form along leaves. We introduce in Section 27 a metric $g_T := \frac{i}{2} \tau \wedge \overline{\tau}$ and we show that the curvature κ of that metric satisfies $\kappa(g_T) = -1$. We also define a finite measure $\mu_T := i\tau \wedge \overline{\tau} \wedge T$. The measure μ_T is invariant under a flow whose integral curves are the level sets of the harmonic conjugates of h_α. We also show that the measures vary continuously with the foliation.

In the last paragraph we give some obstruction to the resolution of the equation $i\partial\overline{\partial} u = f$ along a foliation where f is a smooth $(1,1)$ form.

2. Proof of the Main Theorem

Let T be a harmonic current of mass 1 supported on X and directed by $\mathcal{F}$. In a flow box

$$T = \int h_\alpha [V_\alpha] d\mu(\alpha). \qquad (2)$$

We have to estimate the number of intersection points of a plaque with perturbed plaques near a singularity and also to study the behaviour of the harmonic continuation $\tilde{h}_\alpha$ of h_α along a leaf near a hyperbolic singularity.

This will give us that the geometric intersection is zero and hence $\int T \wedge T = 0$. Since T is arbitrary, the intersection theory of positive harmonic currents implies that T is unique.

After a change of coordinates we do the analysis for the form $\omega_0 = zdw - \lambda wdz, \lambda = a + ib, b \neq 0$, near $(0,0)$.

In order to study positive harmonic currents near 0, we cover a deleted neighborhood of 0 by finitely many "flow boxes" $(B_i)_{1\leq i\leq N}$, with $0 \in \overline{B_i}$ for every i. Each $B_i = S_i \times \Delta$, where S_i is a sector in $\mathbb{C}$ such that the map $\zeta \to e^{\zeta}$ is injective in a strip in the $\zeta-$plane $\gamma_1 < \Im\zeta < \gamma_2$, with values in S_i, Δ is a disc in $\mathbb{C}$, centered at 0. So the leaves in B_i are graphs over all or part of S_i. We will consider them as the local plaques. For the sake of argument we will use the sector S given by $0 < u < 2\pi$.

The strategy for the proof is to choose a family of automorphisms (Φ_ϵ) of $\mathbb{P}^2$ and to estimate the integral (1) in the flow boxes $(B_i)_{i\leq N}$. For that purpose we need to estimate the growth of the harmonic continuation of h_α along the leaves and also the number of intersection points of a plaque L_α, with perturbed plaques L^ϵ_β. The estimates are different close to separatrices and in other regions, this requires a precise subdivision of a polydisc near a singular point. Away from singularities this is just the proof given in [8] for a lamination. In the present case we have to divide the phase space in many regions where the estimates are technically different.

Description of a general leaf:

Consider again the foliation $zdw - \lambda wdz = 0, \lambda = a + ib, b \neq 0$. Notice that if we flip z and w, we replace λ by $1/\lambda = \overline{\lambda}/|\lambda|^2 = a/(a^2 + b^2) - ib/(a^2 + b^2)$. We will assume below that the axes are chosen so that $b > 0$. However, it is important to note that the estimates are also valid if $b < 0$. The point is that it will be seen that the case $a = 1$ is a degeneracy that complicates the estimates. However if we flip coordinates, the constant $a = 1$ becomes $a/(a^2 + b^2) = 1/(1 + b^2) < 1$.

There are two separatrices, $(w = 0), (z = 0)$. Other than that a leaf L_α can be parametrized by

$$\begin{aligned}
(z, w) &= \psi_\alpha(\zeta) \\
z &= e^{i(\zeta+(\log|\alpha|)/b)}, \zeta = u + iv \\
w &= \alpha e^{i\lambda(\zeta+(\log|\alpha|)/b)} \\
|z| &= e^{-v} \\
|w| &= e^{-bu-av}
\end{aligned}$$

Notice that as we follow z once counterclockwise around the origin u increases by 2π, so the absolute value of $|w|$ decreases by the multiplicative factor of $e^{-2\pi b}$. Hence we cover all leaves by restricting the values of α so that $e^{-2\pi b} \leq |\alpha| < 1$. We observe that the intersection with the unit bidisc of the leaf is given by $v > 0, u > -av/b$ independently of α. In the (u, v)-plane this corresponds to a sector $S = S_\lambda$ with corner at 0 and given by $0 < \theta < \arctan(-b/a)$ where the arctan is chosen to have values in $(0, \pi)$. Let $\gamma := \frac{\pi}{\arctan(-b/a)}$. Then the map $\phi : \tau \to \tau^\gamma$ maps this sector to the upper half plane with coordinates (x, y).

Let h_α denote the harmonic function associated to the current T on the leaf L_α. The local leaf clusters on both separatrices. To investigate the clustering on the $z-$ axis, we use a transversal $D_{z_0} := \{(z_0, w); |w| < 1\}$ for some $|z_0| = 1$. We can normalize so that $h_\alpha(z_0, w) = 1$ where (z_0, w) is the point on the local leaf with $e^{-2\pi b} \leq |w| < 1$. So $(z_0, w) = \psi_\alpha(\zeta_0) = \psi_\alpha(u_0 + iv_0)$ with $v_0 = 0$

and $0 < u_0 \leq 2\pi$ determined by the equations $|z_0| = e^{-v_0} = 1$ and $e^{-2\pi b} \leq |w| = e^{-bu_0 - av_0} < 1$. Let $\tilde{h}_\alpha$ denote the harmonic continuation along L_α. Define $H_\alpha(\zeta) := \tilde{h}_\alpha(e^{i(\zeta + (\log|\alpha|)/b)}, \alpha e^{i\lambda(\zeta + (\log|\alpha|)/b)})$ on S_λ.

Proposition 1. *The harmonic function $\tilde{H}_\alpha := H_\alpha \circ \phi^{-1}$ is the Poisson integral of its boundary values. So in the upper half plane $\{U + iV; V > 0\}$,*

$$\tilde{H}_\alpha(U + iV) = \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{H}_\alpha(x) \frac{V}{V^2 + (x - U)^2} dx$$

[for a.e. $\alpha, d\mu$]. Moreover,

$$\int_{e^{-2\pi b} \leq |\alpha| < 1} \int_{-\infty}^{\infty} \tilde{H}_\alpha(x) |x|^{\frac{1}{\gamma} - 1} dx d\mu(\alpha) < \infty.$$

Proof. Let $A_n := \{(z_0, w); e^{-2\pi b(n+1)} \leq |w| < e^{-2\pi bn},\ n = 0, 1, \ldots\}$. The holonomy map around $(z = 0)$ as described above gives a map

$$A_n \to A_{n+1}.$$

The transverse masses of these sets are $\int_{A_0} H_\alpha(\zeta_0 + 2\pi n) d\mu(\alpha) = B_n(\zeta_0)$. The functions $B_n(\zeta)$ are harmonic on $\{v > 0, u > -av/b - 2\pi n\}$. Since the transverse mass is finite on $(z = z_0)$ and since the annuli A_n are disjoint we get,

$$\sum_{n=0}^{\infty} B_n(\zeta_0) < \infty. \qquad (*)$$

We get a similar estimate along the other separatrix. It follows that

$$\int_{A_0} \left(\int_{\partial S_\lambda} H_\alpha \right) d\mu(\alpha) < \infty. \qquad (3)$$

We show now that for almost every α, $\tilde{H}_\alpha(x, y)$ is equal to the Poisson integral of its restriction to $y = 0$. Every positive harmonic function on the upper half plane can be written as a sum of a Poisson integral and $cy, c \geq 0$. The problem is to show that $c = 0$.

We consider the restriction L'_α of L_α to the bidisc $\Delta^2(0, e^{-1})$. The leaf L'_α equals $\psi_\alpha(S'_\lambda)$ where $S'_\lambda := \{v > 1, u > -av/b + 1/b\}$. The image of this sector under ϕ is a domain of the form $\Delta'_{\lambda,\alpha} = \{x + iy; y > \gamma_\alpha(x)\}$ where γ_α is a continuous strictly positive function so that $\gamma_\alpha \to +\infty$ when $|x| \to \infty$. The function B_1 is bounded on the edges of S'_λ. So $\tilde{B}_1 := B_1 \circ \phi^{-1}$ is bounded on the graph of γ_α and hence there is no term $cy, c > 0$ in the canonical representation of $\tilde{B}_1$. The same argument is valid for the functions $\tilde{H}_\alpha$ at least for μ almost every α.

It follows that the representation as a Poisson integral is valid. On the other hand estimate (3) can be read as

$$\int_{e^{-2\pi b} \leq |\alpha| < 1} \int_0^{\infty} H_\alpha(u) du d\mu(\alpha) < \infty \text{ and}$$

$$\int_{e^{-2\pi b} \leq |\alpha| < 1} \int_0^{\infty} H_\alpha(u e^{i \arctan(-b/a)}) du d\mu(\alpha) < \infty,$$

which gives after a change of variables the estimate on the growth of $\tilde{H}_\alpha$.

□

Remark 1. It is convenient in some later calculations to replace $|x|^{1/\gamma-1}$ by $(|x|+1)^{1/\gamma-1}$ in the integral of Proposition 1. By Harnack, this doesn't effect the order of magnitude of the integral.

We decompose a leaf L_α into plaques $L_{\alpha,n}$ where $2n\pi < u < 2(n+1)\pi$. Here n is an integer. [Note that if $a \leq 0$, these n must be positive to have a nonempty intersection with the bidisc.] In this way $L_{\alpha,n}$ is a graph over some part of the $z-$axis.

We let (z,w) be a point in L_α parametrized by a point (u,v). We write in polar coordinates, $u+iv = re^{i\theta}$ with $r = \sqrt{u^2+v^2}, \theta = \arctan(v/u)$. Then in the (U,V) plane this point corresponds to $U+iV = \phi(u+iv) = (u+iv)^\gamma$,

$$U+iV = r^\gamma e^{i\gamma\theta} = r^\gamma \cos(\gamma\theta) + ir^\gamma \sin(\gamma\theta).$$

We hence get the following formula for the function $H_\alpha(u+iv)$:

Lemma 1.

$$H_\alpha(u+iv) = \frac{1}{\pi}\int \tilde{H}_\alpha(x)\frac{r^\gamma \sin(\gamma\theta)}{(r^\gamma\sin(\gamma\theta))^2+(x-r^\gamma\cos(\gamma\theta))^2}dx$$

Now we write the formula for the perturbed foliation $\mathcal{F}_\epsilon = (\Phi_\epsilon)_*\mathcal{F}$ where Φ_ϵ is a family of automorphisms of $\mathbb{P}^2$. We will need as in [8] that all our estimates stay valid when composing Φ_ϵ with Ψ in a neighborhood of the identity in $U(3)$ (depending on ϵ). We will need that Φ_ϵ moves the singular point in a direction away from the separatrices near all the hyperbolic points. We also need the Φ_ϵ to have a common fixed point p in the support of T and that the tangent space of the leaf through p moves to first order with ϵ. So we write in $\mathbb{C}^2$

$$\Phi_\epsilon(z,w) = (\alpha(\epsilon),\beta(\epsilon)) + (z,w) + \epsilon\mathcal{O}(z,w)$$

with $\alpha'(0), \beta'(0) \neq 0$. We will also need that $\lambda \neq \beta'(0)/\alpha'(0)$.

Suppose that (z,w) is a point in the perturbed bidisc $\Phi_\epsilon(\Delta^2)$, not on an indicatrix of $\mathcal{F}_\epsilon$. Then $\Phi_\epsilon^{-1}(z,w)$is on some plaque $L_{\beta,m}$ with parameters (u',v'). We ignore the problem that we need $u' \neq 2\pi$ because we can also use other flow boxes. The original point (z,w) is on a plaque $L^\epsilon_{\beta,m}$ and we get:

Lemma 2.

$$H^\epsilon_\beta(u'+iv') = \frac{1}{\pi}\int \tilde{H}^\epsilon_\beta(y)\frac{(r')^\gamma\sin(\gamma\theta')}{((r')^\gamma\sin(\gamma\theta'))^2+(y-(r')^\gamma\cos(\gamma\theta'))^2}dy$$

Next, for each $(\alpha,\beta,m,n,\epsilon)$, let $I_{\alpha,\beta,m,n,\epsilon}$ denote the set of points p in a slightly smaller bidisc which belong to $L_{\alpha,n}\cap L^\epsilon_{\beta,m}$. We prove:

Theorem 7.

$$\lim_{\epsilon\to 0}\int\sum_{m,n}\sum_{p\in I_{\alpha,\beta,m,n,\epsilon}}\tilde{h}_{\alpha,n}(p)\tilde{h}^\epsilon_{\beta,m}(p)d\mu(\alpha)d\mu(\beta) = 0.$$

Proof. We will decompose the integral, I, into pieces. The first piece I_1 is when $p \in D_1 = \{|w| < c\epsilon, |z| < c\epsilon\}$ We let D_2 denote the region $c|\epsilon| < |z| < C|\epsilon|, |w| < r|\epsilon|$ where $C >> 1$ is some constant and $0 < r < c$ depends on the choice of C. for a suitable fixed constant $0 < c << 1$. The second piece of the integral is I_2 over D_2.

3. Proof of theorem 7 for the intersection points in D_1 (close to the singularity)

Lemma 3. *Let $\delta > 0$. Then for all small enough c, $|\epsilon|$, the slopes of the leaves of $\mathcal{F}_\epsilon$, $dw/dz \in \Delta(\lambda\frac{\beta'(0)}{\alpha'(0)};\delta)$ at all points in D_1.*

Proof. We estimate ω_ϵ in D_1.

$$\begin{aligned}
\omega_\epsilon &= (\Phi_\epsilon)_*(\omega_0)\\
&= \mathcal{O}(\epsilon^2) + [(z-\alpha(\epsilon))(1+A\epsilon) + B\epsilon(w-\beta(\epsilon))]\,dw\\
&+ [(z-\beta(\epsilon))(-\lambda + C\epsilon) + D\epsilon(z-\alpha(\epsilon))]\,dz\\
&= \mathcal{O}(\epsilon^2) + (z-\alpha(\epsilon))dw + (z-\beta(\epsilon))(-\lambda)dz\\
&= (z-\alpha'(0)\epsilon + \mathcal{O}(\epsilon^2))dw - \lambda(w-\beta'(0)\epsilon + \mathcal{O}(\epsilon^2))dz\\
dw/dz &= \frac{\lambda(w-\beta'(0)\epsilon+\mathcal{O}(\epsilon^2))}{z-\alpha'(0)\epsilon+\mathcal{O}(\epsilon^2)}\\
&= \lambda\frac{-\beta'(0)\epsilon+\cdots}{-\alpha'(0)\epsilon+\cdots}\\
&= \lambda\frac{\beta'(0)}{\alpha'(0)}+\cdots
\end{aligned}$$

The Lemma follows immediately.

□

The following lemma decribes the lamination of ω_ϵ near D_1 after possibly shrinking c further and is an immediate consequence of Lemma 3.

Lemma 4. *The plaques of $\mathcal{F}_\epsilon$ near D_1 are of the form $w = f_\eta(z)$ where $f_\eta(\eta) = 0$ and $f'_\eta \in \Delta(\lambda\frac{\beta'(0)}{\alpha'(0)};\delta)$.*

To estimate the geometric wedge product we will consider three types of points in a plaque $L^\epsilon_{\beta,m}$, namely if they are close to where the plaque crosses the $z-$ axis (Case 1) or $w-$ axis or otherwise (Case 2). The estimates for $\tilde{h}^\epsilon_\beta$ are fairly independent of which case we are in, but h_α is very sensitive to the cases.

We estimate the function $\tilde{h}^\epsilon_\beta$ on these plaques. First observe that the points in $D_2 := \Delta^2((-\alpha'(0)\epsilon, -\beta'(0)\epsilon); 2c|\epsilon|)$ are mapped by Φ_ϵ to a region covering D_1.

Lemma 5. *There is a constant $C > 0$ so that if some leaf L^ϵ_β intersects D_1 for a parameter value $u + iv$ then*

$$\frac{1-a}{b}\log(1/|\epsilon|) - C < u < \frac{1-a}{b}\log(1/|\epsilon|) + C, \log(1/|\epsilon|) - C < v < \log(1/|\epsilon|) + C.$$

Proof. First recall that $z = e^{i(u+iv+(\log|\beta|/b))}$. Hence $|z| = e^{-v}$. But $(z,w) \in D_2$. Hence

$$(|\alpha'(0)| - 2c)||\epsilon| < |z| = e^{-v} < (|\alpha'(0)| + 2c)||\epsilon.|$$

So

$$\log|\epsilon| - C < -v < \log|\epsilon| + C$$

which gives the estimate on v.

$$
\begin{array}{rcl}
|w| &=& e^{-bu-av} \\
(|\beta'(0)|-2c)|\epsilon| &<& |w| = e^{-bu-av} < (|\beta'(0)|+2c)|\epsilon| \\
\log|\epsilon| - C' &<& -bu-av < \log|\epsilon| + C' \\
\log(1/|\epsilon|) - C' &<& bu+av < \log(1/|\epsilon|) + C' \\
\frac{1}{b}\log(1/|\epsilon|) - C'' &<& u + av/b < \frac{1}{b}\log(1/|\epsilon|) + C'' \\
\frac{1}{b}\log(1/|\epsilon|) - av/b - C'' &<& u < \frac{1}{b}\log(1/|\epsilon|) - av/b + C'' \\
u &<& \frac{1}{b}\log(1/|\epsilon|) + \frac{a}{b}[-\log(1/|\epsilon|)+C] + C''[a>0] \\
&<& \frac{1-a}{b}\log(1/|\epsilon|) + C''' \\
u &<& \frac{1}{b}\log(1/|\epsilon|) - \frac{a}{b}[\log(1/|\epsilon|)+C] + C''[a \leq 0] \\
&<& \frac{1-a}{b}\log(1/|\epsilon|) + C''' \\
u &>& \frac{1}{b}\log(1/|\epsilon|) + \frac{a}{b}[-\log(1/|\epsilon|)-C] - C''[a>0] \\
&>& \frac{1-a}{b}\log(1/|\epsilon|) - C''' \\
u &>& \frac{1}{b}\log(1/|\epsilon|) - \frac{a}{b}[\log(1/|\epsilon|)-C] - C''[a \leq 0] \\
&>& \frac{1-a}{b}\log(1/|\epsilon|) - C'''
\end{array}
$$

□

Next we estimate the value of $\tilde{h}^{\epsilon}_{\beta}$ for a point (u, v) as in the previous Lemma. Let $\theta, \tan\theta = v/u$ be the argument. By Lemma 5, it follows that for all small ϵ, $\tan\theta \sim b/(1-a) \neq b/(-a)$ so that the angle θ is uniformly inside the sector S_λ for all small ϵ. It follows that $\gamma\theta$ is strictly inside a sector $0 < d < \gamma\theta < \pi - d < \pi$ for some fixed $d > 0$ and all small enough ϵ. This implies that $\sin\gamma\theta > k > 0$ uniformly. This allows us to estimate the kernel for $H^{\epsilon}_{\beta}(u+iv)$ as in Lemma 2:

Lemma 6. *Suppose $(u+iv)$ is such that the corresponding point on the leaf L^{ϵ}_{β} is in D_1, then if $|y| < 2(\log(1/|\epsilon|))^{\gamma}$,*

$$\frac{(r)^{\gamma}\sin(\gamma\theta)}{((r)^{\gamma}\sin(\gamma\theta))^2 + (y-(r)^{\gamma}\cos(\gamma\theta))^2} \sim \frac{1}{(\log(1/|\epsilon|))^{\gamma}}$$

On the other hand if $|y| \geq 2(\log(1/|\epsilon|))^{\gamma}$ then

$$\frac{(r)^{\gamma}\sin(\gamma\theta)}{((r)^{\gamma}\sin(\gamma\theta))^2 + (y-(r)^{\gamma}\cos(\gamma\theta))^2} \sim \frac{(\log(1/|\epsilon|))^{\gamma}}{y^2}$$

Hence we get

Lemma 7. *We have the following estimate of H^{ϵ}_{β} for points in D_1:*

$$H^{\epsilon}_{\beta} \sim \frac{1}{(\log(1/|\epsilon|))^{\gamma}}\int_{|y|<2(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_{\beta}(y)dy + (\log(1/|\epsilon|))^{\gamma}\int_{|y|\geq 2(\log(1/|\epsilon|))^{\gamma}} \frac{\tilde{H}_{\beta}(y)}{y^2}dy$$

Next we fix α, β and plaques $L_{\alpha,n}, L^{\epsilon}_{\beta,m}$ and assume they intersect in D_1. By Lemma 5, there are conditions on m for this to happen:

$$\begin{aligned}
2m\pi &< u' < 2(m+1)\pi \\
\frac{1-a}{b}\log(1/|\epsilon|) - C &< u' < \frac{1-a}{b}\log(1/|\epsilon|) + C \\
\frac{1-a}{b}\log(1/|\epsilon|) - C &< 2(m+1)\pi < \frac{1-a}{b}\log(1/|\epsilon|) + C + 2\pi \\
\frac{1-a}{b}\log(1/|\epsilon|) - C - 2\pi &< 2m\pi < \frac{1-a}{b}\log(1/|\epsilon|) + C
\end{aligned}$$

We pick a plaque $L^{\epsilon}_{\beta,m}$ with an intersection point in D_1. Then this plaque is of the form $w = f(z) = f_\eta(z)$ where $f_\eta(\eta) = 0$ and f' is as in Lemma 4. i.e. close to $\lambda\frac{\beta'(0)}{\alpha'(0)}$. Next consider a plaque $L_{\alpha,n}$

$$\begin{aligned}
z &= e^{i(u+(\log|\alpha|/b))-v} \\
w &= \alpha e^{i\lambda(\zeta+(\log|\alpha|)/b)} \\
2n\pi &< u < 2(n+1)\pi. \\
|w| &= e^{-bu-av}
\end{aligned}$$

Case 1: $|z-\eta| < d|\eta|, 0 < d << 1$.

We estimate the parameter values (u, v) for $L_{\alpha,n}$.

Since $|\eta|(1-d) < |z| = e^{-v} < |\eta|(1+d)$, $\log(1/|\eta|) - 2d < v < \log(1/|\eta|) + 2d$. Note that also, for the point (z, w) to be on $L^{\epsilon}_{\beta,m}$ with $|z-\eta| < d|\eta|$ we must have that $|w| < 2|\lambda|\frac{|\beta'(0)|}{|\alpha'(0)|}d|\eta|$.

Lemma 8. *For (z, w) to be an intersection point between $L_{\alpha,n}$ and L^{ϵ}_{β} in D_1 with $|z-\eta| < d|\eta|$, we must have*
(i) $2n\pi < u < 2(n+1)\pi$
(ii) $2n\pi > \frac{1-a}{b}\log(1/|\eta|) - C$
(iii) $\log(1/|\eta|) - 2d < v < \log(1/|\eta|) + 2d$.
Moreover there is at most one such intersection point.

Proof. We have already proved (iii) and (i) is given. To prove (ii):

$$\begin{aligned}
|w| &= e^{-bu-av} \\
&< 2|\lambda|\frac{|\beta'(0)|}{|\alpha'(0)|}d|\eta| \\
-bu - av &< \log|\eta| + C \\
u &> (-a/b)v - (\log|\eta|)/b - C/b \\
u &> (-a/b)\log(1/|\eta|) - (\log|\eta|)/b - C'/b \\
u &> ((1-a)/b)\log(1/|\eta|) - C''
\end{aligned}$$

To prove the last part, notice that the slope of L^ϵ_β is about λ while the slope of L_α is $\lambda w/z$ so is at most $|\lambda|(2|\lambda|\frac{|\beta'(0)|}{|\alpha'(0)|}d||\eta|)/(|\eta|(1-d)) << |\lambda|$ if we just make d small enough.

□

Lemma 9. *We estimate the value of H_α at intersection points between $L_{\alpha,n}$ and L^ϵ_β in D_1 with $|z-\eta|<d|\eta|$.*
(i) $\frac{1-a}{b}\log(1/|\eta|)-C<2\pi n<C\log(1/|\eta|)$:

$$H_\alpha(u+iv) \quad \sim \quad \int_{|x|<2(\log(1/|\eta|))^\gamma}\frac{\tilde{H}_\alpha(x)}{(\log(1/|\eta|))^\gamma}+\int_{|x|>2(\log(1/|\eta|))^\gamma}\tilde{H}_\alpha(x)\frac{(\log(1/|\eta|))^\gamma}{x^2}$$

(ii) $2\pi n\geq C\log(1/|\eta|)$. *Then* $U+iV\sim n^\gamma+in^{\gamma-1}\log(1/|\eta|)$ *and*

$$\begin{aligned} H_\alpha(u+iv) \quad &\sim \quad \int_{|x-U|<n^{\gamma-1}\log(1/|\eta|)}\frac{\tilde{H}_\alpha(x)}{n^{\gamma-1}\log(1/|\eta|)}dx \\ &+ \quad \int_{|x-U|>n^{\gamma-1}\log(1/|\eta|)}\frac{\tilde{H}_\alpha(x)n^{\gamma-1}\log(1/|\eta|)}{(x-U)^2}dx \end{aligned}$$

Proof. Case (i): We use that $\sin(\gamma\theta)$ is bounded below by a strictly positive constant. Case (ii) is clear.

□

Case 2: Our next step is to discuss intersection points of $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ in D_1 for which $|z-\eta|>d|\eta|$. Note that $L^\epsilon_{\beta,m}$ intersects the $w-$ axis close to $(0,-\lambda\frac{\beta'(0)}{\alpha'(0)}\eta)$ and the above argument applies as well to the region $|w+\lambda\frac{\beta'(0)}{\alpha'(0)}\eta|<d|\eta|$. Hence we only need to consider intersections of $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ when $|w+\lambda\frac{\beta'(0)}{\alpha'(0)}\eta|>d|\eta|$ and also $|z-\eta|>d|\eta|$, call this set S.

Note: This is the place in the argument where we will assume that $a\neq 1$.

Since we are excluding the points near where $L^\epsilon_{\beta,m}$ crosses the two axes, we have the following estimate on points in $L^\epsilon_{\beta,m}$: For some fixed constant $R>1$ we have that

$$\frac{1}{R}|w|<|z|<R|w|$$

for points in S.

Hence

$$\begin{aligned}
\frac{1}{R}e^{-av-bu} &< e^{-v} < Re^{-av-bu} \\
-av-bu-\log R &< -v < -av-bu+\log R \\
bu-\log R &< (1-a)v < bu+\log R \\
2n\pi b-\log R &< (1-a)v < 2(n+1)\pi b+\log R \\
2n\pi - C &< \frac{1-a}{b}v < 2n\pi + C \\
2nb\pi/(1-a)-C' &< v < 2nb\pi/(1-a)+C'
\end{aligned}$$

□

Lemma 10. *(Intersection Lemma) There is a constant* $N>1$ *so that if we cover the rectangle* $2n\pi < u < (2n+1)\pi, 2nb\pi/(1-a)-C' < v < 2nb\pi/(1-a)+C'$ *with* N *equal squares, then there are at most two intersection points in each square.*

Proof. In each square, the slope of $L_{\alpha,n}$ is almost constant and will produce at most one intersection point. The exception is when the slope is close to $\lambda\frac{\beta'(0)}{\alpha'(0}$. Then there might be a tangency between $L_{\alpha,n}$ and L_β. Hence there might be two or more intersection points counted with multiplicity. We will show there are at most 2. Note that the slope S of $L_{\alpha,n}$ is given by the quotient $\lambda w/z$.

$$\begin{aligned}
dw/dz &= \lambda w/z \\
&= \lambda\frac{\alpha e^{i\lambda(\zeta+(\log|\alpha|/b))}}{e^{i(u+(\log|\alpha|/b))-v}} \\
&= \frac{\lambda\alpha e^{((\log|\alpha|)/b)(-b+ia)}}{e^{i(\log|\alpha|)/b}}\frac{e^{i\lambda\zeta}}{e^{i\zeta}} \\
S &= Ce^{i(\lambda-1)\zeta} \\
\frac{\partial S}{\partial\zeta} &= i(\lambda-1)S \\
&\sim i(\lambda-1)\lambda\frac{\beta'(0)}{\alpha'(0)} \\
&\sim 1
\end{aligned}$$

This says that the slope of $L_{\alpha,n}$ near intersection points vary very rapidly, while we also see from Lemma 4 that the slope of $L^\epsilon_{\beta,m}$ varies slowly. This implies that near tangential intersection points there are at most two of them.

□

We estimate the value of H_α at points p where $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ intersect in D_1 away from the axes $(|z-\eta|>d|\eta|, |w+\lambda\frac{\beta'(0)}{\alpha'(0)}\eta|>d|\eta|)$.

Lemma 11. *For the intersection point to be in* D_1 *we need* $|n|>[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C$. *Then*

$$H_\alpha(p) \sim \int_{|x|<C|n|^\gamma}\frac{\tilde{H}_\alpha(x)dx}{|n|^\gamma}+\int_{|x|>C|n|^\gamma}\frac{\tilde{H}_\alpha(x)|n|^\gamma}{x^2}dx$$

Proof. For the first estimate, recall that $|z| = e^{-v} < c|\epsilon|$ and that $2nb\pi/(1-a)-C' < v < 2nb\pi/(1-a)+C'$. For the integral estimate we see that $(u+iv)^\gamma = U+iV$ with $V \sim |n|^\gamma$ and $|U| <\sim |n|^\gamma$. Then the estimate is immediate from the Poisson kernel.

□

We finish the estimate for D_1.

Theorem 8. *The contribution to the geometric wedge product of T and T_ϵ from intersection points in D_1 goes to zero when $\epsilon \to 0$.*

Proof. Let $I = I_\epsilon$ consist of all intersection points p in D_1. They are labeled $p = p_{\alpha,\beta,n,m,\ell}$ if they belong to the plaques $L_{\alpha,n}, L^\epsilon_{\beta,m}$ and ℓ lists them (with multiplicity) if there are more than one. By Lemma 5,

$$\frac{(1-a)\log(1/|\epsilon|)}{2\pi b} - C < m < \frac{(1-a)\log(1/|\epsilon|)}{2\pi b} + C$$

so in particular there are at most finitely many values of m and there is a uniform upper bound on the number of them. We can hence restrict to one fixed value of m. Next recall that from Lemma 7 we have the estimate on the value of H^ϵ_β at each intersection point:

$$\begin{aligned} H^\epsilon_\beta(p) \quad &\sim \quad \frac{1}{(\log(1/|\epsilon|))^\gamma}\int_{|y|<2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)dy \\ &+ \quad (\log(1/|\epsilon|))^\gamma \int_{|y|>2(\log(1/|\epsilon|))^\gamma} \frac{\tilde{H}_\beta(y)}{y^2}dy. \end{aligned}$$

By Lemmas 8 and 10 there is at most a uniformly bounded number of intersection points with $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ in D_1. Hence when we estimate the geometric wedge product we can factor out the contribution from β and we get an upper bound of

$$\begin{aligned} \int \left(\sum_p H^\epsilon_\beta\right) d\mu(\beta) \quad &<\sim \quad \frac{1}{(\log(1/|\epsilon|))^\gamma}\int_{|y|<2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)dyd\mu(\beta) \\ &+ \quad (\log(1/|\epsilon|))^\gamma \int_{|y|>2(\log(1/|\epsilon|))^\gamma} \frac{\tilde{H}_\beta(y)}{y^2}dyd\mu(\beta). \end{aligned}$$

We collect a few equalities that will be used repeatedly.

Lemma 12.

$$
\begin{aligned}
(I) \quad & \frac{1}{(\log(1/|\epsilon|))^\gamma}\int_{|y|<2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)dy \\
= \quad & \frac{1}{(\log(1/|\epsilon|))^\gamma}\int_{|y|<2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}|y|^{1-1/\gamma}dy \\
\sim \quad & \int_{|y|<2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}\frac{|y|}{(\log(1/|\epsilon|))^\gamma}\frac{1}{(|y|+1)^{1/\gamma}}dy \\
(II) \quad & (\log(1/|\epsilon|))^\gamma \int_{|y|>2(\log(1/|\epsilon|))^\gamma} \frac{\tilde{H}_\beta(y)}{y^2}dy \\
= \quad & \int_{|y|>2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}\frac{(\log(1/|\epsilon|))^\gamma}{|y|}|y|^{-1/\gamma}dy \\
(III) \quad & \textit{If } U \sim n^\gamma \\
& \int_{|x-U|<n^{\gamma-1}\log(1/|\eta|)} \frac{\tilde{H}_\alpha(x)}{n^{\gamma-1}\log(1/|\eta|)}dx \\
\sim \quad & \int_{|x-U|<n^{\gamma-1}\log(1/|\eta|)} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\frac{dx}{\log(1/|\eta|)}
\end{aligned}
$$

By the Lebesgue dominated convergence theorem and Lemma 12, we get that

$$\int \sum_p H_\beta^\epsilon d\mu(\beta) \to 0.$$

□

We estimate the value of H_α at one of the intersection points $p \in D_1$. From Lemma 1 we have:

$$H_\alpha(p) = \frac{1}{\pi}\int \tilde{H}_\alpha(x)\frac{r^\gamma \sin(\gamma\theta)}{(r^\gamma \sin(\gamma\theta))^2 + (x - r^\gamma\cos(\gamma\theta))^2}dx$$

Case (i): $|z-\eta| < d|\eta|, |n| < C\log(1/|\eta|)$. By Lemma 8 it follows that $V = r^\gamma \sin(\gamma\theta) \sim (\log(1/|\eta|))^\gamma$ and $|U| <\sim (\log(1/|\eta|))^\gamma$.

$$
\begin{aligned}
H_\alpha(p) &\sim \int_{|x|<C(\log(1/|\eta|))^\gamma} \frac{\tilde{H}_\alpha(x)}{(\log(1/|\eta|))^\gamma} \\
&+ \int_{|x|>C(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x)\frac{(\log(1/|\eta|))^\gamma}{x^2} \\
\sum_{|n|<\log(1/|\eta|)} h_{\alpha,n}(p_n) &\sim \int_{|x|<C(\log(1/|\eta|)^\gamma} \frac{\tilde{H}_\alpha(x)}{(\log(1/|\eta|))^{\gamma-1}} \\
&+ \int_{|x|>C(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x)\frac{(\log(1/|\eta|))^{\gamma+1}}{x^2} \\
&\sim \int_{|x|<C(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{|x|}{(\log(1/|\eta|))^\gamma}\right)^{1-1/\gamma} \\
&+ \int_{|x|>C(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{(\log(1/|\eta|))^\gamma}{|x|}\right)^{1+1/\gamma} \\
& \int \sum_{|n|<C\log(1/|\eta|)} H_{\alpha,n}(p_n)d\mu(\alpha) \\
&\to 0
\end{aligned}
$$

Case (ii): $|z-\eta| < d|\eta|, |n| > C\log(1/|\eta|)$. Then by Lemma 8, $n>0$ and we have $U_n \sim n^\gamma, V \sim n^{\gamma-1}\log(1/|\eta|)$. From Lemma 9 we have:

$$
\begin{aligned}
H_\alpha(p) &\sim \int_{|x-U|<n^{\gamma-1}\log(1/|\eta|)} \frac{\tilde{H}_\alpha(x)}{n^{\gamma-1}\log(1/|\eta|)}dx \\
&+ \int_{|x-U>n^{\gamma-1}\log(1/|\eta|)} \tilde{H}_\alpha(x)\frac{n^{\gamma-1}\log(1/|\eta|)}{|x-U|^2}dx \\
H_\alpha(p) &\sim \int_{|x-U|<n^{\gamma-1}\log(1/|\eta|)} \frac{\tilde{H}_\alpha(x)}{\log(1/|\eta|)}x^{1/\gamma-1}dx \\
&+ \int_{|x-U|>n^{\gamma-1}\log(1/|\eta|)} \tilde{H}_\alpha(x)\frac{n^{\gamma-1}\log(1/|\eta|)}{|x-U|^2}dx \\
\sum_{n>C\log(1/|\eta|)} H_{\alpha,n}(p) &\sim \sum_{n>C\log(1/|\eta|)} \int_{|x-U_n|<n^{\gamma-1}\log(1/|\eta|)} \frac{\tilde{H}_\alpha(x)}{\log(1/|\eta|)}x^{1/\gamma-1}dx \\
&+ \sum_{n>C\log(1/|\eta|)} \int_{|x-U_n|>n^{\gamma-1}\log(1/|\eta|)} \tilde{H}_\alpha(x)\frac{n^{\gamma-1}\log(1/|\eta|)}{|x-U_n|^2}dx \\
&= I + II
\end{aligned}
$$

Note that for a given x, the number of integers n for which $U_n - n^{\gamma-1}\log(1/|\eta|) < x < U_n + n^{\gamma-1}\log(1/|\eta|)$ is bounded above by a multiple of $\log(1/|\eta|)$. It follows that $I <\sim \int_{(\log(1/|\eta|))^\gamma/C}^\infty \tilde{H}_\alpha(x)x^{1/\gamma-1}dx$. This contribution goes to zero as $|\epsilon| \to 0$ since $|\eta| < |\epsilon|$.

We estimate U_n more precisely.

$$\begin{aligned} 2n\pi &< u_n < 2(n+1)\pi \\ \log(1/|\eta|) - 2d &< v_n < \log(1/|\eta|) + 2d \\ (u_n + iv_n)^\gamma &= u_n^\gamma(1 + iv_n/u_n)^\gamma \\ &= u_n^\gamma + \gamma u_n^{\gamma-1} iv_n + E_n \\ |E_n| &<\sim u_n^\gamma (v_n/u_n)^2 \\ &\sim n^{\gamma-2}(\log(1/|\eta|))^2 \end{aligned}$$

Hence $|U_n - (2n\pi)^\gamma| << n^{\gamma-1}\log(1/|\eta|)$. We can hence replace U_n by $(2n\pi)^\gamma$ in II without changing the order of magnitude of the expression. We divide II into pieces II_A, II_B, II_C. In II_A, x is such that $n > C\log(1/|\eta|)$. In II_B, n has a range of the form $n > x^{1/\gamma} + r(x)\log(1/|\eta|), r(x) \sim 1$ and in II_C, $C\log(1/|\eta|) < n < x^{1/\gamma} - s(x)\log(1/|\eta|), s(x) \sim 1$.

$$\begin{aligned} II &= II_A + II_B + II_C \\ II_A &= \int_{x=-\infty}^{C_1(\log(1/|\eta|))^\gamma} \sum_{n > C\log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{n^{\gamma-1}\log(1/|\eta|)}{|x - n^\gamma|^2} dx \\ &\sim \int_{|x| < C_1(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x) \frac{\log(1/|\eta|)}{[10\log(1/|\eta)|]^\gamma - x} dx \\ &+ \int_{x=-\infty}^{-C_1(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x) \frac{\log(1/|\eta|)}{[10\log(1/|\eta)|]^\gamma - x} dx \\ &\sim \int_{|x| < C_1(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x) \frac{1}{[\log(1/|\eta)|]^{\gamma-1}} dx \\ &+ \int_{x=-\infty}^{-C_1(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x) \frac{\log(1/|\eta|)}{|x|} dx \\ &\sim \int_{|x| < C_1(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} \left(\frac{|x|}{[\log(1/|\eta|)]^\gamma}\right)^{1-1/\gamma} dx \\ &+ \int_{x=-\infty}^{-C_1(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} \left(\frac{(\log(1/|\eta|))^\gamma}{|x|}\right)^{1/\gamma} dx \\ II_B &\sim \int_{x=C_1(\log(1/|\eta|))^\gamma}^{\infty} \sum_{n > x^{1/\gamma} + r(x)\log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{n^{\gamma-1}\log(1/|\eta|)}{|x - n^\gamma|^2} dx \\ &\sim \int_{x=C_1(\log(1/|\eta|))^\gamma}^{\infty} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} dx \\ II_C &\sim \int_{x=C_2(\log(1/|\eta|))^\gamma}^{\infty} \sum_{C\log(1/|\eta|) < n < x^{1/\gamma} - s(x)\log(1/|\eta|)} \tilde{H}_\alpha(x) \frac{n^{\gamma-1}\log(1/|\eta|)}{|x - n^\gamma|^2} dx \\ II_C &\sim \int_{x=C_2(\log(1/|\eta|))^\gamma}^{\infty} \tilde{H}_\alpha(x) x^{1/\gamma-1} dx \end{aligned}$$

$$
\begin{aligned}
\int II d\mu(\alpha) &\sim \int_\alpha II_A d\mu(\alpha) + \int_\alpha II_B d\mu(\alpha) + \int_\alpha II_C d\mu(\alpha) \\
&<\sim \int_\alpha \int_{|x|>(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} dx d\mu(\alpha) \\
&+ \int_\alpha \int_{|x|<C_1(\log(1/|\eta|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} \left(\frac{|x|}{(\log(1/|\eta|))^\gamma}\right)^{1-1/\gamma} dx d\mu(\alpha) \\
&\to 0
\end{aligned}
$$

by the Lebesgue dominated convergence theorem.

Case (iii): $|w+\lambda\eta| < d|\eta|$. This case is symmetric to cases (i) and (ii), so done.

Case (iv): $|z|, |w| < c|\epsilon|$; $|z-\eta|, |w+\lambda\eta| > d|\eta|$. We recall the estimate of $H_{\alpha,n}(p)$ at intersection points from Lemma 11. The contribution W to the geometric wedge product is:

$$
\int_\alpha \left[\sum_{|n|>[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C} \int_{|x|<2|n|^\gamma} \frac{\tilde{H}_\alpha(x)dx}{|n|^\gamma} + \int_{|x|>2|n|^\gamma} \frac{\tilde{H}_\alpha(x)|n|^\gamma}{x^2} dx \right] d\mu(\alpha)
$$

We divide the first integral into two pieces, so $W = W_A + W_B + W_C$.

$$
\begin{aligned}
W_A &\sim \int_\alpha \left[\int_{|x|<2[[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C|^\gamma} \sum_{|n|>[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C} \frac{\tilde{H}_\alpha(x)dx}{|n|^\gamma} \right] d\mu(\alpha) \\
&\sim \int_\alpha \left[\int_{|x|<2[[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C|^\gamma} \frac{\tilde{H}_\alpha(x)dx}{(\log(1/|\epsilon|))^{\gamma-1}} \right] d\mu(\alpha) \\
&\sim \int_\alpha \left[\int_{|x|<2[[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C|^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} dx \left(\frac{|x|}{(\log(1/|\epsilon|))^\gamma}\right)^{1-1/\gamma} \right] d\mu(\alpha) \\
&\to 0 \\
W_B &\sim \int_\alpha \left[\int_{|x|>2[[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C|^\gamma} \sum_{(|x|/2)^{1/\gamma}}^{\infty} \frac{\tilde{H}_\alpha(x)dx}{|n|^\gamma} \right] d\mu(\alpha) \\
W_B &\sim \int_\alpha \left[\int_{|x|>2[[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C|^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} dx \right] d\mu(\alpha) \\
&\to 0 \\
W_C &\sim \int_\alpha \left[\int_{|x|>2[[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C|^\gamma} \sum_{|n|=[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C}^{(|x|/2)^{1/\gamma}} \frac{\tilde{H}_\alpha(x)|n|^\gamma dx}{x^2} \right] d\mu(\alpha) \\
W_C &\sim \int_\alpha \left[\int_{|x|>2[[|1-a|\log(1/|\epsilon|)]/[2\pi b]-C|^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} dx \right] d\mu(\alpha) \\
&\to 0
\end{aligned}
$$

Hence we have finished the part of the proof of Theorem 7 where we consider intersection points in $D_1 = \{|z|, |w| < c|\epsilon|\}$.

4. Proof of Theorem 7 for intersection points in $D_2 \subset B(0, C|\epsilon|)$ close to the separatrices.

Recall that D_2 denotes the region $c|\epsilon| < |z| < C|\epsilon|, |w| < r|\epsilon|$ where $C >> 1$ is some constant and $0 < r < c$ depends on the choice of C.

We consider intersection points of $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ in D_2. We parametrize L_α with $(u + iv)$ and L^ϵ_β with $u' + iv'$.

$$
\begin{array}{rcl}
L_{\alpha,n}: & & c|\epsilon| < |z| < C|\epsilon| \\
\log(1/|\epsilon|) - C & < & v < \log(1/|\epsilon) + C \\
|w| & = & e^{-bu-av} < r|\epsilon| \\
bu + av & > & \log(1/r) + \log(1/|\epsilon|) \\
u & > & \frac{1-a}{b}\log(1/|\epsilon|) - C \\
n & > & \frac{1-a}{2\pi b}\log(1/|\epsilon|) - C \\
L^\epsilon_{\beta,m}: & & |z'| < C|\epsilon| \\
v' & > & \log(1/|\epsilon|) - C \\
|\epsilon|(1-2c) & < & |w'| < |\epsilon|(1+2c) \\
\log(1/|\epsilon| - 2c & < & bu' + av' < \log(1/|\epsilon|) + 2c \\
\frac{1}{b}\log(1/|\epsilon|) - 2c/b - av'/b & < & u' < \frac{1}{b}\log(1/|\epsilon|) - av'/b + 2c/b
\end{array}
$$

Lemma 13. *If $a \neq 0$, there is an integer N so that for small r, there is at most N intersection points between any pair $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$.*

Proof. This follows from considering the slopes of the plaques, given by the forms ω, ω_ϵ. Namely the slope of the $L_{\alpha,n}$ is very small and the slope of $L^\epsilon_{\beta,m}$ has close to constant larger modulus and close to constant argument on each of N small squares where there might be an intersection. □

Next we estimate $h_{\alpha,n}$ at an intersection point.

Case (i): $n < \log(1/|\epsilon|)$:

Then $V \sim (\log(1/|\epsilon|))^\gamma, |U| <\sim (\log(1/|\epsilon|))^\gamma$.

$$
\begin{aligned}
h_{\alpha,n}(p) &\sim \int_{|x|<C(\log(1/|\epsilon|))^\gamma} H_\alpha(x)\frac{1}{(\log(1/|\epsilon|))^\gamma}dx \\
&+ \int_{|x|>C(\log(1/|\epsilon|))^\gamma} H_\alpha(x)\frac{(\log(1/|\epsilon|))^\gamma}{x^2}dx \\
&\sim \int_{|x|<C(\log(1/|\epsilon|))^\gamma} H_\alpha(x)|x|^{1/\gamma-1}\frac{|x|}{(\log(1/|\epsilon|))^\gamma}|x|^{-1/\gamma}dx \\
&+ \int_{|x|>C(\log(1/|\epsilon|))^\gamma} H_\alpha(x)|x|^{1/\gamma-1}\frac{(\log(1/|\epsilon|))^\gamma}{|x|}|x|^{-1/\gamma}dx
\end{aligned}
$$

Case (ii): $n > \log(1/|\epsilon|)$

Then $U \sim n^\gamma, V \sim n^{\gamma-1}\log(1/|\epsilon|)$.

$$
h_{\alpha,n}(p) \sim \int H_\alpha(x)\frac{n^{\gamma-1}\log(1/|\epsilon|)}{(n^{\gamma-1}\log(1/|\epsilon|))^2+|x-n^\gamma|^2}
$$

We observe that this integral has already been estimated above. Namely see Case (ii), integrals I+II.

So we get

$$
\begin{aligned}
\sum_{n>10\log(1/|\epsilon|}h_{\alpha,n}(p) &<\sim \int_{|x|>(\log(1/|\epsilon|)^\gamma} H_\alpha(x)|x|^{1/\gamma-1}dx \\
&+ \int_{|x|<C(\log(1/|\epsilon|))^\gamma} H_\alpha(x)|x|^{1/\gamma-1}\left(\frac{|x|}{\log(1/|\epsilon|)}\right)^\gamma dx
\end{aligned}
$$

We estimate next $h^\epsilon_{\beta,m}(p)$. $V' \sim (v')^\gamma$, $|U'| <\sim (v')^\gamma$.

$$
\begin{aligned}
h^\epsilon_{\beta,m}(p) &\sim \int_{|y|<C(v')^\gamma} H_\beta(y)\frac{1}{(v')^\gamma}dy \\
&+ \int_{|y|>Cv')^\gamma} H_\beta(y)\frac{(v')^\gamma}{y^2}dy \\
h^\epsilon_{\beta,m}(p) &\sim \int_{|y|<C(v')^\gamma} H_\beta(y)|y|^{1/\gamma-1}\left(\frac{|y|}{(v')^\gamma}\right)^{1-1/\gamma}\frac{1}{v'}dy \\
&+ \int_{|y|>Cv')^\gamma} H_\beta(y)|y|^{1/\gamma-1}\left(\frac{(v')^\gamma}{|y|}\right)^{1+1/\gamma}\frac{1}{v'}dy
\end{aligned}
$$

Note that for $a \neq 0$, we have that

$$
\begin{aligned}
\log(1/|\epsilon|)/a - bu'/a - 2c/|a| &< v' < \log(1/|\epsilon|)/a - bu'/a + 2c/|b| \\
\log(1/|\epsilon|)/a - 2m\pi b/a - C &< v' < \log(1/|\epsilon|)/a - 2m\pi b/a + C \\
v' &> \log(1/|\epsilon|) \\
m/a &< \frac{1}{2\pi b}\log(1/|\epsilon|)(1/a-1) + C
\end{aligned}
$$

$$
\begin{aligned}
\Sigma &:= \sum_{m/a<\frac{1}{2\pi b}\log(1/|\epsilon|)(1/a-1)+C} h^{\epsilon}_{\beta,m} \\
\Sigma &<\sim \sum_m \int_{|y|<C(\log(1/|\epsilon|)/a-b2m\pi/a)^\gamma} H_\beta(y)|y|^{1/\gamma-1} \\
&* \left(\frac{|y|}{(\log(1/|\epsilon|)/a-b2m\pi/a)^\gamma}\right)^{1-1/\gamma} \\
&* \frac{1}{|\log(1/|\epsilon|)/a-b2m\pi/a|}dy \\
&+ \sum_m \int_{|y|>C(\log(1/|\epsilon|)/a-b2m\pi/a)^\gamma} H_\beta(y)|y|^{1/\gamma-1} \\
&* \left(\frac{(\log(1/|\epsilon|)/a-b2m\pi/a)^\gamma}{|y|}\right)^{1+1/\gamma} \\
&* \frac{1}{\log(1/|\epsilon|)/a-b2m\pi/a}dy \\
&= I+II \\
I &= I_A+I_B \\
I_A &= \int_{|y|<C(\log(1/|\epsilon|))^\gamma} \sum_{m/a<\frac{1}{2\pi b}\log(1/|\epsilon|)(1/a-1)+C} H_\beta(y)|y|^{1/\gamma-1} \\
&* \left(\frac{|y|}{(\log(1/|\epsilon|)/a-b2m\pi/a)^\gamma}\right)^{1-1/\gamma} \\
&* \frac{1}{|\log(1/|\epsilon|)/a-b2m\pi/a|}dy \\
&\sim \int_{|y|<C(\log(1/|\epsilon|))^\gamma} H_\beta(y)|y|^{1/\gamma-1}\left(\frac{|y|}{(\log(1/|\epsilon|))^\gamma}\right)^{1-1/\gamma} \\
I_B &= \int_{|y|=C(\log(1/|\epsilon|))^\gamma}^{\infty} \sum_{m/a<\frac{\log(1/|\epsilon|)}{2\pi ab}-(|y|/C)^{1/\gamma}} H_\beta(y)|y|^{1/\gamma-1} \\
&* \left(\frac{|y|}{(\log(1/|\epsilon|)/a-b2m\pi/a)^\gamma}\right)^{1-1/\gamma} \\
&* \frac{1}{|\log(1/|\epsilon|)/a-b2m\pi/a|}dy \\
&\sim \int_{|y|=C(\log(1/|\epsilon|))^\gamma}^{\infty} H_\beta(y)|y|^{1/\gamma-1}dy \\
II &\sim \sum_{\frac{1}{2\pi b}\log(1/|\epsilon|)(1/a-1)>m/a>\frac{\log(1/|\epsilon|)}{2\pi ab}-(|y|/C)^{1/\gamma}} \int_{|y|=C(\log(1/|\epsilon|))^\gamma}^{\infty} H_\beta(y)|y|^{1/\gamma-1} \\
&* \left(\frac{(\log(1/|\epsilon|)/a-b2m\pi/a)^\gamma}{|y|}\right)^{1+1/\gamma} \\
&* \frac{1}{\log(1/|\epsilon|)/a-b2m\pi/a}dy \\
&<\sim \int_{|y|=C(\log(1/|\epsilon|))^\gamma}^{\infty} H_\beta(y)|y|^{1/\gamma-1}dy
\end{aligned}
$$

With these estimates it follows that Theorem 7 is proved for the region D_2 close to the separatrices, in the ball $B(0, C\epsilon)$ provided that $a \neq 0$.

The case $a = 0$:

We fix (α, n) and (β, m) and investigate intersection points. Note that since $a = 0$, we need $(\log(1/|\epsilon|)/b - 2c/b < u' < (\log(1/|\epsilon|)/b + 2c/b$. Hence there are at most finitely many possible values for $m \sim (\log(1/|\epsilon|))/(2\pi b)$. We proceed as if there is at most one. This will suffice. Also note that since we assume that $|z'| < C|\epsilon|$ we also need $v' > \log(1/|\epsilon|) - C'$. For every integer $k > 0$ we might have an intersection point $p_{n,k}$ between $L_{\alpha,n}$ and $L^{\epsilon}_{\beta,m}$ with $\log(1/|\epsilon|) - C' + k\pi < v' \leq \log(1/|\epsilon|) + (k+1)\pi$.

We estimate $h^{\epsilon}_{\beta,m}(p_{n,k})$.

Lemma 14. *When $a = 0$, then $\gamma = 2$.*

Proof. The inequalities $|z| < 1, |w| < 1$ lead to $u, v > 0$. □

We have $u' \sim \log(1/|\epsilon|), v' \sim \log(1/|\epsilon|) + k$ so $U' + iV' = (u')^2 - (v')^2 + 2iu'v'$ Hence if $0 < k < C'' \log(1/|\epsilon)$ we have the estimate $|U'| <\sim (\log(1/|\epsilon))^2 \sim V'$. If $k > C'' \log(1/|\epsilon|)$ we have $U' \sim -k^2, V' \sim k \log(1/|\epsilon|)$.

(i) $0 < k < C'' \log(1/|\epsilon|)$:

$$\begin{aligned}
h^{\epsilon}_{\beta,m}(p_{n,k}) &\sim \int_{|y|<C(\log(1/|\epsilon|))^2} H_\beta(y) \frac{1}{(\log(1/|\epsilon|))^2} \\
&+ \int_{|y|>C(\log(1/|\epsilon|))^2} H_\beta(y) \frac{(\log(1/|\epsilon|))^2}{y^2} dy \\
\sum_{k=0}^{C'' \log(1/|\epsilon)} h^{\epsilon}_{\beta,m}(p_{n,k}) &\sim \int_{|y|<C(\log(1/|\epsilon|))^2} H_\beta(y) \frac{1}{(\log(1/|\epsilon|))} \\
&+ \sum_{k=0}^{C'' \log(1/|\epsilon)} \int_{|y|>C(\log(1/|\epsilon|))^2} H_\beta(y) \frac{(\log(1/|\epsilon|))^3}{y^2} dy \\
&\sim \int_{|y|<C(\log(1/|\epsilon|))^2} H_\beta(y)|y|^{-1/2} \left(\frac{|y|}{(\log(1/|\epsilon|))^2}\right)^{1/2} \\
&+ \sum_{k=0}^{C'' \log(1/|\epsilon)} \int_{|y|>C(\log(1/|\epsilon|))^2} H_\beta(y)|y|^{-1/2} \left(\frac{(\log(1/|\epsilon|))^2}{|y|}\right)^{3/2} dy
\end{aligned}$$

(ii) $k > C'' \log(1/|\epsilon|)$:

$$
\begin{aligned}
h^\epsilon_{\beta,m}(p_{n,k}) &\sim \int H_\beta(y)\frac{k\log(1/|\epsilon|)}{(k\log(1/|\epsilon|))^2+(y+k^2)^2}dy \\
&\sim \int_{|y+k^2|<k\log(1/|\epsilon|)} H_\beta(y)\frac{1}{k\log(1/|\epsilon|)}dy \\
&+ \int_{|y+k^2|>k\log(1/|\epsilon|)} H_\beta(y)\frac{k\log(1/|\epsilon|)}{(y+k^2)^2}dy \\
\Sigma &:= \sum_{k>C''\log(1/|\epsilon|)} h^\epsilon_{\beta,m}(p_{n,k}) \\
&= I_A+I_B+II_A+II_B+II_C \\
I_A &\sim \int_{y=(-(C'')^2-C'')(\log(1/|\epsilon|))^2}^{(-(C'')^2+C'')\log(1/|\epsilon|)} \sum_{k=C''\log(1/|\epsilon|)}^{\sqrt{-y+r(y)\log(1/|\epsilon|)}} H_\beta(y)\frac{1}{k\log(1/|\epsilon|)}dy \\
&[r(y)\sim\sqrt{|y|}] \\
I_A &\sim \int_{y=(-(C'')^2-C'')(\log(1/|\epsilon|))^2}^{(-(C'')^2+C'')\log(1/|\epsilon|)} H_\beta(y)\frac{1}{\log(1/|\epsilon|)}dy \\
I_A &\sim \int_{y=(-(C'')^2-C'')(\log(1/|\epsilon|))^2}^{(-(C'')^2+C'')\log(1/|\epsilon|)} H_\beta(y)|y|^{-1/2}dy \\
I_B &\sim \int_{y=-\infty}^{(-(C'')^2-C'')(\log(1/|\epsilon|))^2} \sum_{\sqrt{-y-s(y)\log(1/|\epsilon|)}}^{\sqrt{-y+r(y)\log(1/|\epsilon|)}} H_\beta(y)\frac{1}{k\log(1/|\epsilon|)} \\
&[s(y)\sim\sqrt{|y|}] \\
I_B &\sim \int_{y=-\infty}^{(-(C'')^2-C'')(\log(1/|\epsilon|))^2} H_\beta(y)\frac{1}{\log(1/|\epsilon|)} \\
I_B &\sim \int_{y=-\infty}^{(-(C'')^2-C'')(\log(1/|\epsilon|))^2} H_\beta(y)|y|^{-1/2}\left(\frac{|y|}{(\log(1/|\epsilon|))^2}\right)^{1/2} \\
II_A &\sim \int_{y=-\infty}^{((-(C'')^2-C''))(\log(1/|\epsilon|))^2} \sum_{k=C''\log(1/|\epsilon|)}^{\sqrt{-y-s(y)\log(1/|\epsilon|)}} H_\beta(y)\frac{k\log(1/|\epsilon|)}{(y+k^2)^2}dy \\
II_A &\sim \int_{y=-\infty}^{((-(C'')^2-C''))(\log(1/|\epsilon|))^2} H_\beta(y)|y|^{-1/2}dy \\
II_B &\sim \int_{y=-\infty}^{((-(C'')^2+C''))(\log(1/|\epsilon|))^2} \sum_{k=\sqrt{-y+r(y)\log(1/|\epsilon|)}}^{\infty} H_\beta(y)\frac{k\log(1/|\epsilon|)}{(y+k^2)^2}dy \\
II_B &\sim \int_{y=-\infty}^{((-(C'')^2+C''))(\log(1/|\epsilon|))^2} H_\beta(y)|y|^{-1/2}dy
\end{aligned}
$$

$$
\begin{aligned}
II_C &\sim \int_{((-(C'')^2+C''))(\log(1/|\epsilon|))^2}^{\infty} \sum_{k=C''\log(1/|\epsilon|)}^{\infty} H_\beta(y)\frac{k\log(1/|\epsilon|)}{(y+k^2)^2}dy \\
II_C &\sim \int_{((-(C'')^2+C''))(\log(1/|\epsilon|))^2}^{\infty} H_\beta(y)\frac{\log(1/|\epsilon|)}{y+(C''\log(1/|\epsilon|))^2}dy \\
II_C &\sim \int_{((-(C'')^2+C''))(\log(1/|\epsilon|))^2}^{(\log(1/|\epsilon|))^2} H_\beta(y)\frac{1}{\log(1/|\epsilon|)}dy \\
&+ \int_{(\log(1/|\epsilon|))^2}^{\infty} H_\beta(y)\frac{\log(1/|\epsilon|)}{y}dy \\
II_C &\sim \int_{((-(C'')^2+C''))(\log(1/|\epsilon|))^2}^{(\log(1/|\epsilon|))^2} H_\beta(y)|y|^{-1/2}\frac{|y|^{1/2}}{\log(1/|\epsilon|)}dy \\
&+ \int_{(\log(1/|\epsilon|))^2}^{\infty} H_\beta(y)|y|^{-1/2}\frac{|y|^{-1/2}}{\log(1/|\epsilon|)}dy
\end{aligned}
$$

Next we estimate $h_{\alpha,n}(p_{n,k})$. Note however, that the estimates for the case $a \neq 0$ still applies to $h_{\alpha,n}$. This condition was not used to estimate h_α. Hence we are done with the proof of Theorem 7 for the case of intersection points in D_2.

We next let D_3 denote the points in $B(0, C|\epsilon|)$ which are at distance at least $r|\epsilon|$ from all separatrices.

5. Proof of Theorem 7 for points in D_3, i.e. points in $B(0, C|\epsilon|)$ which are at distance at least $r|\epsilon|$ from the separatices.

Recall that by a scaling argument, there is, see Lemma 10, an integer N independent of ϵ so that if we take any two plaques of two leaves $L_\alpha, L_\beta^\epsilon$, then they intersect in D_3 in at most N points.

We estimate H_α on $L_{\alpha,n} \cap D_3$. We can assume $a \neq 1$, otherwise flip the axes.

$$
\begin{aligned}
r|\epsilon| &< |z| < C|\epsilon| \\
r|z| &< e^{-v} < C|\epsilon| \\
\log(1/|\epsilon|) - C' &< v < \log(1/|\epsilon|) + C' \\
r|\epsilon| &< |w| < C|\epsilon| \\
r|\epsilon| &< e^{-bu-av} < C|\epsilon| \\
\log|\epsilon| - C'' &< -bu - av < \log|\epsilon| + C'' \\
\log(1/|\epsilon|) - C'' - av &< bu < -av + \log(1/|\epsilon|) + C'' \\
(1-a)\log(1/|\epsilon|) - C &< bu < (1-a)\log(1/|\epsilon|) + C \\
\frac{1-a}{b}\log(1/|\epsilon|) - C &< b < \frac{1-a}{b}\log(1/|\epsilon|) + C \\
(u+iv)^\gamma &= U + iV \\
V &\sim (\log(1/|\epsilon|)^\gamma \\
|U| &<\sim (\log(1/|\epsilon|))^\gamma
\end{aligned}
$$

$$
\begin{aligned}
h_{\alpha,n} &\sim \int H_\alpha(x)\frac{(\log(1/|\epsilon|))^\gamma}{(\log(1/|\epsilon|))^{2\gamma}+(x-U)^2}dx \\
&\sim \int_{|x|<C(\log(1/|\epsilon|))^\gamma} H_\alpha(x)\frac{1}{(\log(1/|\epsilon|))^\gamma}dx \\
&+ \int_{|x|>(\log(1/|\epsilon|)^\gamma} H_\alpha(x)\frac{(\log(1/|\epsilon|))^\gamma}{x^2}dx \\
&\sim \int_{|x|<C(\log(1/|\epsilon|))^\gamma} H_\alpha(x)(|x|+1)^{1/\gamma-1}\frac{|x|+1}{(\log(1/|\epsilon|))^\gamma}(|x|+1)^{-1/\gamma}dx \\
&+ \int_{|x|>(\log(1/|\epsilon|)^\gamma} H_\alpha(x)(|x|+1)^{1/\gamma-1}\frac{(\log(1/|\epsilon|))^\gamma}{|x|+1}(|x|+1)^{-1/\gamma}dx
\end{aligned}
$$

It follows from these estimate applied to H_β as well, that Theorem 7 is valid for intersection points in D_3.

6. Theorem 7 for $D_4 = \Delta^2(0,\delta) \setminus \Delta^2(0, C|\epsilon|)$

There are 3 regions to consider:

$$
\begin{aligned}
D_4 &= R_1 \cup R_2 \cup R_3 \\
R_1 &= \{C|\epsilon| < |z|, |w| < \delta\} \\
R_2 &= \{C|\epsilon| < |z| < \delta, |w| < C|\epsilon|\} \\
R_3 &= \{C|\epsilon| < |w| < \delta, |z| < C|\epsilon|\}
\end{aligned}
$$

Note that since we have assumed $a \neq 1$, the cases of R_2 and R_3 are not completely symmetric. We will leave it to the reader to verify that the estimates we do later for R_2 nevertheless hold for R_3.

7. Theorem 7 for R_1, the diagonal part of D_4

We first outline our approach. Fix parameters α, β and corresponding plaques $L_{\alpha,n}, L^\epsilon_{\beta,m}$. Next we divide R_1 into dyadic components, rings, $\{R(p)\}$ in the $z-$direction, $e^{-p-1} < |z| < e^{-p}, C|\epsilon| < |w| < \delta$ Then we estimate h_α and h_β on $L_{\alpha,n} \cap R(p)$ and $L^\epsilon_{\beta,m} \cap R(p)$ respectively. Next, for fixed α, β, n, m we estimate the values of p where the leaves $L_{\alpha,n}, L^\epsilon_{\beta,m}$ might intersect, and the number of intersection points for each such p. Putting this information together we can estimate the contribution from R_1 to the geometric wedge product.

Pick a plaque $L_{\alpha,n}$ and a point (z,w) in $L_{\alpha,n} \cap R(p)$ parametrized by (u,v). Then

$$\begin{array}{rcl} e^{-p-1} & < & |z| = e^{-v} < e^{-p} \\ \log(1/\delta) & < & v < \log(1/|\epsilon|) - C \\ \log(1/\delta) & < & p < \log(1/|\epsilon|) - C \\ 2n\pi & < & u < 2(n+1)\pi \\ C|\epsilon| & < & |w| < \delta \\ \log(1/\delta) & < & bu + av < \log(1/|\epsilon|) - C \\ \dfrac{\log(1/\delta)}{b} - av/b & < & u < \dfrac{\log(1/|\epsilon|) - C}{b} - av/b \end{array}$$

Case (i): $a \neq 0, n < p$

$$\begin{array}{rcl} (u+iv)^\gamma & = & U + iV \\ & \sim & U + ip^\gamma, |U| <\sim p^\gamma \\ H_{\alpha,n} & \sim & \displaystyle\int_{|x|<Cp^\gamma} \tilde{H}_\alpha(x)/p^\gamma dx + \int_{|x|>Cp^\gamma} \tilde{H}_\alpha(x) p^\gamma / x^2 dx \\ & \sim & \displaystyle\int_{|x|<Cp^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} \left(\frac{|x|}{p^\gamma}\right)^{1-1/\gamma} \frac{1}{p} dx \\ & + & \displaystyle\int_{|x|>Cp^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} \left(\frac{p^\gamma}{|x|}\right)^{1+1/\gamma} \frac{1}{p} dx \end{array}$$

Case (ii) $a \neq 0, n > p$

$$\begin{array}{rcl} (u+iv)^\gamma & = & U + iV \\ & \sim & n^\gamma + ipn^{\gamma-1} \\ H_{\alpha,n} & \sim & \displaystyle\int_{|x-n^\gamma|\leq pn^{\gamma-1}} \tilde{H}_\alpha(x) \frac{1}{pn^{\gamma-1}} dx \\ & + & \displaystyle\int_{n^\gamma/2>|x-n^\gamma|>pn^{\gamma-1}} \tilde{H}_\alpha(x) \frac{pn^{\gamma-1}}{|x-n^\gamma|^2} dx \\ & + & \displaystyle\int_{n^\gamma/2<|x-n^\gamma|<2n^\gamma} \tilde{H}_\alpha(x) \frac{pn^{\gamma-1}}{n^{2\gamma}} dx \\ & + & \displaystyle\int_{|x-n^\gamma|>2n^\gamma} \tilde{H}_\alpha(x) \frac{pn^{\gamma-1}}{x^2} dx \\ & = & I + II + III + IV \end{array}$$

We will usually leave the case $a = 0$ to the reader.
Case (iii) $a = 0$

$$
\begin{aligned}
\gamma &= 2 \\
(u+iv)^2 &= u^2 - v^2 + 2iuv \\
h_{\alpha,n} &= \int H_{\alpha,n}(x)\frac{uv}{(uv)^2 + (x-(u^2-v^2))^2}dx \\
&\sim \int H_{\alpha,n}(x)\frac{uv}{(uv)^2 + (x+v^2-u^2)^2}dx \\
&\sim \int_{|x+v^2-u^2|<uv} H_{\alpha,n}(x)\frac{uv}{(uv)^2 + (x+v^2-u^2)^2}dx \\
&+ \int_{|x+v^2-u^2|>uv} H_{\alpha,n}(x)\frac{uv}{(uv)^2 + (x+v^2-u^2)^2}dx \\
&\sim \int_{|x+v^2-u^2|<uv} H_{\alpha,n}(x)\frac{1}{uv}dx \\
&+ \int_{|x+v^2-u^2|>uv} H_{\alpha,n}(x)\frac{uv}{(x+v^2-u^2)^2}dx
\end{aligned}
$$

$$
\begin{aligned}
u &\sim v: \\
&\sim \int_{|x|<\sim u^2} H_{\alpha,n}(x)\frac{1}{u^2}dx \\
&+ \int_{|x|>\sim u^2} H_{\alpha,n}(x)\frac{u^2}{x^2}dx \\
&\sim \int_{|x|<\sim u^2} H_{\alpha,n}(x)|x|^{-1/2}\left(\frac{|x|}{u^2}\right)^{1/2}\frac{1}{u}dx \\
&+ \int_{|x|>\sim u^2} H_{\alpha,n}(x)|x|^{-1/2}\frac{u^2}{|x|}\frac{1}{|x|^{1/2}}dx
\end{aligned}
$$

$$
\begin{aligned}
u &<< v \\
&\sim \int_{|x+v^2|<\sim uv} H_{\alpha,n}(x)\frac{1}{uv}dx \\
&+ \int_{|x+v^2|>\sim uv} H_{\alpha,n}(x)\frac{uv}{(x+v^2-u^2)^2}dx \\
&\sim \int_{|x+v^2|<\sim uv} H_{\alpha,n}(x)|x|^{-1/2}\frac{1}{u}dx \\
&+ \int_{|x+v^2|>\sim uv} H_{\alpha,n}(x)\frac{uv}{(x+v^2)^2}dx \\
&\sim \int_{|x+v^2|<\sim uv} H_{\alpha,n}(x)|x|^{-1/2}\frac{1}{u}dx \\
&+ \int_{|x+v^2|>\sim uv} H_{\alpha,n}(x)|x|^{-1/2}\frac{uv|x|^{1/2}}{(x+v^2)^2}dx
\end{aligned}
$$

$$\begin{aligned} u & >> v \\ & \sim \int_{|x-u^2|<\sim uv} H_{\alpha,n}(x)|x|^{-1/2}\frac{1}{v}dx \\ & + \int_{|x-u^2|>\sim uv} H_{\alpha,n}(x)|x|^{-1/2}\frac{uv|x|^{1/2}}{(x-u^2)^2}dx \end{aligned}$$

So now we have estimated h_α on $L_{\alpha,n} \cap R(p)$. The analogue estimates are valid for H_β on $L^\epsilon_{\beta,m}$. The reason is that $e^{-p} >> |\epsilon|$. Our next step is to locate for which $R(p)$ there is an intersection between $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$.

Fix $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ and assume $(z,w) \in L_{\alpha,n} \cap L^\epsilon_{\beta,m}$. We can write

$$\begin{aligned} z &= e^{i(\zeta+(\log|\alpha|)/b} \\ \zeta &= u+iv \\ 2n\pi &< u < 2(n+1)\pi \\ |z| &= e^{-v} \end{aligned}$$

Also $(z,w) = \Phi_\epsilon(z',w'), (z',w') \in L_{\beta,m}$.

$$\begin{aligned} z' &= e^{i(\zeta'+(\log|\alpha|)/b} \\ \zeta' &= u'+iv' \\ 2m\pi &< u' < 2(m+1)\pi \\ |z'| &= e^{-v'} \\ z &= \alpha(\epsilon) + e^{i(\zeta'+(\log|\beta|)/b)} + \epsilon\mathcal{O}(z',w') \\ w &= \beta(\epsilon) + \beta e^{i\lambda(\zeta'+(\log|\beta|)/b)} + \epsilon\mathcal{O}(z',w') \end{aligned}$$

Our goal is to locate which $R(p)$ the point (z,w) can belong to. So we need to find p so that $e^{-p-1} < |z| = e^{-v} < e^{-p}$, i.e. we need to get a good estimate for v in terms of α, β, n, m.

There are 4 unknowns, u, v, u', v'. However, $u \sim 2n\pi, u' \sim 2m\pi$, so we only have v, v' left. Also we have two equations for the z and w coordinates respectively. (In fact, since these are complex equations, we have 4 real equations for the two real unknowns v, v'.)

Before we proceed we show at first that for there to be an intersection, we actually must require that n and m are very close.

Lemma 15. *If $L_{\alpha,n}$ and $L^\epsilon_{\beta,m}$ intersect in R_1, it follows that $|m-n| \leq 1$.*

Proof. Recall that

$$\Phi_\epsilon(z,w) = (\alpha(\epsilon), \beta(\epsilon)) + (z,w) + \epsilon\mathcal{O}(z,w).$$

If δ is chosen small enough, this implies that $|\epsilon\mathcal{O}(z,w)| \leq \sigma|\epsilon|$ for any given $0 < \sigma << 1$.

We pick two plaques, $L_{\alpha,n}, L^\epsilon_{\beta,m}$ and consider intersection points in R_1. Let $S > 0$ be such that $|\epsilon|/S < |\alpha(\epsilon)| - \sigma|\epsilon|, |\beta(\epsilon)| - \sigma|\epsilon| < |\alpha(\epsilon)| + \sigma|\epsilon|, |\beta(\epsilon)| + \sigma|\epsilon| < S$. Note that if we increase the constant C used in the definition of D_4, we can still use the same S.

$$\begin{array}{rcl}
L_{\alpha,n} & : & \\
z & = & e^{i(\zeta+(\log|\alpha|)/b} \\
|z| & = & e^{-v} \\
\log(1/|\delta|) & < v & < \log(1/|\epsilon|) - C \\
w & = & \alpha e^{i\lambda(\zeta+(\log|\alpha|)/b)} \\
|w| & = & e^{-bu-av} \\
L_{\beta,m} & : & \\
z' & = & e^{i(\zeta'+(\log|\beta|)/b)} \\
|z'| & = & e^{-v'} \\
\log(1/|\delta|) < & v' & < \log(1/|\epsilon|) - C \\
w' & = & \beta e^{i\lambda(\zeta'+(\log|\beta|)/b)} \\
|w'| & = & e^{-bu'-av'} \\
L^{\epsilon}_{\beta,n} & = & \Phi_\epsilon(L_{\beta,m}) \\
Z & = & \alpha(\epsilon) + e^{i(\zeta'+(\log|\beta|)/b)} + \epsilon\mathcal{O}(z',w') \\
W & = & \beta(\epsilon) + \beta e^{i\lambda(\zeta'+(\log|\beta|)/b)} + \epsilon\mathcal{O}(z',w')
\end{array}$$

Consider an intersection point in R_1 and set $\zeta' = \zeta + c + id$.

$$\begin{array}{rcl}
z & = & Z \\
e^{-v-d} - S|\epsilon| & < & e^{-v} < e^{-v-d} + S|\epsilon| \\
e^{-d} - Se^{v}|\epsilon| & < & 1 < e^{-d} + Se^{v}|\epsilon| \\
Se^{v}|\epsilon| & < & S(1/(C|\epsilon|)|\epsilon| = S/C << 1. \\
|d| & < & 2Se^{v}|\epsilon| < 2S/C
\end{array}$$

$$
\begin{aligned}
w &= W \\
e^{-bu-bc-av-ad} - S|\epsilon| &< e^{-bu-av} < e^{-bu-bc-av-ad} + S|\epsilon| \\
e^{-bc-ad} - Se^{bu+av}|\epsilon| &< 1 < e^{-bc-ad} + Se^{bu+av}|\epsilon| \\
Se^{bu+av}|\epsilon| &< S/C << 1. \\
|bc+ad| &< 2Se^{bu+av}|\epsilon| < 2S/C \\
|bc| &< |bc+ad| + |a||d| \\
&< 2Se^{bu+av}|\epsilon| + |a|2Se^{v}|\epsilon| \\
|c| &< \frac{1}{|b|}\left(2Se^{bu+av}|\epsilon| + |a|2Se^{v}|\epsilon|\right) \\
&< 2S\frac{1+|a|}{C|b|} \\
|c+id| &< \frac{2S}{C}\left(1+\frac{1+|a|}{|b|}\right) << 1
\end{aligned}
$$

□

It is also convenient to show that α and β must be very close if there is an intersection. We estimate first the modulus and next the angle and finally we combine them.

Lemma 16. *Suppose $L_{\alpha,n}$ intersects $L^{\epsilon}_{\beta,m}$ in R_1. Then*

$$
|\log(|\beta|/|\alpha|)| \leq 2S|\epsilon|\left[e^{v}\left(b+|a|\right) + e^{bu+av}\right].
$$

Proof.

$$
\begin{aligned}
z &= Z \\
e^{i(\zeta+(\log|\alpha|)/b)} &= \alpha(\epsilon) + e^{i\zeta+c+id+(\log|\beta|)/b} + \epsilon\mathcal{O} \\
e^{i(\zeta+(\log|\alpha|)/b)}\left[1-e^{ic-d+i(\log(|\beta|/|\alpha|)/b)}\right] &= \alpha(\epsilon) + \epsilon\mathcal{O} \\
e^{-v}\left|1-e^{ic-d+i(\log(|\beta|/|\alpha|)/b)}\right| &\leq S|\epsilon| \\
\left|1-e^{ic-d+i(\log|\beta|/|\alpha|)/b)}\right| &\leq Se^{v}|\epsilon| << 1 \\
|i(c+(\log|\beta|/|\alpha)/b)-d| &\leq 2Se^{v}|\epsilon| \\
|\log(|\beta|/|\alpha|)/b| &\leq 2Se^{v}|\epsilon| + 2Se^{bu+av}|\epsilon|/b + 2S(|a|/b)e^{v}|\epsilon|
\end{aligned}
$$

The Lemma follows.

□

We remark that the lemma as stated is slightly inaccurate. We only can conclude the estimate modulo 2π. However, the parameters $e^{-2\pi b} \leq |\alpha|, |\beta| < 1$ so this problem arises when say $|\alpha|$ is close to 1 and $|\beta|$ is close to $e^{-2\pi b}$. We ignore this technicality which just means that $|\alpha|$ and $|\beta|$ get close after we follow the leaf L_α once around 0 counterclockwise.

Lemma 17. *Write $\beta/\alpha = |\beta/\alpha|e^{i\theta}$. If there are intersection points in R_1, θ is close to 0 mod 2π. More precisely:*

$$|\theta| \leq 2Se^{bu+av}|\epsilon|\left[|a|/b + |a|/b + 1\right] + 2S|\epsilon|e^{v}\left[|a|^2/b + b + (|a| + |a|^2/b)\right].$$

Proof.

$$\begin{aligned}
w &= W \\
\alpha e^{i\lambda(\zeta + (\log|\alpha|/b))} &= \beta(\epsilon) + \beta e^{i\lambda(\zeta + c + id + (\log|\beta|/b))} + \epsilon\mathcal{O} \\
\beta(\epsilon) + \mathcal{O} &= \alpha e^{i\lambda(\zeta + (\log|\alpha|/b))}\left[1 - \frac{\beta}{\alpha}e^{i\lambda(c + id + (\log|\beta|/b))}\right] \\
Se^{bu+av}|\epsilon| &\geq \left|1 - \frac{\beta}{\alpha}e^{i\lambda(c + id + (\log|\beta|/b))}\right| \\
Se^{bu+av}|\epsilon| &\geq \left|1 - \frac{\beta}{\alpha}e^{[-bc - ad - (\log|\alpha|/b)] + i[ac - bd + a(\log(|\beta|/|\alpha|))/b]}\right| \\
1. >> Se^{bu+av}|\epsilon| &\geq \left|1 - e^{[-bc - ad] + i[\theta + ac - bd + a(\log(|\beta|/|\alpha|))/b]}\right| \\
2Se^{bu+av}|\epsilon| &\geq |\theta + ac - bd + a(\log(|\beta|/|\alpha|))/b| \\
|\theta| &\leq |ac| + |bd| + |a||\log(|\beta|/|\alpha|)|/b + 2Se^{bu+av}|\epsilon| \\
&\leq Se^{bu+av}|\epsilon|\left[2|a|/b + 2|a|/b + 2\right] \\
&+ S|\epsilon|e^{v}\left[2|a|^2/b + 2b + 2(|a| + |a|^2/b)\right]
\end{aligned}$$

□

Next we locate more precisely the intersections of $L_{\alpha,n}$ and $L^{\epsilon}_{\beta,m}$ in R_1.

$$
\begin{aligned}
z &= Z \\
e^{i\zeta+i(\log|\alpha|)/b} &= \alpha(\epsilon)+e^{i\zeta'+i(\log|\beta|)/b}+\epsilon\mathcal{O} \\
\zeta' &= \zeta+\Delta \\
e^{i\zeta+i(\log|\alpha|)/b}-e^{i\zeta+i\Delta+i(\log|\beta|)/b} &= \alpha(\epsilon)+\epsilon\mathcal{O} \\
e^{i\zeta+i(\log|\alpha|)/b}\left[1-e^{i\Delta+i(\log(|\beta|/|\alpha|)/b}\right] &= \alpha(\epsilon)+\epsilon\mathcal{O} \\
1-e^{i\Delta+i(\log(|\beta|/|\alpha|)/b} &= e^{-i\zeta-i(\log|\alpha|)/b}\left[\alpha(\epsilon)+\epsilon\mathcal{O}\right] \\
2k\pi+\Delta+(\log(|\beta|/|\alpha|)/b &= ie^{-i\zeta-i(\log|\alpha|)/b}\left[\alpha(\epsilon)+\epsilon\mathcal{O}\right] \\
&+ \mathcal{O}(\epsilon e^{-i\zeta})^2 \\
w &= W \\
\alpha e^{i\lambda(\zeta+(\log|\alpha|)/b)} &= \beta(\epsilon)+\beta e^{i\lambda(\zeta+\Delta+(\log|\beta|)/b)}+\epsilon\mathcal{O} \\
e^{i\lambda\zeta}\left[\alpha e^{i\lambda(\log|\alpha|)/b)}-\beta e^{i\lambda(\Delta+(\log|\beta|)/b)}\right] &= \beta(\epsilon)+\epsilon\mathcal{O} \\
&= e^{i\lambda\zeta}e^{i\lambda(\log|\alpha|)/b)} \\
&* \left[\alpha-\beta e^{i\lambda(ie^{-i\zeta-i(\log|\alpha|)/b}[\alpha(\epsilon)+\epsilon\mathcal{O}])}\right] \\
1-\frac{\beta}{\alpha}e^{i\lambda(\Delta+(\log|\beta|/|\alpha|)/b)} &= e^{-i\lambda\zeta}\frac{e^{-i\lambda(\log|\alpha|)/b)}}{\alpha}\left[\beta(\epsilon)+\epsilon\mathcal{O}\right] \\
-\log\left(\frac{\beta}{\alpha}\right)+i\lambda(\Delta+(\log|\beta|/|\alpha|)/b) &\sim e^{-i\lambda\zeta}\frac{e^{-i\lambda(\log|\alpha|)/b)}}{\alpha}\left[\beta(\epsilon)+\epsilon\mathcal{O}\right] \\
\Delta+\log(|\beta|/|\alpha|)/b-\frac{1}{i\lambda}\log\left(\frac{\beta}{\alpha}\right) &= \frac{1}{i\lambda}e^{-i\lambda\zeta}\frac{e^{-i\lambda(\log|\alpha|)/b)}}{\alpha}\left[\beta(\epsilon)+\epsilon\mathcal{O}\right] \\
&+ \mathcal{O}(e^{-i\lambda\zeta}\epsilon)^2
\end{aligned}
$$

Adding the two expressions with Δ :

Lemma 18. *Suppose that* $L_{\alpha,n}\cap L^{\epsilon}_{\beta,m}\cap R_1\neq\emptyset$. *Then:*

$$
\begin{aligned}
-\frac{1}{i\lambda}\log\left(\frac{\beta}{\alpha}\right) &= ie^{-i\zeta-i(\log|\alpha|)/b}\left[\alpha(\epsilon)+\epsilon\mathcal{O}\right]+\frac{1}{i\lambda}e^{-i\lambda\zeta}\frac{e^{-i\lambda(\log|\alpha|)/b)}}{\alpha}\left[\beta(\epsilon)+\epsilon\mathcal{O}\right] \\
&+ \mathcal{O}(\epsilon e^{-i\zeta})^2+\mathcal{O}(e^{-i\lambda\zeta}\epsilon)^2
\end{aligned}
$$

To continue the search for intersection points of $L_{\alpha,n}, L^{\epsilon}_{\beta,m}$ in R_1, we divide R_1 into 3 pieces.

$$
\begin{aligned}
R_{1A} &= \{C|\epsilon|<|z|,|w|<\delta,|w|<<|z|\} \\
R_{1B} &= \{C|\epsilon|<|z|,|w|<\delta,|z|<<|w|\} \\
R_{1A} &= \{C|\epsilon|<|z|,|w|<\delta,|z|\sim|w|\}
\end{aligned}
$$

Observe that R_{1A} and R_{1B} are similar. We will leave it up to the reader to verify the estimates for R_{1B}.

8. Theorem 7 for R_{1A}, the part of R_1 close to the $z-$axis

We will assume that $a \neq 0$ and leave the verification of the case $a = 0$ to the reader. If $|w| << |z|$, then the second term in the expression for $\log(\beta/\alpha)$ in Lemma 18 on the right dominates and we get

$$\begin{aligned} e^{av+bu}|\epsilon| &\sim |(\beta/\alpha)-1| \\ 2n\pi &< u < 2(n+1)\pi \\ av &\sim \log|(\beta/\alpha)-1| + \log(1/|\epsilon|) - 2nb\pi \\ |v - \frac{\log|(\beta/\alpha)-1| + \log(1/|\epsilon|) - 2nb\pi}{a}| &< C \\ C|\epsilon| &< e^{-v} < \delta \\ \log 1/\delta &< v < \log 1/|\epsilon| - C \\ p &< v < p+1 \end{aligned}$$

Lemma 19. *For intersection points in $R_{1,A}$, we have*

$$\frac{C|\epsilon|}{\delta} < |\beta - \alpha| < \frac{1}{C}.$$

Proof. Since $e^{av+bu}|\epsilon| \sim |\frac{\beta}{\alpha} - 1| \sim |\beta - \alpha|$ and $e^{av+bu} = 1/|w|$ we have $|\beta - \alpha| \sim |\epsilon|/|w|$. But $C|\epsilon| < |w| < |z|/C < \delta/C$. The lemma follows. □

Lemma 20. *Suppose that $L_{\alpha,n}$ intersects $L^{\epsilon}_{\beta,m}$ in R_{1A}. Then the intersection points must by in $R(p)$ for some*

$$|p - \frac{\log|(\beta/\alpha)-1| + \log(1/|\epsilon|) - 2nb\pi}{a}| < C.$$

For the plaque to enter R_1 we further need n to satisfy

$$\log|(\beta/\alpha)-1| + \log(1/|\epsilon|) - 2nb\pi \in I$$

where I is the interval with endpoints $a\log 1/\delta, a\log(1/|\epsilon|) - aC$

Our next step is to verify that there is a uniform bound on the number of intersection points of $L_{\alpha,n}, L^{\epsilon}_{\beta,m}$ in R_{1A}.

In order to study the number of intersections between plaques, we compare their slopes:

Suppose $(z,w) = (Z,W) := \Phi_\epsilon(z',w')$ is an intersection point of $L_{\alpha,n}$ and $L^{\epsilon}_{\beta,m}$ in R_1. The slope S_1 of $L_{\alpha,n}$ is $\lambda w/z$. The slope of the perturbed leaf is S_2.

$$\begin{aligned}
\Phi'_\epsilon(z',w')(z',\lambda w') &= (z'+\epsilon\mathcal{O}(z',w'),\lambda w'+\epsilon\mathcal{O}(z',w')) \\
S_2 &= \frac{\lambda w'+\epsilon\mathcal{O}(z',w')}{z'+\epsilon\mathcal{O}(z',w')} \\
&= \frac{\lambda W-\lambda\beta(\epsilon)+\epsilon\mathcal{O}(Z,W)}{Z-\alpha(\epsilon)+\epsilon\mathcal{O}(Z,W)} \\
S_2 &= \frac{\lambda w-\lambda\beta(\epsilon)+\epsilon\mathcal{O}(z,w)}{z-\alpha(\epsilon)+\epsilon\mathcal{O}(z,w)} \\
S_2-S_1 &= \frac{\lambda w-\lambda\beta(\epsilon)+\epsilon\mathcal{O}(z,w)}{z-\alpha(\epsilon)+\epsilon\mathcal{O}(z,w)}-\lambda w/z \\
&= \frac{-\lambda\beta(\epsilon)z+\lambda w\alpha(\epsilon)+\epsilon\mathcal{O}(z^2,zw,w^2)}{z(z-\alpha(\epsilon)+\epsilon\mathcal{O}(z,w))} \\
|z|\sim|w| &: \quad S_2-S_1\sim\frac{\lambda}{z^2}\left(w\alpha(\epsilon)-z\beta(\epsilon)\right) \\
|w|>>|z| &: \quad S_2-S_1\sim\frac{\epsilon w}{z^2} \\
|w|<<|z| &: \quad S_2-S_1\sim\frac{\epsilon}{z}
\end{aligned}$$

Lemma 21. *There at most a uniformly bounded number of intersection points in R_{1A}.*

Proof. The case of R_{1A}, R_{1B} follows from slope estimates, For the case R_{1C}, note that leaves might be tangent when (w/z) is close to $\beta(\epsilon)/\alpha(\epsilon)$. They both have slope about λ. But since we assume that $\lambda\neq\beta'(0)/\alpha'(0)$, this tangency is at most of order 2. □

We estimate the contribution to $T\wedge_g T^\epsilon$ from R_{1A}. We assume again that $a\neq 0$. By Lemma 18, the parameters α,β are restricted to the values: $e^{-2\pi b}<|\alpha|,|\beta|<1, 1/C>|\beta-\alpha|>C|\epsilon|/\delta$. So fix α,β. Next, by Lemma 15, we can set $n=m$ to be some integer in the interval given by Lemma 19. The case $n=m\pm 1$ is similar. Because of the finiteness of the number of intersection points, see Lemma 20, we can set

$$p=p(n)=\frac{\log|(\beta/\alpha)-1|+\log(1/|\epsilon|)-2nb\pi}{a}$$

and consider only one intersection point. Then we multiply the values of $H_{\alpha,n}$ and $H_{\beta,n}$ using the formulas in Case (i) or (ii) depending on whether $n<p$ or $n>p$. It is convenient to use instead the cases $n<sp$ and $n>sp, 0<s<<1$. We then add these products over n and integrate the result over $d\mu(\alpha)d\mu(\beta)$.

Case (i), $n<sp$:

$$
\begin{aligned}
n &< s\frac{\log|(\beta/\alpha-1|+\log 1/|\epsilon|-2nb\pi}{a}\\
n(1+\frac{2sb\pi}{a}) &< s\frac{\log|(\beta/\alpha-1|+\log 1/|\epsilon|}{a}\\
1/2<1+\frac{2sb\pi}{a} &< 3/2\\
n &< \frac{s}{1+\frac{2sb\pi}{a}}\frac{\log|(\beta/\alpha-1|+\log 1/|\epsilon|}{a}=: n(\alpha,\beta,\epsilon).
\end{aligned}
$$

Hence for $n < n(\alpha,\beta,\epsilon)$ we use the formula for $n < sp$. For $n > n(\alpha,\beta,\epsilon)$, we use the formula for $n > p$. But recall also from Lemma 19 that n is limited by the condition there, i.e. $L_{\alpha,n}$ must intersect R_1.

$$
\begin{aligned}
h_{\alpha,n} &\sim \int_{|x|<Cv^\gamma} H_\alpha(x)|x|^{1/\gamma-1}\left(\frac{|x|}{v^\gamma}\right)^{1-1/\gamma}\frac{1}{v}dx\\
&+ \int_{|x|>Cv^\gamma} H_\alpha(x)|x|^{1/\gamma-1}\left(\frac{v^\gamma}{|x|}\right)^{1+1/\gamma}\frac{1}{v}dx\\
h_{\beta,m} &\sim \int_{|y|<C(v')^\gamma} H_\beta(y)|y|^{1/\gamma-1}\left(\frac{|y|}{(v')^\gamma}\right)^{1-1/\gamma}\frac{1}{v'}dy\\
&+ \int_{|y|>C(v')^\gamma} H_\beta(y)|y|^{1/\gamma-1}\left(\frac{(v')^\gamma}{|y|}\right)^{1+1/\gamma}\frac{1}{v'}dy
\end{aligned}
$$

Here $v,v' \sim \frac{\log|(\beta/\alpha)-1|+\log 1/|\epsilon|-2nb\pi}{a}$. We need to estimate $\sum_v h_{\alpha,n}h_{\beta,n}$ and then integrate the answer over the measure $\mu(\alpha)\mu(\beta)$.

Note we will majorize the sum by the product $\sum_v h_{\alpha,n}\sum_m h^\epsilon_{\beta,m}$ Then we use the dominated convergence theorem. When we sum over n, we can instead sum over v, $\log 1/\delta < v < \log 1/|\epsilon| - C$.

$$
\begin{aligned}
\sum_v h_{\alpha,n} &\sim \sum_{v=\log 1/\delta}^{\log 1/|\epsilon|-C}\left[\int_{|x|<Cv^\gamma}\tilde{H}_\alpha(x)\frac{dx}{v^\gamma}+\int_{|x|>Cv^\gamma}\tilde{H}_\alpha(x)\frac{|v|^\gamma}{|x|^2}dx\right]\\
&\sim \int_{|x|<(\log 1/\delta)^\gamma}\tilde{H}_\alpha(x)\sum_{v=\log 1/\delta}^{\log 1/|\epsilon|-C}\frac{dx}{v^\gamma}\\
&+ \int_{(\log 1/\delta)^\gamma<|x|<(\log 1/|\epsilon|)^\gamma}\tilde{H}_\alpha(x)\sum_{v=|x|^{1/\gamma}}^{\log 1/|\epsilon}\frac{dx}{v^\gamma}\\
&+ \int_{(\log 1/\delta)^\gamma<|x|<(\log 1/|\epsilon|)^\gamma}\tilde{H}_\alpha(x)\frac{1}{x^2}\sum_{v=\log 1/\delta}^{|x|^{1/\gamma}}v^\gamma dx\\
&+ \int_{|x|>(\log 1/|\epsilon|)^\gamma}\tilde{H}_\alpha(x)\frac{1}{x^2}\sum_{v=\log 1/\delta}^{\log 1/|\epsilon}v^\gamma dx\\
&\sim \int_{|x|<(\log 1/\delta)^\gamma}\tilde{H}_\alpha(x)\frac{1}{(\log 1/\delta)^{\gamma-1}}\\
&+ \int_{(\log 1/\delta)^\gamma<|x|<(\log 1/|\epsilon|)^\gamma}\tilde{H}_\alpha(x)\frac{1}{|x|^{1-1/\gamma}}dx\\
&+ \int_{(\log 1/\delta)^\gamma<|x|<(\log 1/|\epsilon|)^\gamma}\tilde{H}_\alpha(x)\frac{1}{x^2}\frac{1}{|x|^{1-1/\gamma}}dx\\
&+ \int_{|x|>(\log 1/|\epsilon|)^\gamma}\tilde{H}_\alpha(x)\frac{1}{x^2}(\log 1/|\epsilon|)^{\gamma+1}dx\\
&\sim \int_{|x|<(\log 1/\delta)^\gamma}\tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{|x|}{(\log 1/\delta)^\gamma}\right)^{1-1/\gamma}\\
&+ \int_{(\log 1/\delta)^\gamma<|x|<(\log 1/|\epsilon|)^\gamma}\tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx\\
&+ \int_{|x|>(\log 1/|\epsilon|)^\gamma}\tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{(\log 1/|\epsilon|)^\gamma}{|x|}\right)^{1+1/\gamma}dx\\
&\to 0, \delta\to 0
\end{aligned}
$$

This finishes the case (i), $n < sp$.

Lemma 22. *The contribution to the geometric wedge product from R_{1A} in case (i), $a \neq 0, n < p$ goes to zero when $\delta \to 0$.*

We next deal with the case $n > sp$. Recall that:
Case (ii) $a \neq 0, n > p$

$$\begin{aligned}
(u+iv)^\gamma &= U+iV \\
&\sim n^\gamma + ip(n)n^{\gamma-1} \\
H_{\alpha,n} &\sim \int_{|x-n^\gamma|\le p(n)n^{\gamma-1}} \tilde{H}_\alpha(x)\frac{1}{p(n)n^{\gamma-1}}dx \\
&+ \int_{n^\gamma/2>|x-n^\gamma|>p(n)n^{\gamma-1}} \tilde{H}_\alpha(x)\frac{p(n)n^{\gamma-1}}{|x-n^\gamma|^2}dx \\
&+ \int_{n^\gamma/2<|x-n^\gamma|<2n^\gamma} \tilde{H}_\alpha(x)\frac{p(n)n^{\gamma-1}}{n^{2\gamma}}dx \\
&+ \int_{|x-n^\gamma|>2n^\gamma} \tilde{H}_\alpha(x)\frac{p(n)n^{\gamma-1}}{x^2}dx \\
&= I+II+III+IV \\
&= I_n+II_n+III_n+IV_n
\end{aligned}$$

For simplicity of notation we assume $a>0$. Then we have the following range for n from Lemma 20.

$$\begin{aligned}
a\log 1/\delta < \log|(\beta/\alpha)-1| + \log(1/|\epsilon|) - 2nb\pi &< a\log 1/|\epsilon| - aC \\
a\log 1/\delta - \log|(\beta/\alpha)-1| - \log(1/|\epsilon|) &< -2nb\pi < \\
-\log|(\beta/\alpha)-1| - \log(1/|\epsilon|) + a\log 1/|\epsilon| - aC & \\
-a\log 1/\delta + \log|(\beta/\alpha)-1| + \log(1/|\epsilon|) &> 2nb\pi > \\
\log|(\beta/\alpha)-1| + \log(1/|\epsilon|) - a\log 1/|\epsilon| - aC & \\
\frac{\log|(\beta/\alpha)-1| + (1-a)\log(1/|\epsilon|) - aC}{2b\pi} &< n < \\
\frac{-a\log 1/\delta + \log|(\beta/\alpha)-1| + \log(1/|\epsilon|)}{2b\pi} &
\end{aligned}$$

However, n is further restricted because $n > sp$ and $p > \log 1/\delta$. If we then estimate IV and sum over n, we get

$$\begin{aligned}
\sum_n IV_n &<\sim \int_{|x|>(\log 1/\delta)^\gamma} \tilde{H}_\alpha(x)\frac{1}{x^2}\sum_{n=\log 1/\delta}^{|x|^{1/\gamma}} n^\gamma \\
&<\sim \int_{|x|>(\log 1/\delta)^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\
&\to 0
\end{aligned}$$

Similarly for $\sum_n III_n$ we get to estimate $\sum 1/n^\gamma <\sim |x|^{1/\gamma-1}$ which again is fine.

Next we handle the terms II_n. For a given x, the range of n is on the order of $2/3|x|^{1/\gamma} < n < |x|^{1/\gamma} - p(x^{1/\gamma})$ and similar for $n > |x|^{1/\gamma}$. Also note that the terms $p(n) <\sim |x|^{1/\gamma}$ since $n \sim |x|^{1/\gamma}$ and $p <\sim n$. So we sum the expressions $\frac{n^{\gamma-1}}{(x-n^\gamma)^2}$

which integrates to $\frac{1}{|x-n^\gamma|}$, so inserting the limits of the summation, we get a bound of the same form as for III.

Finally we sum over the I_n. Here we make the rough estimate that $\log 1/\delta < p(n) < sn$. So we integrate over $|x-n^\gamma| < sn^\gamma$ but in the integrand we replace $p(n)$ by $\log 1/\delta$. With this estimate we get the integral $\tilde{H}_\alpha(x)\frac{1}{\log 1/\delta|x|} << \tilde{H}_\alpha(x)|x|^{1/\gamma-1}$. Hence this also goes to zero with δ.

Hence we have shown the following:

Lemma 23. *The contribution to the geometric wedge product in the case of R_{1A}, case (ii), $a \neq 0, n > p$ goes to zero when $\delta \to 0$.*

9. Theorem 7 for R_{1C}, the diagonal part of R_1

We are in the set $\{C|\epsilon| < |z|, |w| < \delta, |z| \sim |w|\}$.

On $L_{\alpha,n}$, we have

$$\begin{aligned}
2n\pi &< u < 2(n+1)\pi \\
|v - \frac{2n}{1-a}| &< C'' \\
\log 1/\delta &< v < \log(1/|\epsilon|) - C \\
(u+iv)^\gamma &= U + iV \\
V &\sim |n|^\gamma \\
|U| &<\sim |n|^\gamma \\
h_{\alpha,n} &\sim \int \tilde{H}_\alpha(x)\frac{n^\gamma}{n^{2\gamma}+(x-U)^2}dx \\
&\sim \int_{|x|<2n^\gamma} \tilde{H}_\alpha(x)\frac{dx}{n^\gamma} + \int_{|x|>2n^\gamma} \tilde{H}_\alpha(x)\frac{n^\gamma dx}{x^2}
\end{aligned}$$

$$\begin{aligned}
\sum_n h_{\alpha,n} &\sim \int_{|x|<(\log(1/|\delta))^\gamma} \tilde{H}_\alpha(x)\left(\sum_{n=\log 1/\delta}^{\infty} \frac{1}{n^\gamma}\right)dx \\
&+ \int_{|x|>(\log(1/|\delta))^\gamma} \tilde{H}_\alpha(x)\left(\sum_{n=x^{1/\gamma}} \frac{1}{n^\gamma}\right)dx \\
&+ \int_{|x|>(\log(1/|\delta))^\gamma} \tilde{H}_\alpha(x)\left(\sum_{n=\log 1/\delta}^{x^{1/\gamma}} \frac{n^\gamma}{x^2}\right)dx \\
&\sim \int_{|x|<(\log(1/|\delta))^\gamma} \tilde{H}_\alpha(x)\frac{1}{(\log(1/\delta))^\gamma}dx \\
&+ \int_{|x|>(\log(1/|\delta))^\gamma} \tilde{H}_\alpha(x)\frac{1}{(x^{1/\gamma})^{\gamma-1}}dx \\
&+ \int_{|x|>(\log(1/|\delta))^\gamma} \tilde{H}_\alpha(x)\frac{(x^{1/\gamma})^{\gamma+1}}{x^2}dx
\end{aligned}$$

$$\begin{aligned}\sum_n h_{\alpha,n} &\sim \int_{|x|>(\log(1/\delta))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\ &+ \int_{|x|<(\log(1/\delta))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{|x|}{(\log(1/\delta))^\gamma}\right)^{1-1/\gamma} dx\end{aligned}$$

This is arbitrarily small as long as δ is chosen small enough.

10. Theorem 7 for R_2, the part of D_4 close to the $z-$ axis

This case is divided in two subcases depending on whether one is close to one of the indicatrices (R_{2A}) or not (R_{2B}).

11. Theorem 7 for R_{2A} close to an indicatrix

Again we assume that $a \neq 0$. There are two indicatrices, $w = 0$ and w close to $\beta(\epsilon)$. By symmetry it suffices to do one of them. We choose to estimate close to the indicatrix $w = 0$. So we set $R_{2A} = \{C|\epsilon| < |z| < \delta, |w| < s|\epsilon|\}$ for some small constant $s > 0$. Let $L^\epsilon_{\beta,m}$ and $L_{\alpha,n}$ be plaques intersecting at (z,w) in R_{2A} for parameters $(u',v'),(u,v)$.

Since the point (z,w) is about distance $|\beta'(0)||\epsilon|$ away from the indicatrix for the perturbed lamination, we get ($w' = \beta(\epsilon) + \beta e^{i\lambda(u'+(\log|\beta|/b)+iv')} + \cdots$):

$$\begin{aligned}2m\pi &< u' < 2(m+1)\pi \\ C_1 &< av' + 2mb\pi + \log|\epsilon| < C_2 \\ C|\epsilon| < |z| = e^{-v} &= |z'| = |\alpha(\epsilon) + e^{i(u'+\log|\beta|/b)-v'} + \cdots| \Rightarrow \\ C_3 &< v - v' < C_4 \\ C_4 &< av + 2mb\pi + \log|\epsilon| < C_5 \\ 2n\pi &< u < 2(n+1)\pi \\ |w| &< s|\epsilon| \\ e^{-bu-av} &< s|\epsilon| \\ \log(1/s) &< av + 2nb\pi + \log|\epsilon| \\ 2(n-m)b\pi &= (av + 2nb\pi + \log|\epsilon|) - (av + 2m\pi b + \log|\epsilon|) \\ &> \log(1/s) - C_1.\end{aligned}$$

These calculations show that for the given plaques, the pairs $(u,v),(u',v')$ belong to rectangles of uniformly bounded size. Hence the number of intersection points can easily be estimated by using slope estimates for the plaques. We get a uniformly bounded number of intersection points.

We divide this into cases I, II, III. For I, we have $1/C\log(1/|\epsilon|) < 2mb\pi + \log|\epsilon| < C\log(1/|\epsilon|)$. For II we have $2mb\pi + \log|\epsilon| < 1/C\log(1/|\epsilon|)$. For III we have $2mb\pi + \log|\epsilon| > C\log(1/|\epsilon|)$. We note however, that in case III, v' must be very large in comparison with $\log 1/|\epsilon|$. This implies that $|z'| << |\epsilon|$ hence there in no intersection point in this case. So we are left with the two cases R_{2AI}, R_{2AII}.

12. Theorem 7 for R_{2AI} close to an indicatrix.

It follows in this case that $v, v' \sim \log(1/|\epsilon|)$.

$$\begin{aligned}
u' + iv' &\sim 2m\pi + i\log(1/|\epsilon|) \\
U' + iV' &\sim U' + i(\log(1/|\epsilon|))^\gamma \\
|U'| &<\sim (\log(1/|\epsilon|))^\gamma \\
h^\epsilon_{\beta,m} &\sim \int \tilde{H}_\beta(y)\frac{(\log(1/|\epsilon|))^\gamma}{(\log(1/|\epsilon|))^{2\gamma} + (x-U')^2}dy \\
&\sim \int_{|y|<2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)\frac{1}{(\log(1/|\epsilon|))^\gamma}dy \\
&+ \int_{|y|>2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)\frac{(\log(1/|\epsilon|))^\gamma}{y^2}dy \\
\sum_{m\in I} h_{\beta,m} &\sim \int_{|y|<2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}\left(\frac{|y|}{(\log(1/|\epsilon|))^\gamma}\right)^{1-1/\gamma} dy \\
&+ \int_{|y|>2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}\left(\frac{(\log(1/|\epsilon|))^\gamma}{|y|}\right)^{1/\gamma+1} dy
\end{aligned}$$

Next we estimate $h_{\alpha,n}$. There are two cases to consider:

a): $n < C\log(1/|\epsilon|)$

b): $n > C\log(1/|\epsilon|)$

Case R_{2AIa} :
Recall that we have $n > m - C_6$. Hence we have that $|n| < C\log(1/|\epsilon|)$. This means that we can write $u + iv \sim 2n\pi + i(\log(1/|\epsilon|))$. Hence the estimates work as for $h^\epsilon_{\beta,m}$.

$$\begin{aligned}
\sum_{|n|<C\log(1/|\epsilon|)} h_{\alpha,n} &\sim \int_{|x|<2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{|x|}{(\log(1/|\epsilon|))^\gamma}\right)^{1-1/\gamma} dx \\
&+ \int_{|x|>2(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{(\log(1/|\epsilon|))^\gamma}{|x|}\right)^{1/\gamma+1} dx
\end{aligned}$$

Case R_{2AIb} :

$$\begin{aligned}
u + iv &\sim n + i\log(1/|\epsilon|) \\
U + iV &\sim n^\gamma + in^{\gamma-1}\log(1/|\epsilon|) \\
h_{\alpha,n} &\sim \int \tilde{H}_\alpha(x)\frac{n^{\gamma-1}\log(1/|\epsilon|)}{(n^{\gamma-1}\log(1/|\epsilon|))^2 + (x-n^\gamma)^2}dx
\end{aligned}$$

This integral has already been estimated. See the calculations for the set D_1 in the region where $|z-\eta| < d|\eta|$, case (ii) where $n > 10\log(1/|\eta|)$.

13. Theorem 7 for R_{2AII} close to an indicatrix.

We restrict for simplicity to the case $a > 0$. We can divide into three cases:

a) $n > m > v, v'$
b) $n > v, v' > m$
c) $v, v' > n > m$

14. Theorem 7 for R_{2AIIa} close to an indicatrix.

$$\begin{aligned}
(u+iv)^\gamma &= U+iV\\
&\sim n^\gamma + ivn^{\gamma-1}\\
(u'+iv')^\gamma &= U'+iV'\\
&\sim m^\gamma + iv'm^{\gamma-1}\\
&\sim (\log 1/|\epsilon|)^\gamma + iv'(\log(1/|\epsilon|))^{\gamma-1}\\
\log 1/\delta &< v' < \log 1/|\epsilon|
\end{aligned}$$

$$\begin{aligned}
H_\beta &\sim \int \tilde{H}_\beta(y)\frac{v'(\log(1/|\epsilon|))^{\gamma-1}}{[v'(\log(1/|\epsilon|))^{\gamma-1}]^2+(y-m^\gamma)^2}dy\\
&\sim \int_{|y-m^\gamma|<cv'(\log 1/|\epsilon|)^{\gamma-1}} \tilde{H}_\beta(y)\frac{1}{v'(\log(1/|\epsilon|))^{\gamma-1}}dy\\
&+ \int_{(\log 1/|\epsilon|)^\gamma/2>|y-(\log 1/|\epsilon|)^\gamma|>cv'(\log 1/|\epsilon|)^{\gamma-1}} \tilde{H}_\beta(y)\frac{v'(\log(1/|\epsilon|))^{\gamma-1}}{(y-(\log 1/|\epsilon|)^\gamma)^2}dy\\
&+ \int_{|y-(\log 1/|\epsilon|)^\gamma|>(\log 1/|\epsilon|)^\gamma/2} \tilde{H}_\beta(y)\frac{v'(\log(1/|\epsilon|))^{\gamma-1}}{(y-(\log 1/|\epsilon|)^\gamma)^2}dy\\
&\sim \int_{|y-m^\gamma|<cv'(\log 1/|\epsilon|)^{\gamma-1}} \tilde{H}_\beta(y)\frac{y^{1/\gamma-1}}{v'}dy\\
&+ \int_{|y-(\log 1/|\epsilon|)^\gamma|>cv'(\log 1/|\epsilon|)^{\gamma-1}} \tilde{H}_\beta(y)\frac{v'(\log(1/|\epsilon|))^{\gamma-1}}{(y-(\log 1/|\epsilon|)^\gamma)^2}dy\\
&= \beta_{1,v'}+\beta_{2,v'}\\
H_\alpha &\sim \int \tilde{H}_\alpha(x)\frac{vn^{\gamma-1}}{[vn^{\gamma-1}]^2+(x-n^\gamma)^2}dx\\
&\sim \int_{|x-n^\gamma|<cvn^{\gamma-1}} \tilde{H}_\alpha(x)\frac{1}{vn^{\gamma-1}}dx\\
&+ \int_{|x-n^\gamma|>cvn^{\gamma-1}} \tilde{H}_\alpha(x)\frac{vn^{\gamma-1}}{(x-n^\gamma)^2}dx
\end{aligned}$$

To sum up over the intersection points, we note at first that for a given plaque $L_{\beta,m}$ there is a finite range of v' and $v-v'$ is bounded, so we can assume that there is one intersection point with $L_{\alpha,n}$ for each $n > m$. Hence we sum first over the plaques $L_{\alpha,n}$, $m < n < \infty$.

$$\sum_n \int_{|x-n^\gamma|<cvn^{\gamma-1}} \tilde{H}_\alpha(x)\frac{1}{vn^{\gamma-1}}dx \sim \sum_{n=x^{1/\gamma}-v}^{n=x^{1/\gamma}+v} \int \sim \int_{x>m^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx$$

$$\sum_n \int_{|x-n^\gamma|>cvn^{\gamma-1}} \tilde{H}_\alpha(x)\frac{vn^{\gamma-1}}{(x-n^\gamma)^2}dx \sim \int_{x>m^\gamma-cvm^{\gamma-1}} \tilde{H}_\alpha(x)|x|^{1/\gamma-1} + \int_{x<m^\gamma-cvm^{\gamma-1}} \tilde{H}_\alpha(x)\frac{v}{|x-m^\gamma|}dx$$

so we conclude:

$$\sum_{n>m} H_\alpha \sim \int_{x>m^\gamma} \tilde{H}(x)|x|^{1/\gamma-1} + \int_{x<m^\gamma-cvm^{\gamma-1}} \tilde{H}_\alpha(x)\frac{v}{|x-m^\gamma|}dx$$
$$<\sim \int_{|x|>m^\gamma/2} \tilde{H}_\alpha|x|^{1/\gamma-1} + \int_{|x|<m^\gamma/2} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{|x|}{m^\gamma}\right)^{1-1/\gamma}dx$$

In this case m will have approximately the range $(\log 1/|\epsilon|)/2 < m < \log 1/|\epsilon|$, hence we have

$$\sum_{n>m} H_\alpha <\sim \int_{|x|>(\log 1/|\epsilon|)^\gamma} \tilde{H}_\alpha|x|^{1/\gamma-1} + \int_{|x|<(\log 1/|\epsilon|)^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\left(\frac{|x|}{(\log 1/|\epsilon|)^\gamma}\right)^{1-1/\gamma}dx$$

Next we sum H_β over m or equivalently over $v', \log 1/\delta < v' < (\log 1/|\epsilon|)/2$. We sum first over $\beta_{1,v'}$. For a given y, the range of v' is in the interval with endpoints $(1\pm c)\frac{y-(\log 1/|\epsilon|)^\gamma}{(\log 1/|\epsilon|)^{\gamma-1}}$. This part is bounded by

$$\int_{|y-(\log 1/|\epsilon|)^\gamma|<(\log 1/|\epsilon|)^\gamma/2} \tilde{H}_\beta(y)|y|^{1/\gamma-1}dy \to 0.$$

The second part is bounded by

$$\int_{|y|<2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}\left(\frac{|y|}{(\log 1/|\epsilon|)^\gamma}\right)^{1-1/\gamma}dy + \int_{|y|>2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}\left(\frac{(\log 1/|\epsilon|)^\gamma}{|y|}\right)^{1+1/\gamma}dy$$

15. Theorem 7 for R_{2AIIb} close to an indicatrix.

In this case $n > v, v' > m$. First we recall the estimates for H_α which are the same as in the case R_{2AIIa}.

$$
\begin{aligned}
(u+iv)^\gamma &= U+iV \\
&\sim n^\gamma + ivn^{\gamma-1} \\
\log 1/\delta &< v, v' < \log 1/|\epsilon| \\
H_\alpha &\sim \int \tilde{H}_\alpha(x) \frac{vn^{\gamma-1}}{[vn^{\gamma-1}]^2 + (x-n^\gamma)^2} dx \\
&\sim \int_{|x-n^\gamma|<cvn^{\gamma-1}} \tilde{H}_\alpha(x) \frac{1}{vn^{\gamma-1}} dx \\
&+ \int_{|x-n^\gamma|>cvn^{\gamma-1}} \tilde{H}_\alpha(x) \frac{vn^{\gamma-1}}{(x-n^\gamma)^2} dx
\end{aligned}
$$

Next we estimate H_β.

$$
\begin{aligned}
(u'+iv')^\gamma &= U'+iV' \\
(\log 1/|\epsilon|)/2 &< v' < \log 1/|\epsilon| \\
m + v' &= \log 1/|\epsilon| \\
V' &\sim (\log 1/|\epsilon|)^\gamma \\
|U'| &<\sim (\log 1/|\epsilon|)^\gamma \\
H_\beta &\sim \int_{|y|<2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta \frac{1}{(\log 1/|\epsilon|)^\gamma} dy \\
&+ \int_{|y|>2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta \frac{(\log 1/|\epsilon|)^\gamma}{y^2} dy
\end{aligned}
$$

Next we estimate the contribution to the geometric wedge product. So fix α, β. Next fix a plaque $L_{\beta,m},\ v, v' \sim \log 1/|\epsilon| - m$. Next we consider the contribution from H_α for all $n > v$. This is the same estimate as in the previous section, so goes to zero when $\epsilon \to 0$. To sum up over m, notice that we have about $\log 1/|\epsilon|$ terms of the same order of magnitude. From this we get that the contribution goes to zero when $\epsilon \to 0$.

To estimate the geometric wedge product, we sum independently over n, m throwing out the condition that $n > m$. We get as in the previous section that the contribution goes to zero.

16. Theorem 7 for R_{2AIIc} close to an indicatrix.

Here we deal with the case when
$v, v' > n > m$. In this case the same formula as in the last section applies to both H_α and H_β :

$$
\begin{aligned}
H_\alpha &\sim \int_{|x|<2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\alpha \frac{1}{(\log 1/|\epsilon|)^\gamma} dx \\
&+ \int_{|x|>2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\alpha \frac{(\log 1/|\epsilon|)^\gamma}{x^2} dx \\
H_\beta &\sim \int_{|y|<2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta \frac{1}{(\log 1/|\epsilon|)^\gamma} dy \\
&+ \int_{|y|>2(\log 1/|\epsilon|)^\gamma} \tilde{H}_\beta \frac{(\log 1/|\epsilon|)^\gamma}{y^2} dy
\end{aligned}
$$

17. Theorem 7 for R_{2B} away from the indicatrices.

At an intersection point $p = (z, w)$ of $L_{\alpha,n}, L^\epsilon_{\beta,m}$ we have

$$
\begin{aligned}
s|\epsilon| &< |w| < C|\epsilon| \\
s|\epsilon| &< |w - \beta(\epsilon)| < C|\epsilon| \\
\log|\epsilon| - C &< -av - bu < \log|\epsilon| + C \\
\log|\epsilon| - C &< -av' - bu' < \log|\epsilon| + C \\
-C &< v - v' < C \\
-C &< n - m < C \\
\log(1/\delta) &< v, v' < \log(1/|\epsilon|) - C \\
-C\log(1/|\epsilon|) &< u, u', n, m < C\log(1/|\epsilon|)
\end{aligned}
$$

Given (α, β, n, m) we need to estimate the values of v, v' corresponding to an intersection, as well as the number of intersections. The following is immediate. There is no dependence on α, β.

Lemma 24. *At intersection points of $L_{\alpha,n}, L^\epsilon_{\beta,m}$ in R_{2B} away from the indicatrices, we have*

$$-2nb\pi/a + 1/a\log(1/|\epsilon|) - C < v, v' < -2nb\pi/a + 1/a\log(1/|\epsilon|) + C.$$

It follows that intersection points are localized in bounded rectangles. To show finiteness of number of intersection points for given plaques, we use slope estimates.

We divide the estimates in two cases, (i) if $v'v' \sim \log(1/|\epsilon|)$ and (ii) if $\log(1/\delta) < v, v' < 1/C\log(1/|\epsilon|)$.

18. Theorem 7 for R_{2Bi} when $v \sim \log(1/|\epsilon|)$

The estimates for $h_{\alpha,n}$ and $h^\epsilon_{\beta,m}$ are similiar.

$$
\begin{aligned}
U+iV &= (u+iv)^\gamma \\
&\sim U+i(\log(1/|\epsilon|))^\gamma \\
|U| &<\sim (\log(1/|\epsilon|))^\gamma \\
h_{\alpha,n} &\sim \int \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^\gamma}{(\log(1/|\epsilon|))^{2\gamma}+(x-U)^2}dx \\
&\sim \int_{|x|<C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha(x)\frac{1}{(\log(1/|\epsilon|))^\gamma}dx \\
&+ \int |x|>C(\log(1/|\epsilon|))^\gamma \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^\gamma}{x^2}dx \\
&\sim \int_{|x|<C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha |x|^{1/\gamma-1}\left(\frac{|x|}{(\log(1/|\epsilon|))^\gamma}\right)^{1-1/\gamma}\frac{1}{\log(1/|\epsilon|)}dx \\
&+ \int_{|x|>C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha |x|^{1/\gamma-1}\left(\frac{(\log(1/|\epsilon|))^\gamma}{|x|}\right)^{1+1/\gamma}\frac{1}{\log(1/|\epsilon|)}dx \\
\sum_n h_{\alpha,n} &\sim \int_{|x|<C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha |x|^{1/\gamma-1}\left(\frac{|x|}{(\log(1/|\epsilon|))^\gamma}\right)^{1-1/\gamma}dx \\
&+ \int_{|x|>C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha |x|^{1/\gamma-1}\left(\frac{(\log(1/|\epsilon|))^\gamma}{|x|}\right)^{1+1/\gamma}dx
\end{aligned}
$$

19. Theorem 7 for R_{2Bii} when $v << \log(1/|\epsilon|)$

In this case we have $u, u', n, m \sim \log(1/|\epsilon|)$. The estimates for $h_{\alpha,n}, h^\epsilon_{\beta,m}$ are similar. In the following $0 < d << 1$.

$$
\begin{aligned}
(1-d)\log(1/|\epsilon|) &< 2nb\pi < (1+d)\log(1/|\epsilon|) \\
\log|\epsilon| - C &< -av-bu < \log|\epsilon| + C \\
\log|\epsilon| + 2nb\pi - C &< -av < \log|\epsilon| + 2bn\pi + C \\
-d\log(1/|\epsilon|) - C &< -av < d\log(1/|\epsilon|) + C \\
U+iV &= (u+iv)^\gamma \\
&\sim (\log(1/|\epsilon|))^\gamma + i(\log(1/|\epsilon|))^{\gamma-1}v \\
h_{\alpha,n} &\sim \int \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{((\log(1/|\epsilon|))^{\gamma-1}v)^2+(x-U)^2}dx
\end{aligned}
$$

When we sum up over $h_{\alpha,n}, h^\epsilon_{\beta,m}$ we can take $n = m$ and $v = v'$.

$$
\begin{aligned}
h_{\alpha,n}h^{\epsilon}_{\beta,m} &\sim \int \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{((\log(1/|\epsilon|))^{\gamma-1}v)^2+(x-U)^2}dx \\
&* \int \tilde{H}_\beta(y)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{((\log(1/|\epsilon|))^{\gamma-1}v)^2+(y-U)^2}dy \\
&\sim [\int_{|x-U|<(\log(1/|\epsilon|))^{\gamma-1}|v|}\tilde{H}_\alpha(x)\frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v}dx \\
&+ \int_{|x-U|>(\log(1/|\epsilon|))^{\gamma-1}|v|}\tilde{H}\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(x-U)^2}dx] \\
&* [\int_{|y-U|<(\log(1/|\epsilon|))^{\gamma-1}|v|}\tilde{H}_\beta(y)\frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v}dy \\
&+ \int_{|y-U|>(\log(1/|\epsilon|))^{\gamma-1}|v|}\tilde{H}_\beta(y)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(y-U)^2}dy] \\
&= [I+II][III+IV]
\end{aligned}
$$

There are 4 cases to sum over: (I, III), (II, III), (II, IV) and (I, IV). The case (I, IV) is similar to (II, III) so we can skip it without any loss.

20. Theorem 7 for $R_{2Bii(I,III)}$ when $v << \log(1/|\epsilon|)$

$$\begin{aligned}
h_{\alpha,n}h^{\epsilon}_{\beta,m} &\sim \int_{|x-U|<(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\alpha(x)\frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v}dx \\
&* \int_{|y-U|<(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\beta(y)\frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v}dy \\
&<\sim \frac{1}{v^2}\int_{|x-U|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha(x)\frac{1}{(\log(1/|\epsilon|))^{\gamma-1}}dx \\
&* \int_{|y-U|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)\frac{1}{(\log(1/|\epsilon|))^{\gamma-1}}dy \\
&\sim \frac{1}{v^2}\int_{|x-U|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\
&* \int_{|y-U|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}dy \\
\log(1/\delta) &< v < 1/C\log(1/|\epsilon|) \\
\sum h_{\alpha,n}h^{\epsilon}_{\beta,m} &<\sim \int_{\log(1/\delta)}^{1/C\log(1/|\epsilon|)} \frac{1}{v^2}\int_{|x-U|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\
&* \int_{|y-U|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}dy \\
\sum h_{\alpha,n}h^{\epsilon}_{\beta,m} &<\sim \frac{1}{\log(1/\delta)}\int_{|x-U|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\
&* \int_{|y-U|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}dy \\
\sum h_{\alpha,n}h^{\epsilon}_{\beta,m} &<\sim \frac{1}{\log(1/\delta)}\int_{|x-(\log(1/|\epsilon|))^\gamma|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\
&* \int_{|y-(\log(1/|\epsilon|))^\gamma|<1/C(\log(1/|\epsilon|))^\gamma} \tilde{H}_\beta(y)|y|^{1/\gamma-1}dy
\end{aligned}$$

This contribution goes to zero when $\epsilon \to 0$.

21. Theorem 7 for $R_{2Bii(II,III)}$ when $v << \log(1/|\epsilon|)$

$$\begin{aligned}
h_{\alpha,n}h^{\epsilon}_{\beta,m} &\sim \int_{|x-U|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(x-U)^2}dx \\
&* \int_{|y-U|<(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\beta(y)\frac{1}{(\log(1/|\epsilon|))^{\gamma-1}v}dy
\end{aligned}$$

Here $\log(1/\delta) < v < d\log(1/|\epsilon|), 0 < d << 1$ and $-av = \log|\epsilon| + 2bn\pi + \mathcal{O}(1)$. Also we can take $n = m$. When we sum over n, v runs through $\log(1/\delta) < v < d\log(1/|\epsilon|)$. Hence the contribution to the geometric wedge product is

$$\begin{aligned}
\sum_{n,m} h_{\alpha,n} h^{\epsilon}_{\beta,m} &\sim \sum_{v=\log(1/\delta)}^{d\log(1/|\epsilon|)} \int_{|x-U|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\alpha(x) \frac{1}{(x-U)^2} dx \\
&* \int_{|y-U|<(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\beta(y) dy \\
&\sim \sum_{v=\log(1/\delta)}^{d\log(1/|\epsilon|)} \int_{|x-(\log(1/|\epsilon|))^\gamma|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\alpha(x) \frac{1}{(x-(\log(1/|\epsilon|)^\gamma)^2} dx \\
&* \int_{|y-(\log(1/|\epsilon|))^\gamma|<(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\beta(y) dy
\end{aligned}$$

We introduce a counting function, $N(x,y)$, which tells us for a given (x,y) for how many terms of the sum (x,y) is in the domain of integration,

$$\begin{aligned}
|x-(\log(1/|\epsilon|))^\gamma| &> (\log(1/|\epsilon|))^{\gamma-1}|v| \\
|y-(\log(1/|\epsilon|))^\gamma| &< (\log(1/|\epsilon|))^{\gamma-1}|v|
\end{aligned}$$

$$\begin{aligned}
R_1 &= \{|x-(\log(1/|\epsilon|))^\gamma| > d(\log(1/|\epsilon|))^\gamma, \\
&\quad |y-(\log(1/|\epsilon|))^\gamma| < \log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1}\} \\
N_1(x,y) &\sim d\log(1/|\epsilon|) \\
R_2 &= \{|x-(\log(1/|\epsilon|))^\gamma| > d(\log(1/|\epsilon|))^\gamma, \\
&\quad \log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1} < |y-(\log(1/|\epsilon|))^\gamma| < d(\log(1/|\epsilon|))^\gamma\} \\
N_2(x,y) &\sim \frac{d(\log(1/|\epsilon|))^\gamma - |y-(\log(1/|\epsilon|))^\gamma|}{(\log(1/|\epsilon|))^{\gamma-1}} \\
R_3 &= \{\log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1} < |x-(\log(1/|\epsilon|))^\gamma| < d(\log(1/|\epsilon|))^\gamma, \\
&\quad \log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1} < |y-(\log(1/|\epsilon|))^\gamma| < d(\log(1/|\epsilon|))^\gamma\} \\
N_3(x,y) &\sim \frac{|x-(\log(1/|\epsilon|))^\gamma| - |y-(\log(1/|\epsilon|))^\gamma|}{(\log(1/|\epsilon|))^{\gamma-1}} \text{ when positive} \\
N_3(x,y) &\sim \frac{|x-y|}{(\log(1/|\epsilon|))^{\gamma-1}}
\end{aligned}$$

22. Theorem 7 for $R_{2Bii(II,III)R_1}$ when $v << \log(1/|\epsilon|)$

$$\begin{aligned}
\sum_{n,m} h_{\alpha,n}h^{\epsilon}_{\beta,m} &\sim d\log(1/|\epsilon|)\int_{R_1}\frac{\tilde H_\alpha(x)\tilde H_\beta(y)}{(x-(\log(1/|\epsilon|))^\gamma)^2}dxdy\\
&\sim \int_{R_1}\frac{\tilde H_\alpha(x)|x|^{1/\gamma-1}|x|^{1-1/\gamma}\tilde H_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))^\gamma)^2((\log(1/|\epsilon|)^\gamma)^{1/\gamma-1}}\log(1/|\epsilon|)\\
&\sim \int_{R_1}\frac{\tilde H_\alpha(x)|x|^{1/\gamma-1}|x|^{1-1/\gamma}\tilde H_\beta(y)|y|^{1/\gamma-1}}{|x-(\log(1/|\epsilon|))^\gamma|^{1-1/\gamma}((\log(1/|\epsilon|)^\gamma)^{2/\gamma}}\log(1/|\epsilon|)\\
&<\sim \int_{R_1}\tilde H_\alpha(x)|x|^{1/\gamma-1}\tilde H_\beta(y)|y|^{1/\gamma-1}\frac{1}{\log(1/|\epsilon|)}\\
&\to 0
\end{aligned}$$

23. Theorem 7 for $R_{2Bii(II,III)R_2}$ when $v << \log(1/|\epsilon|)$

$$\begin{aligned}
\sum_{n,m} h_{\alpha,n}h^{\epsilon}_{\beta,m} &\sim \int_{R_2}\frac{\tilde H_\alpha(x)\tilde H_\beta(y)}{(x-(\log(1/|\epsilon|))^\gamma)^2}\frac{d(\log(1/|\epsilon|))^\gamma-|y-(\log(1/|\epsilon|))^\gamma|}{(\log(1/|\epsilon|))^{\gamma-1}}dxdy\\
&\sim \int_{R_2}\frac{\tilde H_\alpha(x)|x|^{1/\gamma-1}\tilde H_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))^\gamma)^2}|x|^{1-1/\gamma}|y|^{1-1/\gamma}\\
&* \frac{d(\log(1/|\epsilon|))^\gamma-|y-(\log(1/|\epsilon|))^\gamma|}{(\log(1/|\epsilon|))^{\gamma-1}}dxdy\\
&\sim \int_{R_2}\frac{\tilde H_\alpha(x)|x|^{1/\gamma-1}\tilde H_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))^\gamma)^2}|x|^{1-1/\gamma}\\
&* (d(\log(1/|\epsilon|))^\gamma-|y-(\log(1/|\epsilon|))^\gamma|)dxdy\\
&\sim \int_{R_2}\tilde H_\alpha(x)|x|^{1/\gamma-1}\tilde H_\beta(y)|y|^{1/\gamma-1}\frac{|x|}{|x-(\log(1/|\epsilon|))^\gamma|}|x|^{-1/\gamma}\\
&* \frac{(d(\log(1/|\epsilon|))^\gamma-|y-(\log(1/|\epsilon|))^\gamma|)}{|x-(\log(1/|\epsilon|))^\gamma|}dxdy\\
&<\sim \int_{R_2}\tilde H_\alpha(x)|x|^{1/\gamma-1}\tilde H_\beta(y)|y|^{1/\gamma-1}\frac{1}{\log(1/|\epsilon|)}dxdy\\
&\to 0
\end{aligned}$$

24. Theorem 7 for $R_{2Bii(II,III)R_3}$ when $v << \log(1/|\epsilon|)$

$$
\begin{aligned}
\sum_{n,m} h_{\alpha,n}h^{\epsilon}_{\beta,m} &\sim \int_{R_3} \frac{\tilde{H}_\alpha(x)\tilde{H}_\beta(y)}{(x-(\log(1/|\epsilon|))^\gamma)^2}\frac{|x-y|}{(\log(1/|\epsilon|))^{\gamma-1}}dxdy \\
&\sim \int_{R_3} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1}\tilde{H}_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))^\gamma)^2(((\log(1/|\epsilon|))^\gamma)^{1/\gamma-1})^2} \\
&* \quad \frac{|x-y|}{(\log(1/|\epsilon|))^{\gamma-1}}dxdy \\
&\sim \int_{R_3} \frac{\tilde{H}_\alpha(x)|x|^{1/\gamma-1}\tilde{H}_\beta(y)|y|^{1/\gamma-1}}{(x-(\log(1/|\epsilon|))^\gamma)^2(\log(1/|\epsilon|))^{2-2\gamma}} \\
&* \quad \frac{|x-y|}{(\log(1/|\epsilon|))^{\gamma-1}}dxdy \\
&\sim \int_{R_3} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\tilde{H}_\beta(y)|y|^{1/\gamma-1} \\
&* \quad \frac{|x-y|}{(x-(\log(1/|\epsilon|))^\gamma)^2(\log(1/|\epsilon|))^{1-\gamma}}dxdy \\
&<\sim \int_{R_3} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\tilde{H}_\beta(y)|y|^{1/\gamma-1} \\
&* \quad \frac{1}{|x-(\log(1/|\epsilon|))^\gamma|(\log(1/|\epsilon|))^{1-\gamma}}dxdy \\
&<\sim \int_{R_3} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\tilde{H}_\beta(y)|y|^{1/\gamma-1} \\
&* \quad \frac{1}{\log(1/\delta)(\log(1/|\epsilon|))^{\gamma-1}(\log(1/|\epsilon|))^{1-\gamma}}dxdy \\
&\sim \frac{1}{\log(1/\delta)}\int_{R_3} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}\tilde{H}_\beta(y)|y|^{1/\gamma-1}dxdy \\
&\to 0
\end{aligned}
$$

25. Theorem 7 for $R_{2Bii(II,IV)}$ when $v << \log(1/|\epsilon|)$

Recall from Lemma 24:

$$-2nb\pi/a + 1/a\log(1/|\epsilon|) - C < v, v' < -2nb\pi/a + 1/a\log(1/|\epsilon|) + C.$$

$$\begin{aligned} h_{\alpha,n}h^{\epsilon}_{\beta,m} &\sim \int_{|x-U|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(x-U)^2}dx \\ &* \int_{|y-U|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\beta(y)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(y-U)^2}dy \\ &\sim \int_{|x-(\log(1/|\epsilon|))^{\gamma}|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(x-(\log(1/|\epsilon|))^{\gamma})^2}dx \\ &* \int_{|y-(\log(1/|\epsilon|))^{\gamma}|>(\log(1/|\epsilon|))^{\gamma-1}|v|} \tilde{H}_\beta(y)\frac{(\log(1/|\epsilon|))^{\gamma-1}v}{(y-(\log(1/|\epsilon|))^{\gamma})^2}dy \\ \log(1/\delta) &< v < d\log(1/|\epsilon|) \end{aligned}$$

Note that when we sum over n, v depends linearly on n and ranges from $\log 1/\delta$ to $d\log(1/|\epsilon|)$, $0 < d << 1$.

Hence we need to estimate the expression $I(\alpha,\beta)$for given (α,β):

$$\begin{aligned} I(\alpha,\beta) &:= \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} \int_{|x-(\log(1/|\epsilon|))^{\gamma}|>(\log(1/|\epsilon|))^{\gamma-1}k} \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}k}{(x-(\log(1/|\epsilon|))^{\gamma})^2}dx \\ &* \int_{|y-(\log(1/|\epsilon|))^{\gamma}|>(\log(1/|\epsilon|))^{\gamma-1}k} \tilde{H}_\beta(y)\frac{(\log(1/|\epsilon|))^{\gamma-1}k}{(y-(\log(1/|\epsilon|))^{\gamma})^2}dy \end{aligned}$$

We introduce the integrals

$$\begin{aligned} I_{j,\alpha} &:= \int_{(\log(1/|\epsilon|))^{\gamma-1}j<|x-(\log(1/|\epsilon|))^{\gamma}|<(\log(1/|\epsilon|))^{\gamma-1}(j+1)} \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}}{(x-(\log(1/|\epsilon|))^{\gamma})^2}dx \\ &\sim \int_{(\log(1/|\epsilon|))^{\gamma-1}j<|x-(\log(1/|\epsilon|))^{\gamma}|<(\log(1/|\epsilon|))^{\gamma-1}(j+1)} \tilde{H}_\alpha(x)\frac{1}{j^2(\log(1/|\epsilon|))^{\gamma-1}}dx \\ &\sim \frac{1}{j^2}\int_{(\log(1/|\epsilon|))^{\gamma-1}j<|x-(\log(1/|\epsilon|))^{\gamma}|<(\log(1/|\epsilon|))^{\gamma-1}(j+1)} \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\ &= \frac{1}{j^2}\hat{I}_{j,\alpha} \end{aligned}$$

$$\begin{aligned} I_{\infty,\alpha} &:= \int_{|x-(\log(1/|\epsilon|))^{\gamma}|>d(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}}{(x-(\log(1/|\epsilon|))^{\gamma})^2}dx \\ &\sim \int_{d(\log(1/|\epsilon|))^{\gamma}<|x-(\log(1/|\epsilon|))^{\gamma}|<Cd(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\alpha(x)\frac{1}{(\log(1/|\epsilon|))^{\gamma+1}}dx \\ &+ \int_{|x-(\log(1/|\epsilon|))^{\gamma}|>C(\log(1/|\epsilon|))^{\gamma}} \tilde{H}_\alpha(x)\frac{(\log(1/|\epsilon|))^{\gamma-1}}{x^2}dx \\ &<\sim \frac{1}{(\log(1/|\epsilon|))^2}\int \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \\ &= I^1_{\infty,\alpha} \end{aligned}$$

and similarly for β.

We get:

$$\begin{aligned}
I(\alpha,\beta) &= \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} \left[k\left(\left(\sum_{j=k}^{d\log(1/|\epsilon|)} I_{j,\alpha} \right) + I_{\infty,\alpha} \right) \right] \left[k\left(\left(\sum_{i=k}^{d\log(1/|\epsilon|)} I_{i,\beta} \right) + I_{\infty,\beta} \right) \right] \\
&\sim \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} k^2 \left[\left(\left(\sum_{j=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{j,\alpha}}{j^2} \right) + I_{\infty,\alpha} \right) \right] \left[\left(\left(\sum_{i=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{i,\beta}}{i^2} \right) + I_{\infty,\beta} \right) \right] \\
&= \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} k^2 \left[\sum_{j=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{j,\alpha}}{j^2} \right] \left[\sum_{i=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{i,\beta}}{i^2} \right] \\
&+ \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} k^2 I_{\infty,\alpha} \left[\sum_{i=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{i,\beta}}{i^2} \right] \\
&+ \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} k^2 \left[\sum_{j=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{j,\alpha}}{j^2} \right] I_{\infty,\beta} \\
&+ \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} k^2 I_{\infty,\alpha} I_{\infty,\beta} \\
&= I + II + III + IV
\end{aligned}$$

Here II and III are symmetric. It suffices to estimate II.

We estimate IV first. Since $\sum k^2 \sim (\log(1/|\epsilon|))^3$, this is immediately small. For II, we get:

$$\begin{aligned}
II &= \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} k^2 I_{\infty,\alpha} \left[\sum_{i=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{i,\beta}}{i^2} \right] \\
&< I_{\infty,\alpha} \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} \left[\sum_{i=k}^{d\log(1/|\epsilon|)} \hat{I}_{i,\beta} \right] \\
&< \frac{1}{(\log(1/|\epsilon|))} \int \tilde{H}_\alpha(x)|x|^{1/\gamma-1}dx \int \tilde{H}_\beta(y)|y|^{1/\gamma-1}dy \\
&\to 0
\end{aligned}$$

Finally we estimate I.

$$I = \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} k^2 \left[\sum_{j=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{j,\alpha}}{j^2}\right]\left[\sum_{i=k}^{d\log(1/|\epsilon|)} \frac{\hat{I}_{i,\beta}}{i^2}\right]$$
$$< \sum_{k=\log 1/\delta}^{d\log(1/|\epsilon|)} \frac{1}{k^2}\left[\sum_{j=k}^{d\log(1/|\epsilon|)} \hat{I}_{j,\alpha}\right]\left[\sum_{i=k}^{d\log(1/|\epsilon|)} \hat{I}_{i,\beta}\right]$$

We can make this as small as we wish by choosing δ small.

26. Proof of Theorem 4

Proof. We use the approach in [8].

Let T be a positive harmonic current directed by $\mathcal{F}$. We want to show that $\int T\wedge T = 0$. Let $T_\epsilon = (\Phi_\epsilon)_* T$ and define T_ϵ^δ as the average of T_ϵ using a small neighborhood of identity in $U(3)$. Then since $T_\epsilon \rightharpoonup T$, we have $\int T\wedge T = \lim_{\epsilon\to 0}\int T\wedge T_\epsilon$. On the other hand $T_\epsilon^\delta = \omega + \partial S_\epsilon^\delta + \overline{\partial S_\epsilon^\delta} + i\partial\overline{\partial} u_\epsilon^\delta$ and $S_\epsilon^\delta \to S_\epsilon$ in L^2. So $T\wedge T_\epsilon = \lim_{\delta,\delta'\to 0,\delta,\delta'<<\epsilon}\int T_\epsilon^\delta \wedge T^{\delta'}$. Hence as in [8] it is enough to show that

$$\lim_{\delta,\delta',\epsilon\to 0,|\delta|,|\delta'|<<|\epsilon|}\int T_\epsilon^\delta \wedge T^{\delta'} = 0.$$

We can compute the geometric intersection $T_\epsilon^\delta \wedge T^{\delta'}$ and it is enough to estimate $T_\epsilon \wedge_g T$. If ϕ is a test function supported in B,

Lemma 25. *We have that* $\int T\wedge T_\epsilon = T\wedge_g T_\epsilon$. *The same holds for* $T^\delta, T_\epsilon^{\delta'}$.

$$< T_\epsilon \wedge T, \phi >_g \leq C\|\phi\|_\infty \int \sum_{J_{\alpha,\beta}^\epsilon} H_\alpha(p) H_\beta^\epsilon(p) d\mu(\alpha) d\mu(\beta),$$

where $J_{\alpha,\beta}^\epsilon$ consists of intersection points of Δ_α and Δ_β^ϵ. We know that the number of points in $J_{\alpha,\beta}^\epsilon$ is bounded by a fixed constant independent of ϵ. For p out of a fixed neighborhood of the singularities the integral converges to zero. This is the case considered in [8]. So it is enough to show that for $\delta > 0$ small enough

$$J_\epsilon(\delta) := \int \sum_{J_{\alpha,\beta}^\epsilon} H_\alpha(p) H_\beta^\epsilon(p) d\mu(\alpha) d\mu(\beta)$$

is arbitrarily small. This is precisely the content of Theorem 7, since all estimates are valid after composition by automorphisms in a small neighborhood of $U(3)$.

Consequently if T, T_2 are two such currents then $\int \frac{T_1+T_2}{2} \wedge \frac{T_1+T_2}{2} = 0$. Hence $\int T_1 \wedge T_2 = 0$, therefore T_1, T_2 are proportional.

□

We give a dynamical consequence of the uniqueness of the harmonic current for $\mathcal{F} \in \mathcal{H}(d)$, here $\mathcal{H}(d)$ is the Zariski open set of foliations of degree d, introduced in Theorem 2.

Corollary 1. *Let* $\mathcal{F} \in \mathcal{H}(d)$. *Let* $\phi : \Delta \to L$ *be the universal covering of a leaf* L. *Let* $\tau_r := \frac{\phi_*[\log^+ \frac{r}{|z|}\Delta_r]}{\|\phi_*[\log^+ \frac{r}{|z|}\Delta_r]\|}$. *Then* $\lim_{r\to 1}\tau_r = T$, *where* T *is the unique harmonic current directed by* $\mathcal{F}$.

Here Δ_r denotes the disc of center 0 and radius r. The Corollary which is a consequence of paragraph 5 in [8] says that the normalized images of $[\log^+ \frac{r}{|z|}\Delta_r]$ converge to T. This is similar to the pointwise ergodic theorem, since we are averaging on an orbit.

Recall that the limit set of a leaf L is defined as $\lim(L) = \cap_n \overline{L \setminus K_n}$, where $K_n \subset K_{n+1}$ is an exhaustion of L by compact sets. One of the main questions in foliation theory is to describe the limit set of a foliation $\mathcal{F}$: $\lim(\mathcal{F}) := \overline{\cup_{L\in\mathcal{F}} \lim(L)}$. Corollary 2 implies in particular that for $\mathcal{F} \in \mathcal{H}(d)$, for every leaf $L \in \mathcal{F}$, $\lim(L) = \text{supp}(T)$. This is clear as shown in [8].

$$\|\Phi_* \left[\log^+ \frac{r}{|z|}\Delta_r\right] \| \to \infty$$

as $r \to 1$, hence $\text{supp}(T) \subset \overline{L \setminus K_n}$ for every n.

Corollary 2. *The map $\lambda \to T_\lambda$ is continuous from $\mathcal{H}(d)$ with values in the positive harmonic currents of mass one. Let $\mathcal{F}_\lambda$ be a holomorphic family of foliations in $\mathcal{H}(d)$. Let (T_λ) be the associated currents. If a hyperbolic point $p_0 \in Supp(T_{\lambda_0})$, then the perturbed hyperbolic point p_λ belongs to $Supp(T_\lambda)$.*

Proof. Assume $\mathcal{F}_{\lambda_n} \to \mathcal{F}_{\lambda_0}$ in $\mathcal{H}(d)$. Let (T_{λ_n}) be the normalized positive harmonic currents associated to $\mathcal{F}_{\lambda_n}$. Since $\|T_{\lambda_n}\| = 1$, the sequence (T_{λ_n}) has cluster points. It is clear that any cluster point S is positive harmonic and directed by $\mathcal{F}_{\lambda_0}$. So $S = T_{\lambda_0}$ by uniqueness. Assume the support of T_{λ_0} intersects a ball $B(p_0, r)$ where p_0 is a hyperbolic singular point of $\mathcal{F}_{\lambda_0}$ and the ball is contained in the common domain of linearization of $p_\lambda \in \text{Sing}(\mathcal{F}_\lambda), p_\lambda \to p_0$, p_λ hyperbolic.

From our local study of positive harmonic currents near a hyperbolic singular point $p_0 \in \text{Supp}(T_{\lambda_0})$. Since $T_\lambda \to T_{\lambda_0}$, T_λ gives mass to $B(p_0, r)$, applying again the local study for T_λ we get that $p_\lambda \in \text{Supp}(T_\lambda)$.

□

Remark 2. Let f be a holomorphic endomorphism of $\mathbb{P}^2$. Let $\mathcal{F}$ be a foliation with only hyperbolic singularities. Then $f^*\mathcal{F}$ is a foliaton and its singularities are not necessarily hyperbolic. However there is only one positive harmonic current of mass 1, directed by $f^*\mathcal{F}$. Indeed let τ be any such current. We will show that $\int T \wedge T = 0$ which implies the uniqueness. Observe that f_*T is a current directed by $\mathcal{F}$. Hence $\int f_*T \wedge f_*T = 0$. Since f^* is a finite covering of degree d^2 we have

$$\int T \wedge T \leq \int f^* \left[f_*T \wedge f_*T\right] = d^2 \int f_*T \wedge f_*T = 0.$$

27. Measure associated to a harmonic current

Let $\mathcal{F} \in \mathcal{H}(d)$ be a holomorphic foliation as in Theorem 2. We know that there is a unique positive harmonic current T of mass one directed by $\mathcal{F}$.

We are going to associate to T a conformal, measurable metric along leaves that we will denote by g_T and also a positive finite measure μ_T which is invariant under the harmonic flow associated also to T.

On a flow box B disjoint from $E = \text{Sing}(\mathcal{F})$, the current T can be written

$$T = \int h_\alpha [V_\alpha] d\mu(\alpha)$$

where h_α are positive harmonic functions and μ is a positive measure on a transversal A. The $[V_\alpha]$ are the currents of integration on plaques. On $B, \partial T = \tau \wedge T$ with $\tau = \frac{\partial h_\alpha}{h_\alpha}, ; \mu$ almost everywhere. Observe that τ is independent of the choice of h_α : if we replace h_α by $c_\alpha h_\alpha, c_\alpha \in \mathbb{R}^+$ then τ is unchanged.

We define the metric g_T on leaves by $g_T = \frac{i}{2}\tau \otimes \overline{\tau}$. Along the plaque V_α with a choice of coordinate (z_α) we have

$$g_T = \frac{i}{2}\left|\frac{\partial h_\alpha}{\partial z_\alpha}\right|^2 \frac{1}{h_\alpha^2} dz_\alpha \otimes d\overline{z}_\alpha \qquad (1)$$

Define $\mathcal{C}_T = \{(\alpha, z); \frac{\partial h_\alpha}{\partial z}(\alpha, z) = 0\}$ it's the critical set of the "metric" g_T. We also define the current of bidegree $(2,2)$, μ_T, which we identify with a measure

$$\mu_T := i\tau \wedge \overline{\tau} \wedge T.$$

In local coordinates in a flow box B, we have:

$$\mu_T = \int d\nu(\alpha) \int_{[V_\alpha]} \left|\frac{\partial h_\alpha}{\partial z_\alpha}\right|^2 \frac{1}{h_\alpha}(idz_\alpha \wedge d\overline{z}\alpha). \qquad (2)$$

Proposition 2. *Let $\mathcal{F} \in \mathcal{H}_d$. The metric g_T has constant negative curvature out of the set $\mathcal{C}_T$ where the metric vanishes.*

Proof. Since the current T is unique, every measurable set of leaves $\mathcal{A}$ has zero or full measure with respect to $\|T\|$. Define $\mathcal{N}_g := \{\text{leaves on which } g_T \text{ vanishes identically}\}$. Since h_α is measurable, then $\mathcal{N}_g$ is measurable. So $\mathcal{N}_g$ is of zero or full measure. But if $\mathcal{N}_g$ is of full measure, $\partial T = 0$ and by conjugation $\overline{\partial} T = 0$, hence T is closed. A foliation $\mathcal{F}$ in $\mathcal{H}_d$ admits no positive closed current directed by $\mathcal{F}$ since all singularities are hyperbolic. So $\mathcal{N}_g$ is of zero $\|T\|$ measure.

From (1) it is clear that the metric is conformal. On a flow box B, the curvature $\kappa(g)$ has the following expression out of C_T. According to Kobayashi,

$$\kappa(g) = -\frac{1}{4}\frac{\Delta \log g}{g} = \frac{1}{2}\frac{\Delta \log h_\alpha}{\left|\frac{\partial h_\alpha}{\partial z_\alpha}\right|^2 \frac{1}{h_\alpha^2}}.$$

So

$$\kappa(g_T) = \frac{h_\alpha^2}{|h_{\alpha,z}|^2}\left(\frac{\partial}{\partial \overline{z}}\left(\frac{h_{\alpha,z}}{h_\alpha}\right)\right).$$

Since h_α is harmonic we get $\kappa(g_T) = -1$.

□

Proposition 3. *Let T be the harmonic current associated to $\mathcal{F} \in \mathcal{H}(d)$. If g_T is the associated metric on leaves, then $g_T \leq g$.*

Proof. We have normalized the metric g_T so that on each leaf L_α, g_T has curvature -1 on $L_\alpha \setminus \mathcal{C}(T)$. he Ahlfor's Schwarz lemma, applied to the abstract Riemann surface $L_\alpha \setminus \mathcal{C}_T$ implies that $g_T \leq g$. □

Because of the nature of the singularities, the leaves are uniformized by the unit disc Δ. Let g denote the Poincaré metric on leaves. We choose a normalization so that the curvature $\kappa(g)$ of g on leaves is -1. We will denote by $\Phi_\alpha : \Delta \to \mathcal{L}_\alpha$, the uniformizing map from Δ to $\mathcal{L}_\alpha$. When we fix a transversal A in a flow box we can choose for each $\alpha \in A$ a uniformizing map $\Phi_\alpha(0) = \alpha$, then Φ_α vary measurably. We will denote by Γ_α the group of deck transformations for the map Φ_α.

We want to define a vector field χ on $\mathcal{F}$ associated to the current T. The vector field will be defined as the metric g_T only $\|T\|$ a.e. On L_α, χ_α is collinear with the gradient field of h_α. We define χ_α on a flow box with local coordinates $z_\alpha = x_\alpha + iy_\alpha$ by

$$\chi_\alpha := c\frac{h_\alpha}{|h_z|^2}(h_{x_\alpha}, h_{y_\alpha}).$$

We choose c so that $g_T(\chi_\alpha, \chi_\alpha) = 1$. The vector field χ_α is independent of the choice of h. It blows up at every point of $\mathcal{C}_T$. Which means that the integral curves of χ_α approach these points at infinite speed. It is clear that the integral curves of χ_α are along the level sets of the harmonic conjugates of h_α such that $f_\alpha = h_\alpha + iv_\alpha$ is holomorphic.

Theorem 9. *Let T be the positive harmonic current associated to $\mathcal{F} \in \mathcal{H}(d)$. Then the measure μ_T is finite and the flow ψ_T of the vector field χ preserves μ_T. Moreover, if $\mathcal{F}_\lambda$ is a holomorphic family of foliations in $\mathcal{H}(d)$, $\lambda \in \Delta(\lambda_0, r)$, then the mass of μ_{T_λ} near hyperbolic singularities is uniformly small in a fixed neighborhood of the singularities.*

Proof. For a flow box B away from the singularities, it is clear that μ_T has finite mass. Indeed the functions h_α are positive harmonic, and by Harnack $\frac{h_\alpha}{|\partial h_\alpha|} \leq c$, hence μ_T has finite mas in B. It is enough to show that μ_T has finite mass in a flow box B_i near a hyperbolic singularity given by $\omega = zdw - \lambda wdz, \lambda = a + ib, b \neq 0$. We use the parametrization

$$\psi_\alpha(\zeta) = (e^{i(\zeta+(\log|\alpha|)/b)}, \alpha e^{i\lambda(\zeta+(\log|\alpha|)/b)})$$

by a sector near the hyperbolic singularity. Since $\psi_\alpha^* h_\alpha = H_\alpha$ is a positive harmonic function and μ a.e. $H_\alpha(\zeta) \to 0$ when $\Im\zeta \to -\infty$, then again by Harnack $\psi_\alpha^*(\tau)$ is bounded. The total mass of μ_T in B_i satisfies

$$\int_{B_i} \mu_T \leq \int_{D(w_0,r)\times\Pi_\lambda} i\psi_\alpha^*(\tau) \wedge \psi_\alpha^*(\overline{\tau}) \wedge \psi_\alpha^*[V_\alpha] H_\alpha d\mu(\alpha)$$

$\psi_\alpha^*[V_\alpha]$ is a graph in the flow box. It is of bounded area and $\int_{D(w_0,r)} H_\alpha d\mu(\alpha)$ defines a bounded harmonic function. So the mass μ_T is bounded near the origin.

From the expression of g_T, we get that $g_T(\chi_\alpha, \chi_\alpha) = 1$. So the flows is leafwise volume preserving.

From the expression (2) of the measure μ_T in B, we get since $|\chi_\alpha| = \frac{h_\alpha}{|h_{\alpha,z}|}$ that ψ_T preserves μ_T. Basically the slicing of μ_T along the leaves gives the area measure on leaves associated to the metric g_T. Let T_λ be the current associated to $\mathcal{F}_\lambda$, and let μ^λ denote the corresponding measure on a transversal. The linearizations associated to a holomorphically varying hyperbolic singularity vary holomorphically. Then $\int_\lambda H_\alpha^\lambda d\mu^\lambda(\alpha) \to 0$ when $\Im\zeta \to -\infty$, uniformly when λ is near λ_0. (We don't say that H_α^λ vary holomorphically.) So the mass of μ_{T_λ} is uniformly small in a fixed neighborhood of the singularities if λ is close enough to λ_0.

□

Remark 3. Since μ_T is finite, the Poincaré recurrence theorem applies: For μ_T every p the orbit of p intersects any set of positive measure infinitely many times.

This gives a strong recurrence property for the leaves of $\mathcal{F}$. Not only the leaves are recurrent but the flow ψ_T is recurrent for μ_T.

Theorem 10. *Let $\lambda \to \mathcal{F}_\lambda$ be a holomorphic family of foliations in $\mathcal{H}(d)$, parametrized by a disc Δ. Then $\lambda \to \mu_\lambda$ is a continuous family of measures.*

Proof. Let (T_λ) be the family of the positive harmonic currents directed by $\mathcal{F}_\lambda$. Recall that $\mu_{T_\lambda} = i\tau_\lambda \wedge \overline{\tau}_\lambda \wedge T_\lambda$.

Fix a flow box B for $\mathcal{F}_{\lambda_0}$ away from the singularities. We can consider (ϕ_λ) local biholomorphisms straightening $\mathcal{F}_\lambda$ in B, when $\lambda \to \lambda_0$. We know that the currents $S_\lambda := (\phi_\lambda)_* T_\lambda$ depend continuously on λ. We can write in B,

$$S_\lambda = \int [w = \alpha] h^\lambda_\alpha(z) d\mu_\lambda(\alpha)$$

where μ_λ is the measure on a fixed transversal $(z = z_0)$. We can assume that $h^\lambda_\alpha(z_0) = 1$ for all α, λ.

Since $S_\lambda \to S_{\lambda_0}$ then for every z we have $h^\lambda_\alpha(z)\mu_\lambda(\alpha) \to h^{\lambda_0}_\alpha \mu_{\lambda_0}(\alpha)$ weakly when $\lambda \to \lambda_0$.

The $(h^\lambda_\alpha)^2$ also vary slowly, by Harnack, so we also get that $\lambda \to (h^\lambda_\alpha(z))^2\mu_\lambda(\alpha)$ is continuous for every z. Define

$$U_\lambda := \int [w = \alpha](h^\lambda_\alpha)^2(z) d\mu_\lambda(\alpha).$$

The family of positive currents U_λ is also continuous because $(h^\lambda_\alpha)^2$ is uniformly bounded. It follows that $\lambda \to i\partial\overline{\partial} U_\lambda$ is continuous i.e.

$$\lambda \to |h^\lambda_{\alpha,z}|^2 [w = \alpha] d\mu_\lambda(\alpha).$$

Using again Harnack inequalities for $\frac{1}{h^\lambda_\alpha}$, we find that
$\lambda \to |h^\lambda_\alpha|^2 [w = \alpha] d\mu_\lambda(\alpha)$ is continuous. Hence $\lambda \to \mu_{T_\lambda}$ is continuous in B.

We have seen in Theorem 9 that μ_{T_λ} has uniformly small mass near the singularities. Hence $\lambda \to \mu_{T_\lambda}$ is continuous.

□

Let $|g^\alpha_T|$ denote the measure induced by the metric g_T on the leaf L_α. We will omit α, most frequently.

We do not address the question whether the flow of χ on L_α is complete. We will say that a set E is invariant if up to a set of μ_T measure zero, it is a union of orbits of χ. For a measurable set E we denote by E_α the intersection $E \cap L_\alpha$.

Theorem 11. *Either there is an invariant set E for χ such that for $\|T\|$ almost every leaf L_α, $|g_T|(E_\alpha) > 0$ and $|g_T|(E^c_\alpha) > 0$ or the measure μ_T is ergodic.*

Proof. Fix a countable family (B_i) of flow boxes such that $\cup_i B_i = \mathbb{P}^2 \setminus (\mathrm{Sing}(\mathcal{F})$. Let E be an invariant set for χ such that $\mu_T(E) > 0$. Define $E_i = \{\alpha; |g_T|(L_\alpha \cap E \cap B_i) = 0\}$. $\mathcal{E} := \cap_i E_i$ is measurable. It is a union of leaves. Since the current T is unique and $\mu_T(\mathcal{E}) = 0$, then $\|T\|$ almost every leaf is in $\mathcal{E}^c$.

For $L_\alpha \in \mathcal{E}^c$, $|g_T|(E_\alpha) > 0$. We can do a similar construction for E^c if $\mu_T(E^c) > 0$. We then get a set of $\|T\|$ full measure of leaves such that $|g_T|(E_\alpha) > 0$ and $|g_T|(E^c_\alpha) > 0$.

□

28. Remarks on $\partial\overline{\partial}$ and the $\overline{\partial}$ equation on a lamination X.

Let $(X, \mathcal{L}, E)$ be a $\mathcal{C}^1$ lamination, possibly with singularities, in a compact Kahler manifold (M, ω). We assume that there is no positive closed current on X directed by $\mathcal{L}$. Let f be a continuous $(1,1)$ form. We address the question of the solvability of the equation

$$i\partial\overline{\partial}u = f. \qquad (1)$$

Here $i\partial\overline{\partial}$ denotes the tangential $i\partial_\tau\overline{\partial}_\tau$ and equation (1) is taken to hold when f is restricted to any leaf, away from the singularities. Moreover u is a continuous function on $X \setminus E$ which is $\mathcal{C}^2$ along any leaf.

If u is a solution of (1) and $u = \lim u_\epsilon$ with u_ϵ smooth on $\mathbb{P}^2$ and the convergence is uniform on compact subsets of $X \setminus E$ in the supnorm together with $\mathcal{C}^2$ norm along leaves, then clearly f satisfies

$$< T, f >= 0$$

for every harmonic current T of order 0 directed by the lamination. at least in the case that $E = \emptyset$.

We have proved in [8] Theorem 3.14 that if T is a positive harmonic current directed by $\mathcal{L}$, which is on an extremal ray in the convex cone of such currents, then any function $u \in L^1(T)$ which satisfies $i\partial\overline{\partial}(uT) = 0$ is constant. Since at least in $\mathbb{P}^2$ we have proved the uniqueness of such currents, it is natural to explore the solvability of equation (1) assuming the moment condition

$$\int T \wedge f = 0 \qquad (2)$$

for a positive harmonic current T directed by $\mathcal{L}$. For simplicity we state our remark for $\mathcal{C}^1$ laminations without singularities, but using [8] Theorems 5.3, 5.7 it can be easily adapted to laminations with a finite number of singularities as considered there.

Proposition 4. *Let $(X, \mathcal{L})$ be a laminated compact set in (M, ω). Assume the lamination is $\mathcal{C}^1$ and that there is no positive closed current on X directed by $\mathcal{L}$. Fix a positive harmonic current T directed by $\mathcal{L}$. Then if there is a smooth $(1,1)$ form f on $(X, \mathcal{L})$ such that equation (1) has a bounded solution, then $\int_0^1 \frac{1}{t}\int_{\Delta_t} \phi^*(f) = \mathcal{O}(1)$ for every parametrization ϕ of a leaf L by the unit disc.*

Proof. We know [8] that the leaf L through p is covered by the unit disc, and that the covering $\phi : \Delta \to L$ satisfies

$$\frac{1}{C}\frac{1}{1-|z|} \leq |\phi'| \leq \frac{C}{1-|z|}$$

for an appropriate constant C. Let $v := u \circ \phi$. Then

$$i\partial\overline{\partial}v = \phi^* f$$

on the unit disc. Since v is bounded, we get by Stokes formula

$$\int_0^1 \frac{1}{t}\int_{\Delta_t} \phi^*(f) = \int_0^1 \frac{1}{t}\int_{\Delta(0,t)} i\partial\overline{\partial}v = \mathcal{O}(1).$$

□

Remark 4.
(i) Since $|\phi^*(f)| \sim \frac{1}{(1-|z|)^2}$, it follows that solvability implies a lot of cancellation on each leaf in order to get the boundedness of the integral.
(ii) Let $(X, \mathcal{L})$ be as in the theorem. Assume that f is a smooth $(0,1)$ form on X. Does the equation

$$\overline{\partial} u = f$$

admit a solution with some regularity? Here $\overline{\partial}$ is to be considered along leaves. If T is a positive harmonic current directed by $\mathcal{L}$, then we should have

$$0 =< T, \partial\overline{\partial} u >=< T, \partial f > .$$

So there is a compatibility condition: $< T, \partial f >= 0$. The regularity required on u to find the obstruction is that $u = \lim u_\epsilon$ with u_ϵ smooth and $\partial\overline{\partial} u_\epsilon \to \partial\overline{\partial} u$ say uniformly.
(iii) Assume X is a Levi flat hypersurface in an algebraic manifold M. By Levi flat we mean that the rank of the Levi form on the tangent space is zero. Then the intersection of X with a subspace or a subvariety V of complex dimension 2 carries a positive harmonic current T_V of bidimension $(1,1)$. Let f be a $\overline{\partial}-$ closed smooth $(0,1)$ form on X. Then if $\overline{\partial} u = f$ along leaves one should have $< T_V, \partial f >= 0$.

Proposition 5. *Let X be a real analytic lamination by Riemann surfaces in a compact manifold M. There is a smooth $(0,1)$ form f such that the equation*

$$\overline{\partial} u = f \qquad (1)$$

has no smooth solution.

Proof. Let T be a harmonic current of order 0 directed by $(X, \mathcal{L})$. if (1) is solvable then

$$\int df \wedge T = \int \partial\overline{\partial} u \wedge T = 0. \qquad (2)$$

We now construct f which does not satisfy (2). Let B be a flow box. Then $T = h_w(z) i dw \wedge d\overline{w} \nu(w)$ in the flow box, $\partial T = \tau \wedge T$. Choose χ_1, χ_2 smooth functions with compact support such that $\int \chi_1(z)\chi_2(w)\tau \wedge T \wedge d\overline{z} \neq 0$. Define $f = \chi_2(w)\chi_1(z) d\overline{z}$. Then $\int df \wedge T = \int \chi_2(w)\chi_1(z) d\overline{z} \wedge \tau \wedge T \neq 0$.

□

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