

HARMONIC CURRENTS OF FINITE ENERGY AND LAMINATIONS

by

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Abstract: We introduce, on a complex Kähler manifold (M, ω) , a notion of energy for harmonic currents of bidegree $(1, 1)$. This allows us to define $\int T \wedge T \wedge \omega^{k-2}$, for positive harmonic currents. We then show that for a lamination with singularities of a compact set in \mathbf{P}^2 there is a unique positive harmonic current which minimizes energy. If X is a compact laminated set in \mathbf{P}^2 of class \mathcal{C}^1 it carries a unique positive harmonic current T of mass 1. The current T can be obtained by an Ahlfors type construction starting with a arbitrary leaf of X .

1. INTRODUCTION

Let X be a compact set in a complex manifold M . If X is laminated by Riemann surfaces, a result due to L. Garnett implies the existence of a positive current T of bidimension $(1, 1)$ which is harmonic i.e. such that $i\partial\bar{\partial}T = 0$. Moreover in a flow box B , the current can be expressed as

$$T = \int h_\alpha[V_\alpha]d\mu(\alpha), \quad (1)$$

the functions h_α are positive and harmonic on the local leaves V_α , and μ is a measure on the transversal. On the other hand there is a well known problem. Does there exist a compact laminated set X in \mathbf{P}^2 which is not a compact Riemann surface? See [CLS], [Gh] and [Z] where the problem is discussed. If such an X does not exist then the closure of any leaf L of a holomorphic lamination in \mathbf{P}^2 will contain a singularity.

In this paper we study positive harmonic currents directed by a laminated set with singularities. More precisely we consider compact sets X laminated by Riemann surfaces out of an exceptional set E . We will assume that E is locally pluripolar and that $\overline{X \setminus E} = X$. We will call such a set (X, \mathcal{L}, E) a laminated compact set with singularities. We consider on such sets harmonic currents T of order 0 and bidegree $(1, 1)$ in \mathbf{P}^2 . The current T can be written in the form

$$T = c\omega + \partial S + \bar{\partial} \bar{S}$$

where ω is the standard Kähler form on \mathbf{P}^2 , $c \in \mathbf{R}$, S is a $(0, 1)$ current. It turns out that $\bar{\partial} \bar{S}$ depends only on T and that for a positive closed current one can define the energy $E(T)$ of T by the following integral

$$E(T) = \int \bar{\partial} \bar{S} \wedge \partial \bar{S}$$

and that $0 \leq E(T) < \infty$. It is hence possible to introduce a Hilbert space of classes of currents of finite energy. With this in hand the integral

$$Q(T) = \int T \wedge T$$

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makes sense for positive harmonic currents (not just the ones associated to (X, \mathcal{L}, E)) and has the usual meaning when T is smooth. We then prove (Theorem 2.22):

THEOREM 1.1. *Let (X, \mathcal{L}, E) be a laminated compact set with singularities. There is a closed positive laminated current on X or there is a unique positive harmonic laminated current T on X minimizing energy.*

We then study the geometric intersection of laminated currents. We show, see Proposition 2.16 and Theorem 5.2:

THEOREM 1.2. *If X is a \mathcal{C}^1 laminated compact set in \mathbf{P}^2 , then X carries a unique laminated positive harmonic current T . The class of the current T is extremal in the cone of positive harmonic currents. Moreover $\int T \wedge T = 0$.*

When X is not \mathcal{C}^1 we have to assume "finite transverse energy" to get the result. The main tool is that the quadratic form Q is negative definite on the hyperplane $\{T; \int T \wedge \omega = 0\}$.

The current can be obtained using a variant of Ahlfors' technique. Let $\Phi : \Delta \rightarrow L$ be the universal covering map from the unit disc to a leaf L . Let $G_r(z) = \frac{1}{2\pi} \log^+ \frac{r}{|z|}$. Define $T_r := (\Phi)_*(G_r[\Delta])$, $r < 1$. If $A(r)$ is the mass of T_r we get that

$$T = \lim_{r \rightarrow 1} \frac{T_r}{A(r)}.$$

For that purpose we need to estimate the derivative of Φ , the estimates are valid for any laminated set.

The main question that is left open in our approach is to estimate

$$c(\mathbf{P}^2) := \inf \left\{ \int T \wedge T; \int T \wedge \omega = 1, T \geq 0, i\partial\bar{\partial}T = 0 \right\}.$$

If $c(\mathbf{P}^2) > 0$, then there is no \mathcal{C}^1 laminated set in \mathbf{P}^2 .

2. HARMONIC CURRENTS

A subset Y of a complex manifold M is laminated by Riemann surfaces if it admits an open covering $\{U_i\}$ and on each U_i there is a homeomorphism $\phi_i = (h_i, \lambda_i) : U_i \rightarrow \Delta \times T_i$ where Δ is the unit disc and T_i is a topological space. The ϕ_i^{-1} are holomorphic in z . Moreover,

$$\phi_{ij}(z, t) = \phi_j \circ \phi_i^{-1}(z, t) = (h_{ij}(z, t), \lambda_{ij}(t))$$

where the $h_{ij}(z, t)$ are holomorphic with respect to z . When T_i is in a Euclidean space and ϕ_i extend to \mathcal{C}^k diffeomorphisms, we say that the lamination is \mathcal{C}^k . We call the U_i flow boxes, $\{\lambda_i = t_0\}$ is a plaque. A leaf is a minimal connected set such that if L intersects a plaque, then L contains the plaque. We only consider the case when we have a compact set X contained in a Kähler manifold M of dimension k . We assume that X contains a closed set E such that $\overline{X \setminus E} = X$ and $X \setminus E$ is a lamination \mathcal{L} by Riemann surfaces. We call such a triple (X, \mathcal{L}, E) a lamination with singularities, E is the singular set. We will say that (X, \mathcal{L}, E) is oriented if there are continuous nonvanishing $(1, 0)$ form $\gamma_j, j = 1, \dots, k-1$ defined on $X \setminus E$ such that $\gamma_j \wedge [\Delta_\alpha] = 0$ for every plaque Δ_α , $[\Delta_\alpha]$ denotes the current of integration on the disc Δ_α . We only consider sets E which are pluripolar, more precisely, for any $p \in E$, there are small balls $B(p, r)$ and u , a plurisubharmonic function on $B(p, r)$

such that $E \cap B(p, r) \subset \{u = -\infty\}$ and u is not identically $-\infty$ on $K \cap B(p, r)$. When $E \cap B(p, r) = \{u = -\infty\}$ we say that E is locally complete pluripolar. A positive current T of bidimension $(1, 1)$ with support in X is said to be directed by \mathcal{L} if on any open set U where \mathcal{L} is defined by nonvanishing continuous $(1, 0)$ forms $\gamma_j, j = 1, \dots, k-1$ i.e. $\gamma_j \wedge [\Delta_\alpha] = 0$, we have

$$T \wedge \gamma_j = 0.$$

To introduce the notion of a minimal set for X we need the following assumption on the family of leaves L .

There is a neighborhood \mathcal{V} of E , so that no leaf is contained in \mathcal{V} . (*)

DEFINITION 2.1. *A minimal set for (X, \mathcal{L}, E) is a compact subset $Y \subset X$ such that Y is not contained in E , moreover $Y \setminus E$ is a union of leaves L and for every leaf $L \subset Y \setminus E, \bar{L} = Y$.*

PROPOSITION 2.2. *Let (X, \mathcal{L}, E) satisfy assumption (*). Then there are minimal sets $Y \subset X$. Two different minimal sets intersect only on E . If M is a surface where the Levi problem is solvable and E is locally contained in a complex hypersurface, then any two minimal sets intersect.*

Proof: Let \mathcal{V} be an open neighborhood of E such that no leaf is contained in \mathcal{V} . Let (X_α) be an ordered decreasing chain saturated for \mathcal{L} . Let $X'_\alpha := X_\alpha \cap (X \setminus \mathcal{V})$. Then $X'_\alpha \neq \emptyset$. Hence $\cap X_\alpha \neq \emptyset$. So Zorn's Lemma applies. This shows that minimal sets exist.

It follows that each $M \setminus Y$ is locally pseudoconvex away from E , and since E is locally contained in a complex hypersurface, $M \setminus Y$ is pseudoconvex [GR]. If the Levi problem is solvable on M , i.e. if pseudoconvex domains are Stein, each $M \setminus Y$ is Stein. Since Stein manifolds cannot have two ends, it follows that any two minimal sets must intersect. ■

Remark 2.3. *Oka solved the Levi problem in \mathbf{P}^k . See [E] for a proof of this and some generalizations. The condition on E in the above Proposition, can be relaxed to assuming that E is meager in the sense of [GR].*

Example 2.4. *Let \mathcal{L}_α be the foliation in \mathbf{P}^2 defined by $w dz - \alpha z dw = 0$, with α irrational. Then $Y_c = \overline{\{|z| = c|w|^\alpha\}}$ is minimal for every c , the closure is in \mathbf{P}^2 . The associated positive closed current T is defined by $\pi^* T = i\partial\bar{\partial}u, u(z, w, t) = \log(\max\{|z||t|^{\alpha-1}, |w|^\alpha\})$ if $\alpha > 1$ and is directed by \mathcal{L}_α .*

DEFINITION 2.5. *Let M be a complex manifold of dimension k . For $0 \leq p, q \leq k$, let T be a (p, q) current of order 0. We say that T is harmonic if $i\partial\bar{\partial}T = 0$.*

Observe that if T is harmonic then \bar{T} , the conjugate, is also harmonic. A current is real if $T = \bar{T}$, in which case $p = q$.

2.1. Decomposition of harmonic currents. We want to prove a representation theorem for real harmonic currents on compact Kähler manifolds.

There is also a notion of \square -harmonic forms in the $\bar{\partial}$ literature [FK]. These are smooth (p, q) forms Ω for which $\square\Omega = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\Omega = 0$. These forms consist of the common null space of $\bar{\partial}$ and $\bar{\partial}^*$. Note that when $p = q$, $\bar{\square} = \square$ so the conjugate of a \square -harmonic form is also \square -harmonic. Since \square -harmonic forms are $\bar{\partial}$ closed they are also harmonic in the sense of currents as defined above. Recall that on a compact Kähler manifold, for a closed current u of bidegree (p, q) , the following are equivalent:

- i) u is exact,
- ii) u is $\bar{\partial}$ exact,
- iii) u is $\partial\bar{\partial}$ exact.

See Demailly [De], p 41 for smooth forms. The proof is the same for currents since cohomology groups for currents and smooth forms are the same. This follows from the deRham Theorem and the fact that for Kähler manifolds the Dolbeault cohomology groups $\oplus_{p+q=k} H^{p,q}(X, \mathbb{C})$ and H_{DR}^k are isomorphic [De] p.42.

PROPOSITION 2.6. *Let T be a harmonic (p, q) current on a compact Kähler manifold M of dimension k . Then*

$$T = \Omega + \partial S + \bar{\partial} R \quad (2)$$

where Ω is a unique closed smooth \square -harmonic form of bidegree (p, q) , and S is a current of bidegree $(p-1, q)$, R is of bidegree $(p, q-1)$. When dT is of order 0, we can choose S, R depending linearly on T and

$$L : T \rightarrow (\Omega, S, R)$$

is continuous in the topology of currents. When T is real we can choose $R = \bar{S}$.

Proof: The current $\bar{\partial}T$ is ∂ -closed, hence d -closed and is $\bar{\partial}$ exact. It follows from Lemma 8.6 in [De] that $\bar{\partial}T$ is $\partial\bar{\partial}$ exact so there is an S_0 of order 0 and of bidegree $(p-1, q)$ such that

$$\bar{\partial}T = \bar{\partial}\partial S_0.$$

Let Ω be a smooth \square -harmonic representative of the Dolbeault cohomology class of $T - \partial S_0$. (See [De] and [V].)

Then $T - \partial S_0 - \Omega$ is $\bar{\partial}$ exact. Hence

$$T = \Omega + \partial S_0 + \bar{\partial} R.$$

The choices of S_0 and R can be made linearly since there is an explicit inverse for $\partial\bar{\partial}$ and hence $\bar{\partial}$, on currents cohomologous to zero. See Dinh-Sibony [DS] where an explicit kernel is given. If T is real we obtain that T can be expressed as claimed.

The current T acts on $H^{n-p, n-q}$, the Dolbeault cohomology group, because of the $\partial\bar{\partial}$ lemma. Hence Ω is uniquely determined by T . ■

PROPOSITION 2.7. *If T is as in (2), then S, R are not unique, but any other S', R' can be obtained as $S' = S + \omega' + \partial v + \bar{\partial} u$, similarly for R' .*

Proof: The cohomology class Ω is defined uniquely. If S', R' is another solution we get $\partial(S - S') + \bar{\partial}(R - R') = 0$. Assume $\partial\sigma + \bar{\partial}\bar{\sigma}' = 0$. Then σ is harmonic. Using the above construction for a harmonic $(p-1, q)$ form, we get

$$S - S' = \sigma = \omega' + \partial v + \bar{\partial}u.$$

Hence

$$S = S' + (\omega' + \partial v + \bar{\partial}u).$$

■

COROLLARY 2.8. *Let T be a harmonic current of degree $(1, p)$ on (M, ω) . Let $T = \Omega + \partial S + \bar{\partial}R$ be any decomposition as in (1). Then $\bar{\partial}S$ is uniquely determined by T . If $p = 1$ and T is real, then T is closed if and only if $\bar{\partial}S = 0$.*

Proof: If S', R' also satisfy $T = \Omega + \partial S' + \bar{\partial}R'$ then $S' = S + \omega' + \bar{\partial}u$ for bidegree reasons. Consequently $\bar{\partial}S' = \bar{\partial}S$. Assume that T is a real $(1, 1)$ current. If T is closed, then $T = \Omega + i\partial\bar{\partial}u$, i.e. $\bar{\partial}S = 0$. Conversely, if $\bar{\partial}S = 0$ and $T = \Omega + \partial S + \bar{\partial}S$,

$$\langle T, \bar{\partial}\theta \rangle = - \langle \bar{\partial}T, \theta \rangle = \langle -\bar{\partial}\partial S, \theta \rangle = 0.$$

Hence T is closed.

■

2.2. Energy of harmonic currents. In this paragraph we introduce a notion of energy of harmonic currents of bidegree $(1, 1)$ on \mathbf{P}^k . On \mathbf{P}^k we consider the standard Kähler form ω normalized so that $\int \omega^k = 1$. Recall that $H^{p,q} = 0$ except when $p = q$ in which case $H^{p,p}$ is generated by ω^p .

We showed above that if T is a real harmonic $(1, 1)$ current on \mathbf{P}^k , then it can be represented as

$$T = c\omega + \partial S + \bar{\partial}S$$

with S of bidegree $(0, 1)$. We have $c \in \mathbb{R}$, $c = \int T \wedge \omega^{k-1}$. We define the energy $E(T) = E(T, T)$ of T as

$$E(T, T) = \int \bar{\partial}S \wedge \partial\bar{S} \wedge \omega^{k-2}$$

when $\bar{\partial}S \in L^2$. Observe that $\bar{\partial}S$ is a $(0, 2)$ form. Hence $0 \leq E(T, T) < \infty$. We have seen in Corollary 2.8 that the energy depends on T only.

We define \mathcal{H}_e to be the space of real harmonic $(1, 1)$ currents on \mathbf{P}^k of finite energy. We consider on \mathcal{H}_e the (real) inner product and semi norm

$$\begin{aligned}
\langle T_1, T_2 \rangle_e &= \left[\int T_1 \wedge \omega^{k-1} \right] \left[\int T_2 \wedge \omega^{k-1} \right] + \frac{1}{2} \int \bar{\partial} S_1 \wedge \partial \bar{S}_2 \wedge \omega^{k-2} \\
&+ \frac{1}{2} \int \bar{\partial} S_2 \wedge \partial \bar{S}_1 \wedge \omega^{k-2} \\
\|T\|_e^2 &= \left| \int T \wedge \omega^{k-1} \right|^2 + \int \bar{\partial} S \wedge \partial \bar{S} \wedge \omega^{k-2}
\end{aligned}$$

LEMMA 2.9. *Let $T = c\omega + \partial S + \bar{\partial} \bar{S}, \bar{\partial} S \in L^2$, a $(1, 1)$ real harmonic current of order 0 in \mathbf{P}^k . Then $\|T\|_e = 0$ if and only if $T = i\partial\bar{\partial}u$, for $u \in L^1$, u real.*

Proof: If $\|T\|_e = 0$, then $\bar{\partial} S = 0$ and $c = 0$. Then T is closed, hence T is exact and therefore $T = i\partial\bar{\partial}u$ [De]. Regularity of the Laplace equation shows that $u \in L^1$. Conversely, suppose that $T = i\partial\bar{\partial}u$, $u \in L^1$, u real, so we can set $S = \frac{1}{2}i\bar{\partial}u$. Then $\bar{\partial} S = 0$, hence $\int \bar{\partial} S \wedge \partial \bar{S} = 0$. Clearly also, $\int T \wedge \omega = 0$ so $\|T\|_e = 0$. ■

LEMMA 2.10. *There is a constant $C = C_k$ so that if T is a real harmonic $(1, 1)$ current of order 0 on \mathbf{P}^k , $k \geq 2$, with finite energy, then there is an element \tilde{T} in the equivalence class of T , i.e. $\|T - \tilde{T}\|_e = 0$, which can be written as $\tilde{T} = c\omega + \partial S + \bar{\partial} \bar{S}$ with $S, \partial S, \bar{\partial} S \in L^2$, $\|S\|_{L^2}, \|\partial S\|_{L^2}, \|\bar{\partial} S\|_{L^2} \leq C\|T\|_e$. Hence $T = \tilde{T} + i\partial\bar{\partial}u$ and $i\partial\bar{\partial}u$ is of order 0.*

Proof: We can write $T = c\omega + \partial S_1 + \bar{\partial} \bar{S}_1$ with $\bar{\partial} S_1 \in L^2$. Since $\bar{\partial} S_1$ is in L^2 , we can find an $S \in L^2_{0,1}$ for which $\bar{\partial} S_1 = \bar{\partial} S$. Moreover, $\partial S \in L^2$ as well since, by Hodge theory we gain one derivative by solving the $\bar{\partial}$ problem $\bar{\partial} S = \bar{\partial} S_1$.

Since $H^{(0,1)}(\mathbf{P}^k) = 0$, there is a distribution v for which $\bar{\partial} v = S_1 - S$. Therefore we have the decomposition

$$\begin{aligned}
T &= c\omega + [\partial(S + \bar{\partial}v)] + \overline{[\partial(S + \bar{\partial}v)]} \\
&= c\omega + \partial S + \bar{\partial} \bar{S} + \partial\bar{\partial}v + \bar{\partial}\partial\bar{v} \\
&= \tilde{T} + i\partial\bar{\partial}\left(\frac{v - \bar{v}}{i}\right)
\end{aligned}$$

The distribution $u := \frac{v - \bar{v}}{i}$ is real. Since T, \tilde{T} have order 0, $i\partial\bar{\partial}u$ is also of order 0.

Since $\|\partial\bar{\partial}u\|_e = 0$, it follows that $\tilde{T} := c\omega + \partial S + \bar{\partial} \bar{S}$ is in the equivalence class of T as desired.

The L^2 estimates are classical [FK]. ■

Let H_e denote the quotient space of equivalence classes $[T]$ in \mathcal{H}_e .

PROPOSITION 2.11. *The space H_e is a real Hilbert space. Every element $[T]$ in H_e can be represented as*

$$T = c\omega + \partial S + \overline{\partial S}$$

where S is a $(0,1)$ form in L^2 , with ∂S and $\overline{\partial S}$ in L^2 . Convergence in H_e implies weak convergence of currents: If $[T_n] \rightarrow 0$ in H_e then there are representatives $\tilde{T}_n \in [T_n]$ such that $\tilde{T}_n \rightarrow 0$ in the weak topology of currents. In fact the mass norms $\|\tilde{T}_n\| \rightarrow 0$.

Proof: We show first that H_e is complete. Let $\{[T_n]\}$ be a Cauchy sequence of equivalence classes, $\lim_{n,m \rightarrow \infty} \|T_n - T_m\|_e = 0$. We can suppose $\|T_{n+1} - T_n\|_e < \frac{1}{2^n}$. Inductively, we can (Lemma 2.10) choose representatives \tilde{T}_n so that

$$\tilde{T}_{n+1} = \tilde{T}_n + c_n\omega + \partial S_n + \overline{\partial S_n}, |c_n|, \|\partial S_n\|_{L^2}, \|S_n\|_{L^2}, \|\overline{\partial S_n}\|_{L^2} \leq C \frac{1}{2^n}.$$

Hence $\{[T_n]\}$ converges in H_e . This shows that H_e is complete. The last statement is similar. ■

We will next introduce a notion of wedge product of real harmonic currents. Let T, T' be representatives of equivalence classes as above,

$$T = c\omega + \partial S + \overline{\partial S}, T' = c'\omega + \partial S' + \overline{\partial S'}.$$

Then a formal calculation gives:

$$\begin{aligned} \int T \wedge T' \wedge \omega^{k-2} &= cc' \int \omega^k + \int \partial S \wedge \overline{\partial S'} \wedge \omega^{k-2} + \int \overline{\partial S} \wedge \partial S' \wedge \omega^{k-2} \\ &= \langle T, \omega^{k-1} \rangle \langle T', \omega^{k-1} \rangle - \int \overline{\partial S} \wedge \partial S' \wedge \omega^{k-2} \\ &\quad - \int \partial \overline{S} \wedge \overline{\partial S'} \wedge \omega^{k-2} \end{aligned}$$

Notice that if T, T' have finite energy, the last expression is well-defined. We define in this case the quadratic form $Q(T, T')$ for currents T, T' of finite energy:

$$Q(T, T') = \langle T, \omega^{k-1} \rangle \langle T', \omega^{k-1} \rangle - \int \overline{\partial S} \wedge \partial S' \wedge \omega^{k-2} - \int \partial \overline{S} \wedge \overline{\partial S'} \wedge \omega^{k-2}$$

and motivated by the formal calculation, we define

$$\int T \wedge T' \wedge \omega^{k-2} := Q(T, T')$$

when T, T' are harmonic $(1,1)$ current on \mathbf{P}^k with finite energy. Recalling the definition of energy, we get

$$\int T \wedge T \wedge \omega^{k-2} = Q(T, T) = \langle T, \omega^{k-1} \rangle^2 - 2E(T, T).$$

Note that $Q(T, T')$ is well defined on equivalence classes in H_e .

THEOREM 2.12. *Any positive harmonic current T of bidegree $(1,1)$ on \mathbf{P}^k is of finite energy. Also $\int T \wedge T \wedge \omega^{k-2} \geq 0$ for these currents. The quadratic form $Q(T, T')$ is continuous on H_e .*

Proof: Assume first that T is smooth, $T = c\omega + \partial S + \bar{\partial}\bar{S}$ with $c \geq 0$ and S smooth. We get after integration by parts:

$$\begin{aligned} \int T \wedge T \wedge \omega^{k-2} &= c^2 \int \omega^k + 2 \int \partial S \wedge \bar{\partial}\bar{S} \wedge \omega^{k-2} \\ &= c^2 \int \omega^k - 2 \int \bar{\partial}\bar{S} \wedge \partial S \wedge \omega^{k-2} \\ &\geq 0 \end{aligned}$$

So

$$2 \int \bar{\partial}\bar{S} \wedge \partial S \wedge \omega^{k-2} \leq |\langle T, \omega^{k-1} \rangle|^2.$$

In general, we can still write $T = c\omega + \partial S + \bar{\partial}\bar{S}$ by Proposition 2.6. Let S_ϵ be a regularization of S and define $T_\epsilon = c\omega + \partial S_\epsilon + \bar{\partial}\bar{S}_\epsilon$. Here we use the classical regularization for a current S ,

$$S_\epsilon = \int_{U(k+1)} \rho_\epsilon(g) g_* S d\nu(g),$$

where ν is the Haar measure on $U(k+1)$ acting as automorphisms of \mathbf{P}^k and ρ_ϵ is an approximation of unity in $U(k+1)$.

Then T_ϵ is still positive and harmonic. We get that

$$2 \int \bar{\partial}\bar{S}_\epsilon \wedge \partial S_\epsilon \wedge \omega^{k-2} \leq |\langle T, \omega^{k-1} \rangle|^2 \quad (3)$$

Since $S_\epsilon \rightarrow S$ weakly, $\bar{\partial}\bar{S}_\epsilon \rightarrow \bar{\partial}\bar{S}$ converges weakly in L^2 , because $\{\bar{\partial}\bar{S}_\epsilon\}_\epsilon$ is bounded in L^2 . Hence $\bar{\partial}\bar{S} \in L^2$ and

$$\int \bar{\partial}\bar{S} \wedge \partial S \wedge \omega^{k-2} \leq \liminf \int \bar{\partial}\bar{S}_\epsilon \wedge \partial S_\epsilon \wedge \omega^{k-2} \leq 1/2 |\langle T, \omega^{k-1} \rangle|^2.$$

Hence T has finite energy and $\int T \wedge T \wedge \omega^{k-2} \geq 0$.

If $T = c\omega + \partial S + \bar{\partial}\bar{S}$ and $T' = c'\omega + \partial S' + \bar{\partial}\bar{S}'$ then $Q(T, T') = cc' \int \omega^k - \int \bar{\partial}\bar{S} \wedge \partial\bar{S}' \wedge \omega^{k-2} - \int \bar{\partial}\bar{S}' \wedge \partial S \wedge \omega^{k-2}$. It is clear that $|Q(T, T')| \leq 2\|T\|_e \|T'\|_e$. Hence Q is continuous on H_e . ■

Remark 2.13. *Since our regularization is a convolution, we also have $\bar{\partial}\bar{S}_\epsilon \rightarrow \bar{\partial}\bar{S}$ in L^2 and hence $E(T_\epsilon, T_\epsilon) \rightarrow E(T, T)$.*

COROLLARY 2.14. *On H_e , the quadratic form*

$$Q(T_1, T_2) = \int T_1 \wedge T_2 \wedge \omega^{k-2}$$

is strictly negative definite on the hyperplane $\mathcal{H} = \{T; \int T \wedge \omega^{k-1} = 0\}$. Consequently, if T, T' are positive harmonic currents, non proportional, then $\int T \wedge T' \wedge \omega^{k-2} < 0$.

$\omega^{k-2} > 0$. If $\int T \wedge \omega^{k-1} = 1$ and $T \geq 0$ then T is non closed if and only if $0 \leq \int T \wedge T \wedge \omega^{k-2} < 1$.

Proof: If $\int T \wedge \omega^{k-1} = 0$, then

$$Q(T, T) := -2 \int \bar{\partial}S \wedge \partial\bar{S} \wedge \omega^{k-2} \leq 0,$$

it is zero only if $T = 0$ in H_e . Hence Q is strictly negative definite on \mathcal{H} . Suppose the space generated by T', T is of dimension 2. There is an $a > 0$, $\int (T' - aT) \wedge \omega^{k-1} = 0$. Hence

$$\begin{aligned} 0 &> Q(T' - aT, T' - aT) \\ &= Q(T', T') + a^2 Q(T, T) - 2aQ(T', T) \end{aligned}$$

Since $Q(T', T'), Q(T, T) \geq 0$ by Theorem 2.12, it follows that $Q(T', T) > 0$.

The last part is an immediate consequence of Corollary 2.8. ■

PROPOSITION 2.15. *The function $T \rightarrow Q(T, T)$ is upper semi continuous in the weak topology on positive harmonic currents and is strictly concave on $\{T \wedge \omega = 1\}$.*

Proof: If $T_n \rightarrow T$ weakly, we have seen as above in the proof of Theorem 2.12 that $\int \bar{\partial}S \wedge \partial\bar{S} \leq \underline{\lim} \int \bar{\partial}S_n \wedge \partial\bar{S}_n$, so Q is upper semi continuous. Concavity is clear because if $\int (T - T') \wedge \omega = 0$, $Q(T - T', T - T') < 0$, so $2Q(T, T') > Q(T, T) + Q(T', T')$. Hence $Q(\frac{T+T'}{2}, \frac{T+T'}{2}) > \frac{1}{2}Q(T, T) + \frac{1}{2}Q(T', T')$. ■

PROPOSITION 2.16. *In $\mathbf{P}^k, k \geq 2$, positive harmonic currents T of bidegree $(1, 1)$ satisfying $Q(T, T) = 0$ are extremal on H_e in the cone of positive harmonic currents.*

Proof: Assume that $0 \leq T' \leq cT$ for some $c > 0$.

$$\begin{aligned} \int T' \wedge T \wedge \omega^{k-2} &= \lim \int T'_\epsilon \wedge T \wedge \omega^{k-2} \\ &\leq \lim \int cT_\epsilon \wedge T \wedge \omega^{k-2} \\ &= c \int T \wedge T \wedge \omega^{k-2} \\ &= 0 \end{aligned}$$

Hence $[T']$ is proportional to $[T]$ by Corollary 2.14. ■

PROPOSITION 2.17. *The map $T \rightarrow \bar{\partial}S$ is not continuous for the weak topology on positive harmonic currents T of bidegree $(1,1)$ and L^2 topology on $\bar{\partial}S$.*

Proof: Let $T = \omega + \epsilon(\partial S + \bar{\partial}S)$ for a smooth $(0,1)$ form S supported in the unit bidisc. For $\epsilon > 0$ small enough, T is positive and $\int T \wedge T = \int \omega \wedge \omega - 2\epsilon^2 \int \bar{\partial}S \wedge \partial\bar{S} < 1$. If the map $T \rightarrow \bar{\partial}S$ with weak topology on T and L^2 topology on $\bar{\partial}S$ were continuous then if $T_n \rightarrow T_0$, $\int T_n \wedge T_n \rightarrow \int T_0 \wedge T_0$. Let f be an endomorphism of \mathbf{P}^2 of algebraic degree d . Then $f^* : \mathcal{H}_e \rightarrow \mathcal{H}_e$ is a linear map of norm d if the algebraic degree of f is d . Indeed

$$\begin{aligned} \left| \int f^* T \wedge \omega \right|^2 &= \left| \int T \wedge f_* \omega \right|^2 \\ &= d^2 \left| \int T \wedge \omega \right|^2 \end{aligned}$$

because $f_* \omega \sim d\omega$. We also have

$$\left| \int f^*(\bar{\partial}S \wedge \partial\bar{S}) \right| = d^2 \left| \int \bar{\partial}S \wedge \partial\bar{S} \right|$$

This can be obtained by smoothing.

Therefore $E(f^*T/d) = E(T)$ so $\int f^*T/d \wedge f^*T/d = \int T \wedge T$.

Let $f[z : w : t] = [z^2 : w^2 : t^2]$ and $T = \omega + \epsilon(\partial S + \bar{\partial}S)$ as above. Then

$$\begin{aligned} 1 &> \int T \wedge T \\ &= \int (f^n)^* T / 2^n \wedge (f^n)^* T / 2^n \\ &= \int (f^n)^* \omega / 2^n \wedge (f^n)^* \omega / 2^n - 2 \int \bar{\partial}(f^n)^*(S) / 2^n \wedge \partial(f^n)^*(\bar{S}) / 2^n. \end{aligned}$$

If we choose $S = a(|z|, |w|)d\bar{z}$ it is easy to check that $(f^n)^* S / 2^n \rightarrow 0$ weakly. Hence $(f^n)^* T / 2^n$ converges weakly to a closed current $A = \lim \frac{(f^n)^* \omega}{2^n}$ whose class in $H^{(1,1)}$ is ω . If $T \rightarrow \bar{\partial}S$ were continuous the second integral would converge to zero. Since $\int \omega \wedge \omega = 1$, we get that the map $T \rightarrow \bar{\partial}S$ is not continuous. ■

This is the justification for introducing the norm on finite energy currents, which gives a different topology than weak topology.

L. Garnett has shown in [G] the existence of positive currents T , satisfying $i\partial\bar{\partial}T = 0$ and directed by foliations. In [BS] a version of this result is given allowing leaves to intersect. Here we are interested in constructing laminar currents for a foliation with singularities that are only holomorphic motions in flow boxes, the holomorphic case is treated in [BS].

THEOREM 2.18. *Let (X, \mathcal{L}, E) be a directed set with singularities in \mathbf{P}^2 . Then there is a laminated harmonic positive current T of the form $T = \int_{\alpha} h_{\alpha}[\Delta_{\alpha}]d\mu(\alpha)$ in flow boxes. Here $\mu(\alpha)$ is a measure on transversals, h_{α} are strictly positive harmonic*

functions, uniformly bounded above and below by strictly positive constants, h_α are Borel measurable with respect to α .

Proof: Let γ be a continuous $(0, 1)$ form such that $\gamma \wedge [\Delta] = 0$ for every plaque Δ . In [BS] Theorem 1.4 a current $T \geq 0$ supported on X satisfying $i\partial\bar{\partial}T = 0$ and $T \wedge \gamma = 0$ is constructed. It is shown that the current is laminar in flow boxes when the foliation is holomorphic. We consider here the general case.

The Theorem follows from the next Lemma.

LEMMA 2.19. *A positive harmonic current, directed by a holomorphic motion in a polydisc is a laminar current.*

Proof of the Lemma: Let B be a flow box. In B , $T = \|T\|i\gamma \wedge \bar{\gamma}$ where $\|T\|$ is a positive measure. Assume $z = 0$ is a transversal and π is the projection along leaves in B , on that transversal. Let (ν_w) be the desintegration of $\|T\|$ along π . If ϕ is a test form of bidegree $(1, 1)$ supported on B ,

$$\langle T, \phi \rangle = \int \langle \nu_w i\gamma \wedge \bar{\gamma}, \phi \rangle d\mu(w)$$

with $\mu = \pi_*(\|T\|)$. This applies when $\phi = i\partial\bar{\partial}f$ and when f has support on an arbitrary open set of plaques, we have: $\langle T, i\partial\bar{\partial}f \rangle = 0$. Hence

$$\begin{aligned} 0 &= \int \langle \nu_w i\gamma \wedge \bar{\gamma}, i\partial\bar{\partial}f \rangle d\mu(w) \\ &= \int \langle \nu'(w)[D_w], \Delta_w f \rangle d\mu(w). \end{aligned}$$

where $\Delta_w f$ denotes the Laplacian along the leaf through w . This extends to test functions which are continuous and \mathcal{C}^2 along leaves. Hence $\langle \nu'(w)[D_w], \Delta_w f \rangle = 0$ μ a.e., so $\nu'(w)$ is a harmonic function on the leaf $[D_w]$. ■

Remark 2.20. *It follows from a Theorem by Skoda [Sk] that no positive harmonic $(1, 1)$ current can have mass on a set of 2-dimensional Hausdorff measure $\Lambda_2 = 0$.*

THEOREM 2.21. *Let (X, \mathcal{L}, E) be a laminated set with singularities in \mathbf{P}^2 . There is a unique equivalence class of harmonic currents directed by \mathcal{L} of mass one and minimal energy.*

Proof: Let $C_1 = \{T; T \geq 0, \int T \wedge \omega = 1, T \text{ } \mathcal{L} \text{ - directed, harmonic}\}$. Then C_1 is compact in the weak topology of currents. From Theorem 2.18 we know that C_1 is nonempty.

The energy is a lower semi continuous function on C_1 by Proposition 2.15. Since C_1 is compact it now follows that $E(T, T)$ takes on a minimum value c on C_1 .

If $E(T, T) = c$ and $\int T \wedge \omega = 1$, then $Q(T, T) = 1 - 2c$. If T, T' are two elements of non colinear equivalence classes of currents where the minimum c is reached, then $Q(\frac{T+T'}{2}, \frac{T+T'}{2}) > 1 - 2c$ by strict concavity of Q (Proposition 2.15), a contradiction. So the minimum is unique.

■

We next show that under mild extra hypotheses, minimal equivalence classes contain only one current. Recall that a current on a laminated compact X which in local flow boxes has the form $\int h_\alpha [V_\alpha] d\mu(\alpha)$ for a positive measure $\mu(\alpha)$ on the space of plaques, and $h_\alpha > 0$, harmonic functions on plaques V_α is said to be a laminated positive harmonic current. The current is closed and laminated if the h_α are constant.

THEOREM 2.22. *Let (X, \mathcal{L}, E) be a directed set with singularities. Suppose E is locally complete pluripolar with $\Lambda_2(E) = 0$ in \mathbf{P}^2 . Assume there is no nonzero positive closed laminated current on X . Consider the convex compact set C of laminated positive harmonic $(1,1)$ currents of mass 1. Then there is a unique element T in C minimizing energy. The current T is extremal in C .*

Proof: We know that C is nonempty. Let $T_1, T_2 \in C$ be two minimizing currents. We know by remark 2.20 that T_j has no mass on E . By Theorem 2.21, $[T_1] = [T_2]$ so $T_1 - T_2 = i\partial\bar{\partial}u$, hence $T_1 - T_2$ is closed.

In a flow box we have $T_j = \int h_j^\alpha [V_\alpha] d\mu_j(\alpha)$, $j = 1, 2$. Let $\nu(\alpha) = \mu_1 + \mu_2$, so $\mu_j = r_j(\alpha)\nu$. Then

$$T_1 - T_2 = \int (h_1^\alpha r_1(\alpha) - h_2^\alpha r_2(\alpha)) [V_\alpha] d\nu(\alpha).$$

Since $d(T_1 - T_2) = 0$ it follows that

$$h_1^\alpha r_1(\alpha) - h_2^\alpha r_2(\alpha) \equiv c(\alpha)$$

We decompose the measure $c(\alpha)\nu(\alpha)$ on the space of plaques, $c(\alpha)\nu(\alpha) = \lambda_1 - \lambda_2$ for positive mutually singular measures λ_j . Then

$$T_1 - T_2 = \int [V_\alpha] \lambda_1(\alpha) - \int [V_\alpha] \lambda_2(\alpha) = T^+ - T^-$$

for positive closed currents T^\pm . These locally defined currents fit together to global positive closed currents on $K \setminus E$. Observe that the mass of T^\pm is bounded by the mass of $T_1 + T_2$.

Since E is locally complete pluripolar the trivial extensions of T^\pm are also closed.

Consequently $T^\pm \equiv 0$ and $T_1 = T_2$. The fact that T is extremal follows from the strict concavity of Q .

■

COROLLARY 2.23. *Let (X, \mathcal{L}, E) be a laminated set with singularities in \mathbf{P}^2 with $\Lambda_2(E) = 0$. One of the following statements holds:*

- i) *There is a closed positive laminated current of mass 1 on X .*
- ii) *There is a unique positive laminated harmonic current of mass one on X .*
- iii) *There is a positive laminated harmonic current T of mass one on X such that $\int T \wedge T > 0$. In particular the current T_0 minimizing energy satisfies $\int T_0 \wedge T_0 > 0$.*

Proof: Assume i) and ii) fail. Then there are two positive harmonic non closed currents T_1, T_2 of mass one which are not colinear. We can assume that $\int (T_1 - T_2) \wedge \omega = 0$. By Corollary 2.14, $Q(T_1 - T_2, T_1 - T_2) < 0$. If $Q(T_1, T_1)$ or

$Q(T_2, T_2)$ is strictly positive, we are done since $\int T \wedge T = Q(T, T)$. If not, then $Q(T_1, T_2) = \int T_1 \wedge T_2 > 0$. But then if $T := \frac{T_1 + T_2}{2}$, $Q(T, T) > 0$. ■

3. INTERSECTIONS OF LAMINAR CURRENTS

3.1. C^1 laminations. Here we assume that $X \subset \mathbb{P}^2$ is a laminated compact covered by finitely many flow boxes B_i . We suppose that X locally extends to a C^1 lamination of an open neighborhood. We can assume that $p := [0 : 0 : 1] \in B_1 \subset X$ and that the leaf L through p has the form $w = \mathcal{O}(z^2)$. So we will assume that the lamination of X is of the form $w = w_0 + f_{w_0}(z)$, $f_{w_0}(0) = 0$, where the map $\Psi(z, w_0) = (z, w_0 + f_{w_0}(z))$ is a C^1 diffeomorphism in a neighborhood of p .

Let $\Phi_\epsilon([z : w : t]) = [z, w + \epsilon z : t]$ denote a family of automorphisms of \mathbb{P}^2 . Notice that each of these automorphisms fixes the w axis.

For two graphs L_1, L_2 given by $w = g_1(z), w = g_2(z), z \in S$, we define the vertical distances over S between the two as $d_S^{\max}(L_1, L_2) = \sup_{z \in S} |f_1(z) - f_2(z)|$ and $d_S^{\min}(L_1, L_2) = \inf_{z \in S} |f_1(z) - f_2(z)|$.

THEOREM 3.1. *There exists an integer N independent of ϵ so that in any of the flow boxes B_i local leaves L_1 and $\Phi_\epsilon(L_2)$ can at most intersect in N points, counted with multiplicity. Moreover there exist neighborhoods U_ϵ of Id in $U(3)$ so that the same conclusion holds for $\Psi_1(L_1)$ and $\Psi_2(\Phi_\epsilon(L_2))$, $\Psi_1, \Psi_2 \in U_\epsilon$.*

Proof: Fix a $\delta > 0$. Let L_w denote the leaf through $[0 : w : 1]$ and let L_w^ϵ denote its image under Φ_ϵ . Say L_{w_0} is given by $w = w_0 + f_{w_0}(z)$ and $L_{w_0}^\epsilon$ is given by $w = w_0 + f_{w_0}(z) + \epsilon z$. Note that the vertical distance d_S^{\max} between L_{w_0} and $L_{w_0}^\epsilon$ is $|\epsilon|\delta$ at the boundary S of the disc $|z| \leq \delta$. Because the lamination is of class C^1 , there exists a constant $C > 1$ so that if $(0, w_0), (0, w_1) \in B_1$, then

$$\sup_{|z| \leq \delta} |w_0 + f_{w_0}(z) - w_1 - f_{w_1}(z)| \leq C \inf_{|z| \leq \delta} |w_0 + f_{w_0}(z) - w_1 - f_{w_1}(z)|$$

If L_1 and $\Phi_\epsilon(L_2)$ intersect in a flow box B_i , then L_1 and L_2 must be at most $a|\epsilon|$ apart in B_i for some fixed a . Let $c > 0$ be any small constant. If $L_1, \Phi_\epsilon(L_2)$ intersect in N points in B_i and N is sufficiently large then L_1 and $\Phi_\epsilon(L_2)$ can be at most a distance $c|\epsilon|$ apart from each other in B_i . Note that the same conclusion holds if the number of intersections is counted for $\Psi_1(L_1)$ and $\Psi_2(\Phi_\epsilon(L_2))$ for a small enough U_ϵ . However, there is a path of at most a fixed length along these leaves ending in the flow box containing $[0 : 0 : 1]$. It follows that continuing these leaves to this neighborhood, they will have to stay at most $bc|\epsilon|$ apart, for a fixed constant b . Choosing c small enough, we get $bc < \frac{\delta}{2C}$. Let $L_1 = \{w = w_1 + f_{w_1}(z)\}$, $L_2 = \{w = w_2 + f_{w_2}(z)\}$. Then $|w_2 + f_{w_2}(z) + \epsilon z - w_1 - f_{w_1}(z)| < \frac{|\epsilon|\delta}{2C}$ when $|z| \leq \delta$. Hence

$$\begin{aligned}
d_{|z|\leq\delta}^{\max}(L_1, L_2) &\leq C d_{|z|\leq\delta}^{\min}(L_1, L_2) \\
&\leq C d_{z=0}^{\min}(L_1, L_2) \\
&= C d_{z=0}^{\max}(L_1, \Phi_\epsilon(L_2)) \\
&\leq C d_{|z|\leq\delta}^{\max}(L_1, \Phi_\epsilon(L_2)) \\
&\leq \frac{|\epsilon|\delta}{2}.
\end{aligned}$$

Applying this estimate when $|z| = \delta$, we get

$$\begin{aligned}
\frac{|\epsilon|\delta}{2} &> \frac{|\epsilon|\delta}{2C} \\
&> |w_2 + f_{w_2}(z) + \epsilon z - w_1 - f_{w_1}(z)| \\
&\geq |\epsilon|\delta - |w_2 + f_{w_2}(z) - w_1 - f_{w_1}(z)| \\
&\geq |\epsilon|\delta - \frac{|\epsilon|\delta}{2} \\
&= \frac{|\epsilon|\delta}{2},
\end{aligned}$$

a contradiction. ■

3.2. Laminations by holomorphic motions. Now we consider the case of laminations which are not \mathcal{C}^1 . We recall the following result by Bers-Royden [BR].

PROPOSITION 3.2. *We are given a lamination of a neighborhood of the unit polydisc in \mathbb{C}^2 . Assume that the leaves are of the following form:*

$$L_t, t \in \mathbb{C}, |t| < C, w = F_t(z), F_t(0) = t, F_0(z) \equiv 0.$$

The map $\Phi(z)(t) = F_t(z)$ is a holomorphic motion and we have the estimate:

$$\frac{1}{K}|t-s|^{\frac{1+|z|}{1-|z|}} \leq |F_t(z) - F_s(z)| \leq K|t-s|^{\frac{1-|z|}{1+|z|}}.$$

THEOREM 3.3. *Let $X \subset \mathbb{P}^2$ be a compact subset laminated by Riemann surfaces. Then there exists a holomorphic family $\Phi_\epsilon : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ for $\epsilon \in \mathbb{C}, \Phi_0 \equiv Id$ with the following properties. There are finitely many flow boxes $\{B_i\}_{i=1, \dots, \ell}$ covering X and an $\epsilon_0 > 0$ and a constant A such that if L, L' are any local leaves in any flow box B_i then:*

If $0 < |\epsilon| < \epsilon_0$, the number of intersection points counted with multiplicity of $L, \Phi_\epsilon(L')$ is at most $A \log \frac{1}{|\epsilon|}$. Moreover there exist neighborhoods U_ϵ of Id in $U(3)$ so that the same conclusion holds for $\Psi_1(L_1)$ and $\Psi_2(\Phi_\epsilon(L_2))$, with $\Psi_1, \Psi_2 \in U_\epsilon$.

Proof: We first choose a finite cover by flow boxes, B_i . We can do this so that for each flow box there is a linear change of coordinates in \mathbb{P}^2 so that $[z : w : t] = [0 : 0 : 1] \in B_i \cap K$. Moreover, we can arrange that if L is any local

leaf intersecting $\Delta(0, 2)$ then $L \cap \Delta(0, 2)$ is contained in a local leaf \tilde{L} of the form $\{w = f_\alpha(z), |z| < 3\}$, $(0, \alpha) \in \tilde{L}$, and $\|f_\alpha\|_\infty < 3$. Moreover we can assume that each $\|f'_\alpha\| < .1$ and that $f'_0(0) = 0$. Redefining the flow boxes, we can let B_i denote the union of those graphs over $|z| < 3$ intersecting $\Delta(0, 2)$. We can assume that the smaller flow boxes B'_i consisting of those graphs over $|z| < 1$ for which the graph is in $\Delta^2(0, 1)$ already cover K .

Next we fix the coordinates z, w, t on \mathbb{P}^2 used for the first flow box B_1 . Define the family Φ_ϵ by

$$\Phi_\epsilon[z : w : t] = [z : w + \epsilon z : t].$$

LEMMA 3.4. *There exists a $\delta > 0$ and $C > 0$ so that if $w = f_\alpha(z), w = f_\beta(z)$ are two local leaves in B_1 , then*

$$\frac{|\alpha - \beta|^2}{C} \leq |f_\alpha(z) - f_\beta(z)| \leq C|\alpha - \beta|^{1/2}, \forall z, |z| \leq \delta.$$

Proof: This is a special case of the Bers-Royden result. ■

LEMMA 3.5. *Let $\epsilon_0 > 0$ be small enough. Then if L_1, L_2 are leaves in the first flow box then $d_{\{|z| \leq \delta\}}^{max}(\Phi_\epsilon(L_1), L_2) \geq |\epsilon|^3$ for all $|\epsilon| \leq \epsilon_0$.*

Proof: Let L_i be given by $w = f_i(z), f_i(0) = w_i$. Then $\Phi_\epsilon(L_1)$ is the graph $w = f_1(z) + \epsilon z$. Suppose that $|f_1(z) + \epsilon z - f_2(z)| \leq |\epsilon|^3$ for all $|z| \leq \delta$. Then, we get that $|w_1 - w_2| \leq |\epsilon|^3$. Hence, by the previous Lemma, we have that $|f_1(z) - f_2(z)| \leq C|\epsilon|^{3/2}$ for all $|z| \leq \delta$. Hence if $|z| = \delta$,

$$\begin{aligned} |\epsilon|^3 &\geq |f_2(z) + \epsilon z - f_1(z)| \\ &\geq |\epsilon|\delta - |f_2(z) - f_1(z)| \\ &\geq |\epsilon|\delta - C|\epsilon|^{3/2} \\ &\geq |\epsilon|(\delta - C\sqrt{|\epsilon|}), \\ &\Rightarrow \\ \epsilon_0^2 &\geq |\epsilon|^2 \geq \delta - C\sqrt{|\epsilon|} \geq \delta - C\sqrt{\epsilon_0} \\ &\Rightarrow \\ \epsilon_0^2 + C\sqrt{\epsilon_0} &\geq \delta, \end{aligned}$$

a contradiction if ϵ_0 is small enough. ■

The following lemma is well known.

LEMMA 3.6. *a) There is a constant $0 < c < 1$ so that the following holds: Let g be a holomorphic function on the unit disc with $|g| < 1$ and suppose that g has N zeroes in $\Delta(0, 1/2)$. Then $|g| \leq c^N$ on $\Delta(0, 1/2)$.*

b) Let g denote a holomorphic function on the unit disc and suppose that $|g| < 1$ and that $|g| < \eta < 1$ on $\Delta(0, 1/4)$. Then $|g| < \sqrt{\eta}$ on $\Delta(0, 1/2)$.

Proof: To prove a) set

$$c = \sup_{|\alpha| \leq 1/2, |z| \leq 1/2} \frac{|z - \alpha|}{|1 - z\bar{\alpha}|} < 1.$$

To prove b) observe that $\log |g| < \log \eta$ when $|z| < \frac{1}{4}$. Hence by subharmonicity

$$\log |g| \leq \max\{\log \eta \frac{\log |z|}{\log 1/4}, \log \eta\}.$$

This implies that if $|z| = 1/2$, then $\log |g| \leq \frac{\log \eta}{2}$. ■

Continuation of the Proof of Theorem 3.4: Pick $\rho > 0$. Let $p \in X$. Since every leaf is dense, there is a (nonunique) continuous curve $\gamma_p(t), 0 \leq t \leq 1$ from $\gamma(0) = p$ to a point $\gamma_p(1) = (0, w_p) \in B'_1$ which is contained in the leaf through p . By continuity, for every $q \in K$ close enough to p , the curve γ_q can be chosen so that $\text{dist}(\gamma_q(t), \gamma_p(t)) \leq \rho$ for all $0 \leq t \leq 1$.

A chain of flow boxes is a finite collection $C = \{B_{i(j)}\}_{j=1}^k$. Let $p \in X$. We say that the leaf through p follows the chain $\{B_{i(j)}\}_{j=1}^k$ if there are local leaves $L_j \subset B_{i(j)}, \hat{L}_j := L_j \cap B'_{i(j)}, p \in \hat{L}_1, \hat{L}_j \cap \hat{L}_{j+1} \neq \emptyset \forall j < k, i(k) = 1$.

By compactness there are finitely many chains of flow boxes C_1, \dots, C_ℓ such that for each $p \in X$, there is an open neighborhood $U(p)$ and a chain C_r so that the leaf through q follows C_r for any $q \in U(p) \cap X$.

We will apply Lemma 3.6 repeatedly along a chain. We need to apply Lemma 3.6 at most a fixed number of times m depending on the length of each chain. Note that every time we switch flow box there is a change of coordinates which distorts distances by at most a factor $C > 1$.

LEMMA 3.7. *Let ϵ be sufficiently small and suppose that $N = N(\epsilon)$ is an integer such that $C^2 c^{\frac{N}{2^k}} \leq |\epsilon|^3$. Then no local leaves of the laminations $L_1, \Phi_\epsilon(L_0)$ can intersect more than N times in any flow box.*

Proof: Suppose that local components of L_1 and $\Phi_\epsilon(L_0)$ intersect in more than N points in some local flow box B'_s . Then these local graphs differ by at most c^N . Using Lemma 3.6 they differ by at most $c^{\frac{N}{2}}$ in a suitable larger flow box. Changing to the coordinates of another flow box might increase the difference to $Cc^{\frac{N}{2}}$. Applying Lemma 3.6 the difference increases to at most $C^{\frac{1}{2}}c^{\frac{N}{4}}$ and after another change of flow box to $C^{\frac{3}{2}}c^{\frac{N}{4}}$. Following the leaves along a chain of flow boxes we see inductively that the distance between continuations of the leaves grows at most like $C^2 c^{\frac{N}{2^k}}$ after k steps. Hence once we are in the first flow box, the leaves differ by at most $|\epsilon|^3$. By the above lemmas, this is impossible for any pair of leaves. ■

There is a constant A so that for all small enough ϵ local leaves $L_1, \Phi_\epsilon(L_0)$ have at most $N_\epsilon := A \log \frac{1}{|\epsilon|}$ intersection points. The construction is stable under small perturbations by Ψ_1, Ψ_2 close to the identity.

4. CONSTRUCTION FROM DISCS. AHLFORS TYPE CONSTRUCTION.

In this paragraph we consider a laminated set (X, \mathcal{L}, E) in a compact complex manifold M .

We want to construct harmonic currents using the Ahlfors exhaustion technique.

4.1. When leaves are not uniformly Kobayashi hyperbolic. We consider only the case when X is not a compact Riemann surface, possibly singular. Consider the universal covering for each leaf. We can assume that the covering is \mathbb{C} or the unit disc Δ . Let $\phi : \Delta \rightarrow L$ be a covering map. If $|\phi'(0)|$ is not uniformly bounded, then using the Brody technique, one can construct an image of \mathbb{C} . The part of the image not in E is locally contained in a leaf of X .

The Ahlfors exhaustion technique furnishes a positive closed current of mass 1 directed by the lamination. So we get the following proposition:

PROPOSITION 4.1. *Let (X, \mathcal{L}, E) be a laminated set with singularities. If there is no positive closed current on X , laminated on $X \setminus E$, then there is a constant C such that $|\phi'(0)| \leq C$.*

When the $|\phi'(0)|$ are uniformly bounded, we say that the leaves are uniformly hyperbolic.

4.2. The case with no positive closed current directed by (X, \mathcal{L}, E) . Let X be a minimal laminated compact set in \mathbf{P}^2 . Suppose that X does not contain any non constant holomorphic image of \mathbb{C} . Let B_i be a covering of $X \setminus E$ by flow boxes which in local coordinates are of the form $w_i = f_\alpha(z_i), |z_i| < 1$ [but the graphs extend uniformly to $|z_i| < 2$]. Let $\Phi : \Delta \rightarrow L$ denote the universal covering of an arbitrary leaf. We say that $x \in \Delta$ is a center point if $\Phi(x) = (0, w_i)$ in some B_i . We can normalize Φ for any center point, say $\Phi_x : \Delta \rightarrow L, \Phi_x(0) = \Phi(x)$. [i.e. we move x to 0 with an automorphism of the unit disc.] Let $w_i = f_x(z_i)$ denote the associated graph in the flow box. Denote by $U_x := \Phi_x^{-1}(\{(z_i, f_x(z_i)); |z_i| < 1\})$. If Φ is a multisheeted covering, we let U_x denote the connected component containing 0. Then $U_x \subset \Delta$ is a relatively compact open subset of Δ containing 0. Let $0 < r_x \leq R_x < 1$ denote the largest, respectively smallest radii such that $\Delta(0, r_x) \subset U_x \subset \Delta(0, R_x)$.

LEMMA 4.2. $R_x \leq 1/2$.

Proof: Since Φ_x^{-1} maps $\Delta(0, 2)$ into $\Delta(0, 1)$ and sends 0 to 0 this follows from the Schwarz' Lemma.

LEMMA 4.3. *Fix a finite number of flow boxes $\overline{B}_i \cap E = \emptyset, i = 1, \dots, \ell$. Then $\inf\{r_x; x \text{ is a center point, } \phi(x) \in \cup_{i=1}^{\ell} B_i\} > 0$. In fact the same holds if we inf over all leaves and all covers of the leaves by discs.*

Proof: Fix an x and a covering $\Phi : \Delta \rightarrow L$ of the leaf through x . Note that $\Phi_x^{-1}(\Delta(0, 1)) \supset \Delta(0, r_x)$ hence by the Koebe 1/4 Theorem, $[\Phi_x^{-1}]'(0) = \alpha, |\alpha| \leq 4r_x$.

Hence $\Phi'_x(0) = \frac{1}{\alpha}, \left| \frac{1}{\alpha} \right| \geq \frac{1}{4r_x}$. If $r_x \rightarrow 0$ then using the Brody technique we construct an image of \mathbb{C} contained in X . ■

Next we prove a density Theorem for the above minimal laminations with only Kobayashi hyperbolic leaves:

THEOREM 4.4. *Assume $E = \emptyset$. Fix a finite cover of X by flow boxes B_i . There are constants R, N so that if $\Phi : \Delta \rightarrow X$ is a covering of any leaf, then $\Phi(\Delta_{kob}(x, R) \cap B_i)$ intersects at most N of the graphs in $\{|z_i| < \frac{3}{2}\}$ and contains at least one complete graph over $|z_i| < 1$.*

Proof: Let $\eta \in \cup B_j$ and L_η the leaf through η . There exist finitely many curves $\gamma_i^\eta \subset L_\eta, \gamma_i^\eta(0) = \eta, \gamma_i^\eta(1) \in B_i$. In fact, for every $\zeta \in U(\eta) \cap X$ there are curves $\gamma_i^\zeta \subset L_\zeta$ depending continuously on ζ and landing in B_i for a small enough neighborhood $U(\eta)$. There exists a finite number M_η so that if $\zeta \in U(\eta) \cap X, t \in \left[\frac{j-1}{M_\eta}, \frac{j}{M_\eta} \right]$, then $\gamma_i^\zeta(t) \in \Delta_{i,\zeta,j}$ where $\Delta_{i,\zeta,j}$ is one of the unit discs in one of our chosen local flow boxes. It follows that if $\Phi : \Delta \rightarrow L$ is any covering of any leaf and $\Phi(x_0) \in U(\eta)$, then $\Phi(\Delta_{kob}(x_0, R_\eta))$ contains a disc in each flow box if R_η is large enough. By compactness of X , there is an $R = \max R_\eta > 0$ so that if $\Phi : \Delta \rightarrow X$ is a covering of any leaf and $x \in \Delta$, then $\Phi(\Delta_{kob}(x, R))$ contains at least one complete graph in each flow box. There is by Lemma 4.2 an $R' > R$ so that if $\zeta \in \Phi(\Delta_{kob}(x, R)) \cap B_i$ (or a slight extension), then the whole graph in B_i is contained in $\Phi(\Delta_{kob}(x, R'))$. It follows from Lemma 4.3 that there is an N so that no $\Phi(\Delta_{kob}(x, R))$ can intersect any flow box in more than N graphs. ■

COROLLARY 4.5. *Let R be sufficiently large. Then if D is any maximal disc of center p contained in an annulus $1 - 1/R^k < |x| < 1 - 1/R^{k+1}$, and assume $\Phi(p) \in \cup B_j$, then the image $\Phi(D)$ always will contain a full disc in each B_i and will intersect at most N graphs in any B_i .*

Proof: This follows since the Poincaré radius of these discs increase to infinity independently of k as $R \rightarrow \infty$. ■

COROLLARY 4.6. *For large R the area of $\Phi(\{1 - 1/R^k < |x| < 1 - 1/R^{k+1}\})$ grows like R^k as $k \rightarrow \infty$. Moreover the length of $\Phi(\{|x| = 1 - 1/R^{k+1}\})$ is on the order of R^k , and $|\Phi'(x)| \sim \frac{1}{1-|x|}$ near $\partial\Delta$, so*

$$\int_{\Delta} (1 - |x|) |\Phi'(x)|^2 d\lambda(x) = \infty$$

Proof: The map Φ is an isometry for the Kobayashi distance. The Kobayashi distance on leaves is comparable to any given Hermitian metric. The map Φ expands a disc of radius ϵ centered at $1 - 2\epsilon$ to a disc of radius about 1 in the Kobayashi

metric. Hence $|\Phi'(x)| \sim \frac{1}{1-|x|}$.

■

THEOREM 4.7. *Let $\phi : \Delta \rightarrow M$ be a holomorphic map where M is a compact complex Hermitian manifold. If*

$$\int_{\Delta} (1 - |x|)|\phi'(x)|^2 d\lambda(x) = \infty,$$

then there is a positive harmonic current T , supported on $\overline{\phi(\Delta)}$. If $\phi(\Delta)$ is contained in a leaf of a lamination \mathcal{L} , then the current T is directed by the lamination.

Proof: Assume that $\phi(0) = p$. Define

$$G_r(x) := \frac{1}{2\pi} \log^+ \frac{r}{|x|}, T_r := (\phi)_*(G_r[\Delta]), r < 1.$$

If θ is a $(1, 1)$ test form on M

$$\langle T_r, \theta \rangle = \frac{1}{2\pi} \int_{\Delta} \log^+ \frac{r}{|x|} \phi^*(\theta).$$

So T_r is positive of bidimension $(1, 1)$. The mass of T_r is comparable to $\int_{\Delta} \log \frac{r}{|x|} |\phi'(x)|^2 \sim \int_{\{|x| < r\}} (r - |x|) |\phi'(x)|^2 =: A(r)$. A direct computation gives

$$i\partial\bar{\partial}T_r = \phi_*(\nu_r) - \delta_p,$$

where δ_p is the Dirac mass at p and ν_r is the Lebesgue measure on the circle of radius r . Let T be a cluster point of $T_r := \frac{T_r}{A(r)}$. Since $A(r) \rightarrow \infty$ we have $i\partial\bar{\partial}T = 0$.

If $\phi(\Delta)$ is a leaf of a lamination \mathcal{L} , there is a $(1, 0)$ form γ such that $[\phi(\Delta)] \wedge \gamma = 0$. Hence $T_r \wedge \gamma = 0$. Hence $\langle T, i\partial_b \bar{\partial}_b f \rangle = 0$ and we have a decomposition of T as in Theorem 2.18.

■

Remark 4.8. *If we assume that $\lim_{r \rightarrow 1} (1 - r) \int_{D_r} |\phi'(z)|^2 = \infty$, it follows from a result of Ahlfors that $\lim_{r \rightarrow 1} \frac{\ell(\phi(\partial\Delta_r))}{\text{Area}(\phi(\Delta_r))} \rightarrow 0$, ℓ represents length. Hence one can choose the current T to be closed.*

In the usual Ahlfors procedure to construct a closed current starting from an image of \mathbb{C} one has to first extract good subsequences from $\frac{\Phi_[\Delta_R]}{\text{Area } \Phi_*[\Delta_R]}$ when $R \rightarrow \infty$. Then cluster points of these give closed currents. In our case, there is no need to first take a subsequence.*

THEOREM 4.9. *There is no nonvanishing holomorphic vector field along leaves on a laminated compact with only hyperbolic leaves.*

Proof: If we pull back the vector field to a disc covering a leaf, we get a holomorphic function on the unit disc going uniformly to infinity at the boundary, since as we have seen $|\phi'(x)| \sim \frac{1}{1-|x|}$ and the vector field is bounded in flow boxes.

■

PROPOSITION 4.10. *If a positive harmonic current gives mass to a leaf, then this leaf is a compact Riemann surface.*

LEMMA 4.11. *Let T be a laminated harmonic current. Let ϕ denote the covering map $\Delta \rightarrow L$. If H denotes the analytic continuation of $h \circ \phi$ in a flow box, then we have the estimate:*

$$c(1 - |x|) \leq H \leq C \frac{1}{(1 - |x|)}.$$

Proof: The estimate follows from Harnack's inequality and the Hopf Lemma. ■

Proof of the Proposition: We assume first that the leaf is hyperbolic. Let ϕ, H be as in the Lemma. Suppose at first that H is unbounded. Then we can choose a sequence $p_n \rightarrow \partial\Delta$ and $H(p_n) \rightarrow \infty$ such that H is uniformly large on $\Delta(p_n, R)$ by Harnack, where R is as in Theorem 4.4. Hence T will have infinite mass on a flow box. If H is bounded and nonconstant, we can choose θ_n so that $\lim_{r \rightarrow 1} H(re^{i\theta_n})$ are different. We can again choose $p_n \rightarrow \partial\Delta$ so that $\phi(\Delta(p_n, R))$ are disjoint and again T will have infinite mass on a flow box. If H is constant, we get a positive closed current and the leaf has an analytic closure. The same argument applies to the case when the leaf is not hyperbolic. ■

5. VANISHING OF $\int T \wedge T$.

DEFINITION 5.1. *The harmonic laminated current has finite transverse energy if in some local flow box $\int \log |\alpha - \beta| d\mu(\alpha) d\mu(\beta) < \infty$.*

Having finite transverse energy is well defined and independent of the choice of flow box.

Recall that $\Phi_\epsilon([z : w : t]) = [z : w + \epsilon z : t]$. If T is a current, let $T_\epsilon := (\Phi_\epsilon)_*(T)$.

THEOREM 5.2. *If a harmonic current for a laminated compact in \mathbf{P}^2 has finite transverse energy, then the geometric intersection $T \wedge T_\epsilon \rightarrow 0$. The same conclusion holds for \mathcal{C}^1 -laminations without the hypothesis of finite transverse energy. In both cases we have $\int T \wedge T = 0$.*

Proof: We calculate the geometric wedge product $T \wedge T_\epsilon$ in a flow box. Set $T = \int h_\alpha[\Delta_\alpha] d\mu(\alpha)$, $T_\epsilon = \int h_\beta^\epsilon[\Delta_\beta^\epsilon] d\mu(\beta)$. Let ϕ be a test function supported in a flow box. To avoid confusion, we index with g when wedge products are geometric during the proof. We have

$$\langle T \wedge T_\epsilon, \phi \rangle_g = \int \sum_{p \in J_{\alpha, \beta}} \phi h_\alpha(p) h_\beta^\epsilon(p) d\mu(\alpha) d\mu(\beta)$$

where $J_{\alpha, \beta}$ consists of the intersection points of Δ_α and Δ_β^ϵ . Assume at first that μ has finite transverse energy. Using the estimate on the size of $J_{\alpha, \beta}$ in Theorem 3.3, we get:

$$\begin{aligned}
|(T \wedge T^\epsilon)_g(\phi)| &\leq C_1 \|\phi\|_\infty \int_{\text{dist}(\Delta_\alpha, \Delta_\beta) \leq C\epsilon} A \log \frac{1}{|\epsilon|} d\mu(\alpha) d\mu(\beta) \\
&\leq C_2 \|\phi\|_\infty \int_{\text{dist}(\Delta_\alpha, \Delta_\beta) \leq C\epsilon} \log \frac{1}{\text{dist}(\Delta_\alpha, \Delta_\beta)} d\mu(\alpha) d\mu(\beta) \\
&\rightarrow 0 \text{ as } \epsilon \rightarrow 0
\end{aligned}$$

In the \mathcal{C}^1 case the number of intersection points is bounded by N independent of ϵ (Theorem 3.1). Hence

$$\begin{aligned}
|(T \wedge T^\epsilon)_g(\phi)| &\leq C \|\phi\|_\infty \int_{\text{dist}(\Delta_\alpha, \Delta_\beta) \leq C\epsilon} N d\mu(\alpha) d\mu(\beta) \\
&\rightarrow 0
\end{aligned}$$

since μ has no pointmasses by Proposition 4.10. Next we show that $Q(T, T) = \int T \wedge T = 0$. It suffices to show by Theorem 2.12 that $Q(T, T_\epsilon) \rightarrow 0$ or even that for smoothings $T^\delta, T_\epsilon^{\delta'}$ that $Q(T^\delta, T_\epsilon^{\delta'}) \rightarrow 0$ when δ, δ' are sufficiently small compared to ϵ and $\delta, \delta', \epsilon \rightarrow 0$.

Note that the estimate on the geometric wedge product is stable under small translations of T, T_ϵ . This is what allows us to smooth.

Let ϕ be a test function supported in some local flow box. As above, the value of the geometric wedge product on ϕ is:

$$\langle T \wedge T^\epsilon, \phi \rangle_g = \int \sum_{p \in J_{\alpha, \beta}} \phi h_\alpha(p) h_\beta^\epsilon(p) d\mu(\alpha) d\mu(\beta)$$

We can write this as

$$\langle T \wedge T^\epsilon, \phi \rangle_g = \int \left(\int_{\Delta_\beta^\epsilon} [\phi h_\alpha h_\beta^\epsilon](p) i\partial\bar{\partial} \log |w - f_\alpha(z)| \right) d\mu(\alpha) d\mu(\beta)$$

The same applies when we do this for slight translations within small neighborhoods $U(\epsilon)$ of the identity in $U(3)$ and their smooth averages T^δ ,

$$\langle T^\delta \wedge T_\epsilon, \phi \rangle_g = \int \left(\int_{\Delta_\beta^\epsilon} [\phi h_\beta^\epsilon](p) T^\delta \right) d\mu(\beta) = \langle T_\epsilon, \phi T^\delta \rangle.$$

Averaging also over small translations of T^ϵ we get

$$\langle T^\delta \wedge T_\epsilon^{\delta'}, \phi \rangle_g = \langle T_\epsilon^{\delta'}, \phi T^\delta \rangle.$$

We still have that $\langle T_\epsilon^{\delta'}, \phi T^\delta \rangle \rightarrow 0$ when $\delta, \delta' \ll \epsilon, \epsilon \rightarrow 0$. If we apply this to $\phi = 1$, we get $\langle T_\epsilon^{\delta'}, T^\delta \rangle = Q(T_\epsilon^{\delta'}, T^\delta) \rightarrow 0$. Hence $Q(T, T) = 0$.

■

COROLLARY 5.3. *If a laminated compact set in \mathbf{P}^2 carries a positive closed laminar current T , then T has infinite transverse energy.*

Proof: If $T \neq 0$ has finite energy, then $0 = \int T \wedge T = |\int T \wedge \omega|^2 - E(T, T)$ but $E(T, T) = 0$ since T is closed. Hence $T = 0$, a contradiction. ■

J. Duval has independently obtained this Corollary. Hurder and Mitsumatsu proved that there is no \mathcal{C}^1 lamination in \mathbf{P}^2 which carries a positive closed current [HM].

COROLLARY 5.4. *If (X, \mathcal{L}) is a \mathcal{C}^1 lamination on \mathbf{P}^2 with only hyperbolic leaves, then*

$$T = \lim_{r \nearrow 1} \phi_* \frac{\left(\log^+ \left(\frac{r}{|z|} \right) [D_r] \right)}{A_r}$$

uniformly with respect to ϕ .

Proof: We know from [HM] that there is no positive closed current directed by \mathcal{L} . It follows from Corollary 2.23 and Theorem 5.2 that there is a unique harmonic current of mass 1 on (X, \mathcal{L}) . Hence the result follows. ■

6. EXAMPLES OF HARMONIC CURRENT

In this section we investigate harmonic currents on \mathbf{P}^2 of the form $T = i\partial u \wedge \overline{\partial u}$. Our main result is that if $u \in \mathcal{C}^2_{\mathbb{R}}(\mathbf{P}^2)$ and $i\partial\overline{\partial}T = 0$, then u is constant, hence $T \equiv 0$. We also compute the energy of some positive harmonic currents.

Let M be a complex manifold of dimension m . For $1 \leq k \leq m$, we define $\mathcal{P}_-^{(k)}(M)$ as the cone of upper semicontinuous real functions v on M such that for every $p \in M$, there is an open neighborhood U of p and $\{v_n\} \subset \mathcal{C}^2(U)$ such that $v_n \searrow v$ in U and $(-1)^k (i\partial\overline{\partial}v_n)^k \leq \epsilon_n \omega^k$, $\epsilon_n \searrow 0$. We say that U is associated to v . Here ω denotes a strictly positive hermitian form. Notice that this condition implies that when $\epsilon_n = 0$, not all eigenvalues of $i\partial\overline{\partial}v_n$ can be strictly negative. We define $\mathcal{P}_+^{(k)} := -\mathcal{P}_-^{(k)}(M)$ and $\mathcal{P}^{(k)} = \mathcal{P}_+^{(k)} \cap \mathcal{P}_-^{(k)}$. In particular, $\mathcal{P}_-^{(1)}$ consists of the plurisubharmonic functions and $\mathcal{P}^{(1)}$ are the pluriharmonic functions. In dimension 2, a smooth function v belongs to $\mathcal{P}_-^{(2)}$ if and only if its Levi form has at most one eigenvalue of each sign. These functions then, also belong to $\mathcal{P}_+^{(2)}$ and hence $\mathcal{P}^{(2)}$. Pseudoconvex domains are usually characterized by plurisubharmonic functions, i.e. $\mathcal{P}_-^{(1)}$. We show here that $\mathcal{P}_-^{(2)}$ works as well, and that there are similar results for $\mathcal{P}_-^{(k)}$, $k > 2$.

Let $\Phi : M \rightarrow N$ be a holomorphic map between complex manifolds. If $v \in \mathcal{P}_-^{(k)}(N)$, then $v \circ \Phi \in \mathcal{P}_-^{(k)}(M)$. In particular, if $k > \dim N$, any upper semicontinuous v is in $\mathcal{P}_-^{(k)}(N)$, hence $v \circ \Phi \in \mathcal{P}_-^{(k)}(M)$.

We give some examples of compact complex manifolds for which $\mathcal{P}^{(2)}(M) \neq \mathbb{R}$:

1. Tori: Let T be a torus. Then $T = \mathbf{C}^k$ mod a lattice generated by $\{v_i\}_{i=1}^{2k}$. Let $\pi : \mathbf{C}^k \rightarrow \mathbb{R}$, $\pi(\sum x_i v_i) = x_i$. Let $v = \phi(x_1)$ where ϕ is a smooth function supported in $]0, 1[$. Then $(i\partial\bar{\partial}v)^2 = 0$.

2. Hopf manifolds. Let $M = \mathbf{C}^2 / \langle \phi \rangle$ where $\langle \phi \rangle$ denotes the group generated by $\phi(z_1, z_2) = (\alpha_1 z_1, \alpha_2 z_2)$ with α_1, α_2 fixed, $0 < |\alpha_1| \leq |\alpha_2| < 1$. Fix r such that $|\alpha_1| = |\alpha_2|^r$. Define

$$v(z_1, z_2) = \frac{|z_1|^2}{|z_1|^2 + |z_2|^r}$$

The function v is well defined on M and $(i\partial\bar{\partial}v)^2 = 0$.

3. For any surface admitting a projection on a Riemann surface $P^{(2)}(M) \neq \mathbb{R}$, for example ruled surfaces. Actually the Hopf surfaces above admit such a projection $\mathbf{C}^2 / \langle \phi \rangle \rightarrow \mathbf{P}^1$, $(z_1, z_2) \rightarrow [z_1^q : z_2^p]$ if $\alpha_1^q = \alpha_2^p$.

For $k > 1$, set $z = (z_1, \dots, z_{k-1})$, $|z| = \max\{|z_1|, \dots, |z_{k-1}|\}$ and let H_{k-1}^r denote the Hartogs figure:

$$H_{k-1}^r := \{(z, w) \in \mathbf{C}^{k-1} \times \mathbf{C} = \mathbf{C}^k; |z| \leq 1 + r, |w| \leq 1\} \setminus \{r < |w| \leq 1, |z| < 1\}.$$

$$\text{Let } \hat{H}_{k-1}^r := \{(z, w) \in \mathbf{C}^k; |z| \leq 1 + r, |w| \leq 1\}.$$

DEFINITION 6.1. Let $2 \leq k \leq m$. We say that an open set $N \subset M$ is $(k-1)$ -pseudoconvex if whenever $\Phi : U \rightarrow M$ is a biholomorphic map of a neighborhood $U \supset \hat{H}_{k-1}^r$ onto its image and $\Phi(H_{k-1}^r) \subset N$, then $\Phi(\hat{H}_{k-1}^r) \subset N$.

Remark 6.2. In the case $k = 2$, the definition is equivalent to N being pseudoconvex.

PROPOSITION 6.3. Let M be a complex manifold of dimension $m \geq 2$. Let N be a connected open set in M . Assume $v \in P^k(M)$, $2 \leq k \leq m$ and $v < 0$ on N , $v|_{\partial N} \equiv 0$, $v \leq 0$ on M . Let $\mathcal{U} = \{U_\alpha\}$ be a cover of M associated to v . If $\Phi(\hat{H}_{k-1}^r) \subset U_\alpha$ and $\Phi(H_{k-1}^r) \subset N$, then $\Phi(\hat{H}_{k-1}^r) \subset N$. In particular if $k = 2$ then N is pseudoconvex in M .

Proof: Assume that $\Phi(\hat{H}_{k-1}^r) \setminus N \neq \emptyset$. Then also, $\Phi(\hat{H}_{k-1}^r) \cap \partial N \neq \emptyset$. There is a biholomorphic map $\Phi : U \rightarrow M$ onto its image, $\hat{H}_{k-1}^r \subset U$, $\Phi(H_{k-1}^r) \subset N$. We can then assume that $M = U$, $\Phi = \text{Id}$, $H_{k-1}^r \subset N$ and that there is an interior point of \hat{H}_{k-1}^r in ∂N . Assume that $v_n \searrow v$, $v_n \in \mathcal{C}^2(U)$, $(-1)^k (i\partial\bar{\partial}v_n)^k \leq \epsilon_n \omega^k$. We also assume that ∂N contains a point (z_0, w_0) , $|z_0|, |w_0| < 1$. Then $v < 0$ on H_{k-1}^r and $v(z_0, w_0) = 0$. This is where we use that $v|_{\partial N} \equiv 0$. We would like to get a contradiction.

Let $X := \{|z| \leq 1, r < |w| \leq 1\}$. Define the function u by

$$u(z, w) := \eta \left(1 - \sum_{j=1}^{k-1} |z_j|^2 \right) - \frac{\epsilon}{|w|^2} + \delta$$

where $\eta \ll \epsilon < \delta \ll 1$. We will choose the constants so that $v < u$ on ∂X and $u(z_0, w_0) < 0$. First observe that if δ is small enough, then automatically $v < u$ on

all of ∂X except possibly where $|w| = 1$ and $|z| < 1$. Fix any such δ . Let $\epsilon < \delta$ be chosen big enough that $-\frac{\epsilon}{|w_0|^2} + \delta < 0$. Since $\epsilon < \delta$ and $v \leq 0$ on $|w| = 1$ and $|z| < 1$, $v < u$ on all of ∂X if we choose $\eta = 0$. To finish the choice of constants, η, ϵ, δ choose $\eta > 0$ small enough that $v < u$ still on ∂X and in, addition $u(z_0, w_0) < 0$. Then $i\partial\bar{\partial}(-u) \geq a\omega$ on X for some constant $a > 0$.

Next choose n large enough so that $v_n < u$ on ∂X . We have $v_n(z_0, w_0) \geq v(z_0, w_0) = 0 > u(z_0, w_0)$. Then if we add a strictly positive constant c_n to u we can assume that

$$\begin{aligned} v_n &< u + c_n \text{ on } \partial X \\ v_n &\leq u + c_n \text{ on } X \\ v_n(z_1^n, w_1^n) &= u(z_1^n, w_1^n) + c_n, (z_1^n, w_1^n) \in X \end{aligned}$$

This implies that $i\partial\bar{\partial}v_n(z_1^n, w_1^n) \leq i\partial\bar{\partial}u(z_1^n, w_1^n)$. Hence $i\partial\bar{\partial}(-v_n)(z_1^n, w_1^n) \geq a\omega$. This implies that

$$\begin{aligned} \epsilon_n \omega^k &\geq (-1)^k (i\partial\bar{\partial}v_n)^k(z_1^n, w_1^n) \\ &= (i\partial\bar{\partial}(-v_n))^k(z_1^n, w_1^n) \\ &\geq a^k \omega^k, \end{aligned}$$

a contradiction. ■

COROLLARY 6.4. *If M is a compact manifold and $v \in P^{(2)}(M)$, then $K(v) := \{p; v(p) = \max_M v\}$ is pseudoconcave, i.e. $M \setminus K(v)$ is pseudoconvex. If $v \in P^{(k)}(M) \cap C^2$, then $M \setminus K(v)$ is $k - 1$ pseudoconvex.*

Remark 6.5. *Let v be a C^2 function on a compact complex manifold of dimension m . Stokes' Theorem implies that if $(-1)^k (i\partial\bar{\partial}v)^k \leq 0$, then $(dd^c v)^k = 0$. In the case of compact Kähler manifolds, Stokes' Theorem applied to $(-1)^k (i\partial\bar{\partial}v)^k \wedge \omega^{m-k}$ shows that the same conclusion holds.*

Remark 6.6. *The proof above shows that if an upper semicontinuous function v is locally a decreasing limit of C^2 functions v_n such that at each point $i\partial\bar{\partial}v_n$ has $m - 1$ nonnegative eigenvalues, then $K(v)$ is pseudoconcave. Namely, we get by the above construction with a Hartogs figure of dimension two:*

$$\begin{aligned} v_n &< u + c \text{ on } \partial X, \\ v_n &\leq u + c \text{ on } X \\ v_n(z_1, w_1) &= u(z_1, w_1) + c \text{ at some point of } X. \text{ Hence} \\ i\partial\bar{\partial}(v_n - u)(z_1, w_1) &\leq 0 \end{aligned}$$

which contradicts that one eigenvalue is nonnegative.

COROLLARY 6.7. *Let v be a continuous function on \mathbf{P}^m such that $v \in \mathcal{P}_-^{(2)}$, then v is constant. In particular there are no nonconstant functions in $\mathcal{C}_{\mathbb{R}}^2$ such that $T = i\partial v \wedge \bar{\partial} v$ satisfies $i\partial\bar{\partial}T = 0$.*

Proof: We know that $K(v)$ is pseudoconcave. We show next that also $K(-v)$ is pseudoconcave. It suffices to show that $-v$ is also a decreasing limit of \mathcal{C}^2 functions w_n , $(i\partial\bar{\partial}w_n)^2 \leq \epsilon_n\omega^2$. For this, let v_n be such a sequence for v . Taking a subsequence if necessary we can assume that $v \leq v_n \leq v + 1/2^n$. Set $w_n = -v_n + 1/n$. Since on \mathbf{P}^m the Levi problem has a positive solution, this implies that the complements of $K(v)$ and $K(-v)$ are both Stein. But then the intersection of the two domains is a Stein manifold of dimension > 1 with two ends, unless $K(v)$ and $K(-v)$ have a nonempty intersection. But then v must be constant. ■

COROLLARY 6.8. *If the Levi problem is solvable on a compact complex manifold M , then $P^{(2)}$ only contains constant functions. Hence there is no nonconstant holomorphic map from M to a manifold with nontrivial $P^{(2)}(M)$.*

Recall [BS] that given a positive closed current S on M , an upper semicontinuous function ϕ defined on $\text{Supp}(S)$ is S -plurisubharmonic if for every $p \in \text{Supp}(S)$ there is an open set U , $p \in U$ and a sequence $\phi_n \in \mathcal{C}^2(U)$, such that $\phi = \lim_{\searrow} \phi_n$ on $U \cap S$ and $i\partial\bar{\partial}\phi_n \wedge S \geq 0$. A function ϕ is S -pluriharmonic if both ϕ and $-\phi$ are S -plurisubharmonic.

THEOREM 6.9. *Let S be a positive closed current of bidegree $(1, 1)$ in \mathbf{P}^2 . Assume ϕ is S -plurisubharmonic and let $K(\phi) = \{p \in \text{Supp}(S); \phi(p) = \max \phi\}$. Then $\mathbf{P}^2 \setminus K(\phi)$ is pseudoconvex. If ϕ is S -pluriharmonic, then ϕ is constant.*

Proof: Recall that for S -plurisubharmonic functions, the local maximum principle is valid [BS], Prop. 3.1. We claim that $\mathbf{P}^2 \setminus K(\phi)$ is pseudoconvex. We modify the proof of Proposition 6.3. Assume that $\mathbf{P}^2 \setminus K(\phi)$ is not pseudoconvex. We can assume in local coordinates that $K(\phi)$ contains a point (z_0, w_0) , $|z_0|, |w_0| < 1$ and that $K(\phi)$ does not intersect the Hartogs figure $H = \{(z, w); |w| \leq r < 1, |z| \leq 1 + \delta, \} \cup \{(z, w); 1 \leq |z| \leq 1 + \delta, |w| \leq 1\}$. We can also assume that on a fixed neighborhood of $\hat{H} = \{(z, w); |z| \leq 1 + \delta, |w| \leq 1\}$ there is a sequence of \mathcal{C}^2 functions $\phi_n \searrow \phi$ on $\text{Supp}(S)$, $i\partial\bar{\partial}\phi_n \wedge S \geq 0$. We can assume $\phi = 0$ on $K(\phi)$. Then $\phi < 0$ on $H \cap \text{Supp}(S)$ and $\phi(z_0, w_0) = 0$.

Let $X := \{|z| \leq 1, r < |w| \leq 1\}$. Define the function u by

$$u(z, w) := \eta(1 - |z|^2) - \frac{\epsilon}{|w|^2} + \delta$$

where $\eta \ll \epsilon < \delta \ll 1$. We will choose the constants so that $\phi < u$ on $\partial X \cap \text{Supp}(S)$. First observe that if δ is small enough, then automatically $\phi < u$ on all of $\partial X \cap \text{Supp}(S)$ except possibly where $|w| = 1$ and $|z| < 1$. Fix any such δ . Let $\epsilon < \delta$ be chosen big enough that $-\frac{\epsilon}{|w_0|^2} + \delta < 0$. Since $\epsilon < \delta$ and $\phi \leq 0$ on $|w| = 1$ and $|z| < 1$, $(z, w) \in \text{Supp}(S)$, $\phi < u$ on all of $\partial X \cap \text{Supp}(S)$ if we choose $\eta = 0$. To finish the choice of constants, η, ϵ, δ choose $\eta > 0$ small enough that $\phi < u$ still on $\partial X \cap \text{Supp}(S)$ and in addition $u(z_0, w_0) < 0$.

Next choose n large enough so that $\phi_n < u$ on $\partial X \cap \text{Supp}(S)$. Then if we add a strictly positive constant c to u we can assume that

$$\begin{aligned}\phi_n &< u + c \text{ on } \partial X \cap \text{Supp}(S) \\ \phi_n &\leq u + c \text{ on } X \cap \text{Supp}(S) \\ \phi_n(z_1, w_1) &= u(z_1, w_1) + c, (z_1, w_1) \in X \cap \text{Supp}(S)\end{aligned}$$

Now, $-u$ is plurisubharmonic, so $i\partial\bar{\partial}(-u) \wedge S \geq 0$. Hence $\phi_n - u$ is S -plurisubharmonic so this contradicts the local maximum modulus principle for S -plurisubharmonic functions.

If ϕ is S -pluriharmonic, then $K(\phi)$ and $K(-\phi)$ intersect, hence ϕ is constant. ■

PROPOSITION 6.10. *If $v \in \mathcal{P}_-^{(k)}(M)$ then v satisfies the local maximum principle.*

Proof: Recall that the local maximum principle says that for every ball $\max_B v \leq \max_{\partial B} v$. This follows, since the Hartogs figure argument is local. In fact, let K denote the compact set at which the maximum is reached. Let $p \in \partial K$ and use a Hartogs figure there. ■

There are positive closed currents T on \mathbf{P}^2 of the form $T = i\partial u \wedge \bar{\partial} \bar{u}$, u continuous except at one point and such that $\int T \wedge T \neq 0$, for example: $u = \log^+ |z|$ in \mathbf{C}^2 if $[z : w : t]$ are the homogeneous coordinates in \mathbf{P}^2 .

PROPOSITION 6.11. *Consider $C = \{T \geq 0, i\partial\bar{\partial}T = 0, \int T \wedge \omega = 1\}$. Then $\inf_{T \in C} \int T \wedge T \leq 1 - \frac{1}{2\pi^2}$.*

Proof: Let $u(|z|^2), v(|w|^2)$ be \mathcal{C}^∞ real valued functions with support in the unit interval. Define $\psi(z, w) := u(|z|^2) + iv(|w|^2)$. Let $T := i\partial\psi \wedge \bar{\partial}\bar{\psi}$ on \mathbf{C}^2 . Then $T \geq 0$ and $T \wedge T = 0$. Moreover, T is pluriharmonic on \mathbf{C}^2 .

We want to decompose T as in Proposition 2.6.

$$\begin{aligned}i\partial\psi \wedge \bar{\partial}\bar{\psi} &= i(u'\bar{z}dz + iv'\bar{w}dw) \wedge (u'zd\bar{z} - iv'wd\bar{w}) \\ &= i(u')^2 z\bar{z}dz \wedge d\bar{z} + i(v')^2 w\bar{w}dw \wedge d\bar{w} \\ &\quad + u'v'z\bar{w}d\bar{z} \wedge dw + u'v'\bar{z}wdz \wedge d\bar{w}\end{aligned}$$

Let

$$\begin{aligned} U(z) &:= \frac{i}{\pi} \int \log |z - x| (u'(x\bar{x}))^2 x\bar{x} dx \wedge d\bar{x} \\ V(w) &:= \frac{i}{\pi} \int \log |w - y| (v'(y\bar{y}))^2 y\bar{y} dy \wedge d\bar{y} \end{aligned}$$

Then

$$\begin{aligned} i\partial\psi \wedge \bar{\partial}\bar{\psi} &= i\partial\bar{\partial}U(z) + i\partial\bar{\partial}V(w) + \bar{\partial}(uv'\bar{w}dw) - uv''w\bar{w}d\bar{w} \wedge dw \\ &\quad - uv'd\bar{w} \wedge dw + \partial(uv'w d\bar{w}) - uv''w\bar{w}dw \wedge d\bar{w} - uv'dw \wedge d\bar{w} \end{aligned}$$

Hence:

LEMMA 6.12. *On \mathbb{C}^2 ,*

$$\begin{aligned} T &= i\partial\psi \wedge \bar{\partial}\bar{\psi} \\ &= i\partial\bar{\partial}U(z) + i\partial\bar{\partial}V(w) + \bar{\partial}(uv'\bar{w}dw) + \partial(uv'w d\bar{w}). \end{aligned}$$

Let $A := i/\pi \int (u')^2 |z|^2 dz \wedge d\bar{z}$, $B := i/\pi \int (v')^2 |w|^2 dw \wedge d\bar{w}$. Then $U(z) = A \log |z|$, $|z| > 1$, $V(w) = B \log |w|$, $|w| > 1$. We decompose T further: Let $h := U(z) + V(w) - \frac{1}{2}(A + B) \log(1 + |z|^2 + |w|^2)$. Then

LEMMA 6.13. *On \mathbb{C}^2 :*

$$\begin{aligned} T &= i\partial\bar{\partial}h(z, w) + \frac{1}{2}(A + B)\omega + \bar{\partial}(uv'\bar{w}dw) + \partial(uv'w d\bar{w}) \\ \omega &:= i\partial\bar{\partial} \log(1 + |z|^2 + |w|^2). \end{aligned}$$

We extend T to \mathbb{P}^2 as \tilde{T} , the trivial extension. We need to know that T has finite mass near $[0 : 1 : 0]$ and $[1 : 0 : 0]$ for \tilde{T} to be well defined. To extend first across the line at infinity, $\eta = 0$ away from $[0 : 1 : 0]$ and $[1 : 0 : 0]$, we extend the three parts individually. First ω extends as the Kähler form, also called ω . The form $uv'w d\bar{w}$ has compact support and extends as S trivially. Next we investigate h near $[0 : 1 : 0]$. We calculate in local coordinates. $[z : w : 1] = [Z : 1 : t]$, to get $h(z, w) = \tilde{h}(Z, t) = U(Z/t) + V(1/t) - \frac{1}{2}(A + B) \log(1 + |Z/t|^2 + |1/t|^2)$. When $|Z/t| > 1$ we have $U(Z/t) = A \log |Z/t|$, $V(1/t) = B \log |1/t|$. So $\tilde{h}(Z, t) = A \log |Z| - A \log |t| - B \log |t| - \frac{1}{2}(A + B) \log(1 + |Z|^2 + |t|^2) + \frac{1}{2}(A + B) \log |t|^2$, $\tilde{h}(Z, t) = A \log |Z| - \frac{1}{2} \log(1 + |t|^2 + |Z|^2)$. Hence $\tilde{h}(Z, t)$ extends smoothly across $\eta = 0$ except possibly at $[0 : 1 : 0]$ and $[1 : 0 : 0]$. In particular $i\partial\bar{\partial}\tilde{h}$ extends trivially at $\eta = 0$ except at $[1 : 0 : 0]$ and $[0 : 1 : 0]$. Next we calculate in a neighborhood of $[0 : 1 : 0]$. We get

$$\tilde{h}(Z, t) = U(Z/t) + A \log |t| - \frac{A + B}{2} \log(1 + |t|^2 + |Z|^2).$$

The function $U(Z/t) + A \log |t| =: \phi(Z, t)$ is plurisubharmonic when $t \neq 0$ and equals $A \log |Z|$ when $|Z/t| > 1$ or $t = 0$, $Z \neq 0$. So $\phi(Z, t)$ is plurisubharmonic away from the origin. Hence ϕ has a well defined plurisubharmonic extension through

$(0, 0)$ by setting $\phi(0, 0) = -\infty$. It follows that \tilde{h} is a global quasisubharmonic function on \mathbb{P}^2 with poles at $[0 : 1 : 0], [1 : 0 : 0]$. Hence,

LEMMA 6.14. *The trivial extension \tilde{T} is given by $\tilde{T} = \frac{A+B}{2}\omega + \partial S + \bar{\partial} S + i\partial\bar{\partial}\tilde{h}, S = uv'wd\bar{w}$.*

It is easy to check that \tilde{T} is pluriharmonic on \mathbf{P}^2 .

End of Proof of Proposition 6.11: This follows since $\partial\bar{\partial}\tilde{h}$ has no mass on the line at infinity. Hence $\int \bar{\partial}S \wedge \partial\bar{S} = AB, \int T \wedge T = \left(\frac{A+B}{2}\right)^2 4\pi^2 - 2AB$ and $\int T \wedge \omega = \frac{A+B}{2}2\pi$ so if we let $T_1 = \frac{T}{\int T \wedge \omega}$ we find

$$\int T_1 \wedge T_1 = 1 - \frac{2AB}{\left(\frac{A+B}{2}\right)^2 4\pi^2}.$$

The minimum is reached for $A = B$ and equals $1 - \frac{1}{2\pi^2}$. ■

PROPOSITION 6.15. *Let M be a complex surface and $\rho \in C_{\mathbb{R}}^2(M)$. Assume that $\partial\rho$ is non vanishing on $X = \{\rho = 0\}$ and that $(i\partial\bar{\partial}\rho)^2 = 0$ on X and also that $i\partial\bar{\partial}\rho \wedge \partial\rho \wedge \bar{\partial}\rho = O(\rho^2)$. Then $T = i\delta_{\{\rho=0\}}\partial\rho \wedge \bar{\partial}\rho$ is a smooth positive harmonic current. Moreover $T \wedge T = 0$.*

Proof: Choose a $\chi \in C_0^\infty(-1, 1), \chi \geq 0, \int \chi = 1$. Let

$$T_\epsilon = i\frac{1}{\epsilon}\chi\left(\frac{\rho}{\epsilon}\right)\partial\rho \wedge \bar{\partial}\rho.$$

Then we have

$$i\partial\bar{\partial}T_\epsilon = -\frac{1}{\epsilon^2}\chi'\left(\frac{\rho}{\epsilon}\right)\partial\rho \wedge \bar{\partial}\partial\rho \wedge \bar{\partial}\rho - \chi\left(\frac{\rho}{\epsilon}\right)\bar{\partial}\partial\rho \wedge \partial\bar{\partial}\rho.$$

Clearly then $i\partial\bar{\partial}T_\epsilon \rightarrow 0$. ■

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