ENERGY AND INVARIANT MEASURES FOR BIRATIONAL SURFACE MAPS

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0. Introduction

Let X be a compact Kähler surface, and let $f : X \to X$ be a bimeromorphic mapping. We consider (f, X) as a dynamical system, which means that we consider the behavior of the iterates $f^n = f \circ \cdots \circ f$ as n tends to infinity. Since f is invertible we may consider both forward and backward dynamics, i.e., f^n as $n \to +\infty$ and $n \to -\infty$. A meromorphic map of a surface is holomorphic outside a finite set, $I(f)$, of points which are blown up to curves. Thus f is not in general a continuous map, so it is not clear to what extent there is a standard category of dynamical systems into which such an object falls.

We consider two bimeromorphic maps to be equivalent if they are bimeromorphically conjugate. Two complex surfaces can be bimeromorphically equivalent, however, without being homeomorphic. One approach that has proved fruitful in complex dynamics is to start with the induced action f^* on the cohomology group $H^{1,1}(X)$. A question that arises when f has points of indeterminacy is whether the passage to cohomology is natural for the dynamics, i.e., whether $(f^n)^* = (f^*)^n$. This happens exactly when the condition

$$\bigcup_{n\geq 0} f^{-n} I(f) \cap \bigcup_{n\geq 0} f^n I(f^{-1}) = \emptyset \tag{1}$$

holds. This condition may be viewed as a separation between the obstructions to forward and backward dynamics. Diller and Favre [DF] showed that any bimeromorphic surface map $f : X \to X$ is bimeromorphically equivalent to a map $\hat{f} : \hat{X} \to \hat{X}$ for which (1) holds. In general, the spectral radius ρ of $\hat{f}^*$ on $H^{1,1}(X)$ is greater than or equal to 1, and it was shown in [DF] that if $\rho = 1$, then either f is a dynamically trivial automorphism, or f preserves a rational or elliptic fibration and exhibits a dynamic which is essentially one-dimensional.

We assume in this paper that $\rho > 1$. In this case there are stable/unstable currents $\mu^\pm$ whose cohomology classes generate the f^* and f_* eigenspaces for ρ, and in fact $f^*\mu^+ = \rho\mu^+$ and $f_*\mu^- = \rho\mu^-$. The currents $\mu^\pm$ carry geometric information of (complex) dimension 1 and are useful in analyzing the dynamics of f.

A natural hope is that the wedge product $\mu := \mu^+ \wedge \mu^-$ might define an invariant measure that serves as a bridge between the action of f^* on $H^{1,1}$ and the ergodic properties of f on X. This was shown to happen for polynomial automorphisms of $\mathbf{C}^2$ in the papers [BS] and [FS]; for automorphisms of K3 surfaces in [C]; and for certain birational maps in [Dil2] and [Gue]. Typically one considers the positive, closed currents $\mu^\pm = dd^c g^\pm$ in terms of local potentials. The operation of wedge product is then interpreted in terms of the so-called complex Monge-Ampère operator $dd^c g^+ \wedge dd^c g^-$. As is well known, this operation is possible if at least one of the potentials g^+ or g^- is locally bounded. And this is what happens in all of the papers cited above. On the other hand, it is possible for both local potentials g^+ and g^- to be locally unbounded at a point, as is the case for the "golden mean" family, which was analyzed in detail in [BD].

The condition

$$\overline{\bigcup_{n\geq 0} f^{-n} I(f)} \cap \overline{\bigcup_{n\geq 0} f^n I(f^{-1})} = \emptyset \tag{2}$$

was introduced in [Dil2], and it was shown to be equivalent to the condition that for each point there is a neighborhood on which one of the local potentials g^+ or g^- is continuous. In this paper

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we employ a quantitative condition stronger than (1) and weaker than (2):

$$\sum_{n\geq 0} \rho^{-n} \log \operatorname{dist}(f^n I(f^{-1}), I(f)) > -\infty. \tag{3}$$

By Theorem 4.3 this is equivalent to $g^+(x) > -\infty$ for all $x \in I(f)$.

Theorem. *If (3) holds, then $\mu := \mu^+ \wedge \mu^-$ is a probability measure that puts no mass on any algebraic set; μ is invariant and mixing for f. Further,*

$$\int |\log ||Df||\,|\;\mu < \infty, \tag{4}$$

and thus the Lyapunov exponents of f with respect to μ are well-defined and finite. Finally, the Lyapunov exponents satisfy

$$\lambda^- \leq -\frac{\log \rho}{8} < 0 < \frac{\log \rho}{8} \leq \lambda^+,$$

and thus μ is a hyperbolic measure of saddle type.

The finiteness of the integral in (4), and thus the finiteness of the Lyapunov exponents, seems to be closely linked with condition (3). On the other hand, Favre [Fav3] has constructed a mapping which does not satisfy (3). Favre's example depends on the existence of an invariant complex line whose rotation number satisfies a delicate number-theoretic property. In any case, condition (3) seems to be generic, cf Proposition 4.5.

We define $\mu^+ \wedge \mu^-$ using an "energy" approach to interpret the complex Monge-Ampère operator, as was done in [BT] and [B]. Specifically, if T is a positive, closed (1,1)-current, then we define the energy of a function φ to be

$$\mathcal{E}_T(\varphi) := \int d\varphi \wedge d^c\varphi \wedge T.$$

The approach from [BT] is that if φ is essentially psh, and if $\mathcal{E}_T(\varphi) < \infty$, then $dd^c\varphi \wedge T$ defines a measure, and φ is integrable with respect to this measure. In the situation at hand, we will show that $\mathcal{E}_T(\varphi) < \infty$ for $T = \mu^-$ and $\varphi = g^+$.

The currents $\mu^\pm$ are obtained dynamically by starting with Kähler forms β_1 and β_2 and taking normalized limits of pullbacks:

$$\mu^+ = c_1 \cdot \lim_{n\to\infty} \rho^{-n} f^{*n}\beta_1, \quad \mu^- = c_2 \cdot \lim_{n\to\infty} \rho^{-n} f_*^n \beta_2. \tag{5}$$

We show in Corollary 4.9 that the measure μ is also obtained as

$$\mu = c \cdot \lim_{n,m\to\infty} \rho^{-n-m} f^{*n}\beta_1 \wedge f_*^m \beta_2. \tag{6}$$

The contents of the paper are as follows. In §1 we discuss the pullbacks of currents and the associated (local) potential functions. The fact of convergence in (5) was established in [DF]. However, in order to pass from (5) to (6), we need to know how the intermediate pullbacks depend on β_1 and β_2. This dependence is clarified in §2. In §3 we discuss properties of the energy integral. In §4 we discuss condition (3); we show that when (3) holds the gradients of the local potentials of $\mu^\pm$ belong to L^2. Thus $\mu := \mu^+ \wedge \mu^-$ is well defined. We show in Theorem 4.11 that μ is invariant. §5 is dedicated to showing that μ is mixing, and §6 gives the estimates on the Lyapunov exponents.

As a final introductory note, the authors would like to thank the referee warmly for his very perceptive comments concerning this paper and particularly for pointing out Theorem 4.5 along with its proof.

1. Pullbacks under Birational Maps

Throughout this paper we let X denote a compact Kähler surface endowed with the hermitian metric associated to a fixed Kähler form β. Let $f : X \to X$ be a bimeromorphic self-map. That is, there is a compact surface Γ (the desingularized graph of f) with proper modifications (i.e. generically injective holomorphic maps) $\pi_1, \pi_2 : \Gamma \to X$ such that $f = \pi_2 \circ \pi_1^{-1}$. The set

$$\mathcal{C}(\pi_j) := \{x \in \Gamma : \#(\pi_j^{-1}\pi_j(x)) > 1\} = \{x \in \Gamma : \dim(\pi_j^{-1}\pi_j(x)) = 1\}$$

is the critical set for π_j. The images $I(f) := \pi_1(\mathcal{C}(\pi_1))$ and $\mathcal{C}(f) := \pi_1(\mathcal{C}(\pi_2))$ are the *indeterminacy* and *critical* sets, respectively, for f. Note that in this case the critical set is actually an exceptional set, since the irreducible components are mapped to points. It is shown in [DF], section 2 that after a finite number of blow-ups we may assume that (1) holds. In this case, $I(f^n) = \bigcup_{j=0}^{n-1} f^{-j} I(f)$.

Since f is ill-defined at points of indeterminacy, it is useful to adopt some conventions concerning images of points and curves under f. Given any $x \in X$, we set $f(x) = \pi_2(\pi_1^{-1}(x))$ with the effect that $f(x)$ is a point if $x \notin I(f)$ and a component of $\mathcal{C}(f^{-1})$ otherwise. Given any curve $V \subset X$, we set $f(V) = \overline{f(V \setminus I(f))}$. For irreducible V, it follows that $f(V)$ is a point if $V \subset \mathcal{C}(f)$ and an irreducible curve if not.

Proposition 1.1. *There exist constants $A, B > 0$ such that*

$$\|D_x f\| \leq A \operatorname{dist}(x, I)^{-B}$$

for all $x \in X$. Further, given a second point $y \in X$, one has

$$\operatorname{dist}(f(x), f(y)) \leq A \operatorname{dist}(\{x, y\}, I)^{-B} \operatorname{dist}(x, y).$$

Proof. Choose any hermitian metric on the graph Γ of f. Then $\|D\pi_2\|$ is uniformly bounded on Γ, so it suffices to prove the first inequality for π_1^{-1} in place of f.

In local coordinates, the entries of $D\pi_1$ are holomorphic functions, so the entries of $(D\pi_1)^{-1}$ are meromorphic functions with poles in $\mathcal{C}(\pi_1)$. Since Γ is compact, there are constants $A, B > 0$ such that

$$\left\|(D_y \pi_1)^{-1}\right\| \leq A \operatorname{dist}(y, \mathcal{C}(\pi_1))^{-B}$$

for all $y \in \Gamma$. But $\|D\pi_1\|$ is uniformly bounded on Γ, so this implies

$$\left\|D_{\pi_1(y)}(\pi_1^{-1})\right\| = \left\|(D_y \pi_1)^{-1}\right\| \leq A \operatorname{dist}(\pi_1(y), I)^{-B}.$$

The first inequality now follows because π_1 is surjective.

The second inequality follows from the first by integrating along a path from x to y. □

We consider the hermitian inner product on the set of smooth (1,1)-forms given by

$$\langle \alpha, \alpha' \rangle := \int_X \alpha \wedge \bar{\alpha}'.$$

It follows that any smooth (1,1)-form defines an element of the dual space of (1,1)-forms, and thus defines a (1,1)-current. The (1,1) cohomology group $H^{1,1}(X)$ may be given as the smooth, closed (1,1)-forms modulo the exact ones. It follows from Stokes' Theorem that the hermitian pairing on (1,1)-forms induces a pairing on $H^{1,1}(X)$. In fact, this pairing is a nondegenerate duality.

If T is a closed (1,1)-current, then $T(d\xi) = 0$, which means that T annihilates all d-exact (1,1)-forms. Thus the restriction of T to the closed forms defines an element of $H^{1,1}(X)^*$, and there is a cohomology class $\{T\} \in H^{1,1}(X)$ which represents this restriction in the sense that $T = \langle \cdot, \{T\} \rangle$.

The "$\partial\bar{\partial}$-Lemma" from Kähler geometry (see [GH, page 149]) asserts that if T_1 and T_2 are closed (1,1) currents which define the same cohomology class, then there there is a current S of degree 0 such that

$$T_1 = T_2 + dd^c S.$$

In particular, if T is a closed (1,1)-current on X, there is a smooth (1,1)-form α defining the cohomology class $\{T\}$, and by the $\partial\bar{\partial}$-Lemma, there is a current h such that $T = \alpha + dd^c h$.

Next we define the pullback of a smooth form. If α is a smooth (1,1) form on X, then $\pi_2^*\alpha$ is a smooth (1,1) form on Γ. By duality, $\pi_2^*\alpha$ defines a current on Γ of bidegree (1,1). Thus

$$f^*\alpha := \pi_{1*}(\pi_2^*\alpha)$$

is a current on X. The pullback f^* commutes with d and with the complex structure, so closed (respectively, exact) forms are pulled back to closed (resp. exact) currents of the same bidegree. This gives a well defined map f^* on $H^{1,1}(X)$. Similarly, we set $f_*\eta := (f^{-1})^*\eta = \pi_{2*}\pi_1^*\eta$. In other words, we set $f_* = (f^{-1})^*$. Note that f^* and f_* are adjoint with respect to the intersection form $\langle \cdot, \cdot \rangle$ on cohomology classes, which is to say

$$\langle f^*\alpha, \alpha' \rangle = \langle \pi_2^*\alpha, \pi_1^*\alpha' \rangle = \langle \alpha, f_*\alpha' \rangle.$$

We can also define the pullback f^*T if T is a positive, closed (1,1)-current on X. By pulling back local potentials of T, we may define f^*T on $X - I(f)$. Now for any $x \in I(f)$, we may choose a pseudoconvex neighborhood U of x with $H^2(U - \{x\}) = 0$. Thus there is a potential p on $U - \{x\}$ such that $f^*T = dd^c p$ on $U - \{x\}$. Since p is psh on $U - \{x\}$, it follows that p has a psh extension $\tilde{p}$ to U. We define $f^*T := dd^c p$ on U.

In order to discuss the singularities caused by pulling back forms and currents, let us recall that the Lelong number of a positive closed $(1,1)$ current T at a point $x \in X$ is the non-negative number

$$\nu(T,x) := \lim_{r\to 0} \frac{C}{r^2} \int_{B_x(r)} \beta \wedge T.$$

If u is a local potential for T in a neighborhood of x, i.e., if $T = dd^c u$, then

$$\nu(T,x) := \sup\{t \geq 0 : u(y) < t \log \operatorname{dist}(x,y) + O(1)\}$$

(see [Dem2, Equation 5.5e]).

We use Proposition 1.1 to gain control over the singularities of pullbacks of smooth (1,1) forms:

Proposition 1.2. *Let ω be a Kähler form on X and ω' be a smooth form cohomologous to $f^*\omega$. Then we can write*

$$f^*\omega = \omega' + dd^c u, \tag{7}$$

where u is smooth and negative on $X \setminus I(f)$ and satisfies

$$A \log \operatorname{dist}(x,I) - B \leq u(x) \leq A' \log \operatorname{dist}(x,I) + B'$$

for some constants $A, B, A', B' > 0$ and every $x \in X$.

Proof. If ω' and $f^*\omega$ represent the same element of $H^{1,1}(X)$, then there exists a u satisfying (7). The current $f^*\omega$ is positive, so u is given locally as the sum of a smooth function u_1 and a plurisubharmonic function u_2. In particular, we can assume that u is negative. The remaining assertion in the proposition only concerns some (any) choice of u_2 in the neighborhood of a point $y \in I(f)$.

For each component V' of $\pi_1^{-1}(y)$ and its image $V = \pi_2(V')$, we have

$$\int_{V'} \pi_2^*\omega = \int_V \omega > 0.$$

The intersection form on $\pi_1^{-1}(y)$ is negative definite, so we can choose a non-trivial effective divisor V' supported on $\pi_1^{-1}(y)$ such that $\pi_2^*\omega + [V']$ is cohomologically trivial near V. In particular, we can write $\pi_2^*\omega + [V'] = dd^c v$ for some function v defined in a neighborhood U' of $\pi_1^{-1}(y)$ and smooth off $\pi_1^{-1}(y)$. Therefore $v \circ \pi_1^{-1}$ is a local potential for

$$\pi_{1*}(\pi_2^*\omega + [V']) = \pi_{1*}\pi_2^*\omega$$

on the neighborhood $U = \pi_1(U')$ of y. The singularities of v come entirely from local potentials for $[V']$. Hence we can arrange

$$v(x') \geq A \log \operatorname{dist}(p', \pi_1^{-1}(y))$$

for some $A > 0$ and all $x' \in U'$. Finally, since π_1 is uniformly Lipschitz, we obtain after adjusting A that

$$u_2 := v \circ \pi_1^{-1}(x) \geq A \log \operatorname{dist}(x,y),$$

which finishes the proof of the lower bound for u_2.

To obtain the upper bound for u_2, we rely on the push-pull formula [DF, Theorem 3.3] applied to π_1. This gives

$$\pi_1^* f^*\omega = \pi_1^*\pi_{1*}(\pi_2^*\omega) = \pi_2^*\omega + [V']$$

where V' is an effective divisor such that $\pi(\operatorname{supp} V') = I(f)$ whose support contains at least one component of $\mathcal{C}(\pi_1) \cap \pi_2^{-1}(V)$ for every irreducible $V \subset \mathcal{C}(f^{-1})$. In particular, the Lelong number of the positive current $\pi_1^* f^*\omega$ is positive at some point in $\pi_1^{-1}(p)$ for every $p \in I(f) = \pi_2\mathcal{C}(f^{-1})$. It follows from [Fav1, Theorem 2] that $f^*\omega$ has a positive Lelong number at each point in $I(f) = \pi_1(\mathcal{C}(\pi_1))$. We conclude that any local potential u_2 for $f^*\omega$ near $y \in I(f)$ must satisfy

$$u_2(x) \leq A' \log \operatorname{dist}(x,y) + B'$$

for some $A', B' > 0$. $\square$

Let $P(X)$ denote the set of upper semicontinuous functions u on X such that $dd^c u \geq -c\beta$ for some $c \in \mathbf{R}$. Such functions are locally the sum of a psh function and a smooth function. (Since X is compact, there are no global psh functions.) Given a finite set $S \subset X$, let $\tilde{P}(X, S)$ denote those functions $u \in P(X) \cap C^\infty(X \setminus S)$ such that

$$u(x) \geq A \log \operatorname{dist}(x, S) - B$$

for some $A, B > 0$ and all $x \in X$.

Proposition 1.3. *Suppose that $S \subset X$ is finite and disjoint from $I(f^{-1})$. Then $u \in \tilde{P}(X, S)$ implies that $u \circ f$ is a difference of functions in $\tilde{P}(X, f^{-1}(S) \cup I(f))$.*

Proof. Because $u \in \tilde{P}(X, S)$, we get

$$0 \leq dd^c u \circ f + cf^*\beta = dd^c(u \circ f + v) + \beta' \leq dd^c(u \circ f + v) + c\beta,$$

where β' is a smooth $(1,1)$ form cohomologous to $f^*\beta$, $v \in \tilde{P}(X, I(f))$, and $c > 0$ is chosen large enough that $c\beta \geq \beta'$. Moreover, since $u \in \tilde{P}(X, S)$, we see from Proposition 1.1 that for $f(x)$ near S, and therefore uniformly far from $I(f^{-1})$,

$$\begin{aligned} u \circ f(x) &\geq A \log \operatorname{dist}(f(x), S) - B \\ &\geq A \log \operatorname{dist}(x, f^{-1}(S)) - B + C \log \operatorname{dist}(f(x), I(f^{-1})) \\ &\geq A \log \operatorname{dist}(x, f^{-1}(S)) - B. \end{aligned}$$

Combining the two displayed inequalities, we see that

$$u \circ f = (u \circ f + v) - v,$$

where $v \in \tilde{P}(X, I(f))$, and $u \circ f + v \in \tilde{P}(X, f^{-1}(S) \cup I(f))$. □

2. Invariant Cohomology Classes and Currents

The condition (1) implies that $(f^n)^* = (f^*)^n$ on $H^{1,1}(X)$ for every $n \in \mathbf{Z}$, (see [FS] and [DF, Theorem 1.14]). In this case the bimeromorphically invariant quantity

$$\rho := \lim_{n \to \infty} \|f^{n*}|_{H^{1,1}}\|^{1/n} \geq 1$$

is the modulus of the largest eigenvalue of f^* on $H^{1,1}(X)$. In this paper, we assume that

$$\rho > 1. \tag{8}$$

An element of $H^{1,1}$ is a Kähler class if it contains a Kähler form. We say that a cohomology class is nef if it is in the closure of the Kähler classes. Alternatively, a class is nef if and only if (see [Lam] or [Bu]) its intersections with the fundamental classes of curves and with the class of the Kähler form β are all non-negative.

The following is [DF] Theorem 5.1.

Theorem 2.1. *If (8) holds, then ρ is the unique (counting multiplicity) eigenvalue of f^* of modulus larger than one; and the associated eigenspace is generated by a nef class θ^+. If θ^- is a nef class generating the corresponding eigenspace for f_*, then $\langle \theta^+, \theta^- \rangle > 0$.*

We note that it is also shown in [DF] that if $\langle \theta^+, \theta^+ \rangle = 0$, then f is birationally conjugate to an automorphism of some complex surface. The results from this paper for that case have already been obtained by Cantat in his thesis [Can1]. So while it does not much simplify the exposition, there is no harm in assuming in what follows that $\langle \theta^+, \theta^+ \rangle > 0$ and similarly for θ^-. Indeed, under this assumption [DF] Theorem 7.2 states that the surface X must be rational.

For convenience, we scale $\theta^\pm$ and β so that

$$\langle \theta^+, \theta^- \rangle = \langle \theta^+, \beta \rangle = \langle \theta^-, \beta \rangle = 1. \tag{9}$$

This completely determines θ^+ and θ^-.

We fix Kähler forms $\omega_1, \ldots, \omega_N$ whose cohomology classes form a basis a for $H^{1,1}(X)$, and we let Ω denote the linear span of these forms. We also assume for convenience that Ω contains the Kähler form β corresponding to the metric on X. We endow Ω with the norm $\|\omega\| = (\sum |c_j|^2)^{1/2}$

where $\omega = \sum c_j \omega_j$. Let $\omega^+, \omega^- \in \Omega$ denote the unique elements representing the classes θ^+ and θ^-, respectively. By Theorem 2.1, an element $\eta \in \Omega$ has a decomposition

$$\eta = \eta^\perp + c\omega^+ \tag{10}$$

where $\eta^\perp$ belongs to the span of the eigenspaces corresponding to eigenvalues other than ρ. The fact that f^* and $(f^{-1})^*$ are adjoint gives $c = \langle \omega, \theta^- \rangle$.

If η is a closed (1,1)-current, then we let $\omega(\eta)$ denote the element of Ω that corresponds to the cohomology class $\{\eta\}$ defined by η. Thus $\omega^\pm = \omega(\theta^\pm)$. It is evident that, as a mapping from currents to Ω, ω is a projection, i.e., $\omega \circ \omega = \omega$. There is a current $p(\eta)$ such that

$$\eta = \omega(\eta) + dd^c p(\eta). \tag{11}$$

Since $\omega(\eta)$ is smooth, it follows that $p(\eta)$ is smooth wherever η is. The potential $p(\eta)$ is uniquely defined modulo an additive constant, and we specify it uniquely by the condition

$$\langle p(\eta), \beta \wedge \beta \rangle = 0.$$

Now we investigate the interplay between the decomposition (11) and f^*. If η is positive, then $p(\eta) \in P(X)$, and we may apply f^* to (11) to obtain $f^*\eta = f^*\omega(\eta) + dd^c f^* p(\eta)$. Then we set

$$\gamma^+ = p f^* \omega$$

and apply the decomposition (11) to obtain

$$f^*\eta = \omega f^* \omega(\eta) + dd^c[\gamma^+(\eta) + f^* p(\eta)].$$

The operators $\eta \mapsto \omega\eta$ and $\eta \mapsto \gamma^+\eta$ are linear in η and depend only on the cohomology class $\{\eta\}$. The map ω induces an isomorphism $\omega : H^{1,1}(X) \to \Omega$. This provides a conjugacy between the action of f^* on $H^{1,1}$ and the action of ωf^* on Ω. Thus we have $\omega f^* = \omega f^* \omega$, and we may iterate the previous equation to obtain

$$f^{n*}\eta = \omega f^{n*}\eta + \rho^n dd^c g_n^+, \tag{12}$$

where we define

$$g_n^+\eta = \frac{1}{\rho^n}\left(f^{*n} p(\eta) + \sum_{j=0}^{n-1} f^{(n-j-1)*}\gamma^+(f^{j*}\eta)\right). \tag{13}$$

The cone $H^{1,1}_{\mathrm{nef}}$ of nef classes is closed convex and 'strict'. The last condition, which follows from the fact that nef classes are represented by positive closed currents, means that if θ and $-\theta$ are nef, then $\theta = 0$. It follows that there is an affine hyperplane $H \subset H^{1,1}$ such that $H \cap H^{1,1}_{\mathrm{nef}}$ is compact, convex and generates $H^{1,1}_{\mathrm{nef}}$ as a real cone. If we set

$$K = \{\eta \in \Omega : \langle \eta, V \rangle \geq 0 \text{ for every irreducible } V \subset \mathcal{C}(f^{-1})\},$$

then K is a cone defined by a finite number of linear inequalities, and it follows from [Lam] that $H^{1,1}_{\mathrm{nef}} \subset K$ (here as in other places we identify K with the corresponding set of cohomology classes).

Lemma 2.2. *Let K be the subset defined above. Then the function*

$$M(\eta) := \sup_{x \in X - I(f)} \gamma^+(\eta) x$$

is finite for $\eta \in K \cap H \cap \Omega$.

Proof. It is enough to show that for each point $x \in I(f)$ there is a neighborhood U and a local potential for $f^*\eta$ that is bounded above. We have $f^*\eta = \pi_{1*}\pi_2^*\eta$, so if $U \cap I(f) = \{x\}$, we may argue as in Proposition 1.3 to conclude that on $\pi_1^{-1}U$ we have

$$\pi_1^* f^* \eta = \pi_2^* \eta + [V]$$

where V is a (possibly trivial) effective divisor supported on a fiber $\pi_1^{-1}(x)$. Hence, $\pi_1^* f^*\eta = dd^c v$ for some function v on $\pi_1^{-1}(U)$ whose singularities come entirely from local potentials for $[V]$. Thus v is bounded above. Now the pushforward, $v \circ \pi_1^{-1}$ is a local potential for $f^*\eta$ on U and is bounded above, as desired. □

Theorem 2.3. *There are positive constants A, B such that for any $\eta \in \Omega$*

$$|\gamma^+\eta(x)| \leq \|\eta\| \left(A + B\left|\log \operatorname{dist}(x, I(f))\right|\right) \tag{14}$$

holds for all $x \in X$. Further, there exists a constant C such that if the cohomology class of η is nef, then $\gamma^+\eta(x), g_n^+\eta(x) \leq C\|\eta\|$ for all $n \in \mathbf{N}$ and $x \in X$.

Proof. The first assertion follows from writing η as a linear combination of the basis elements $\omega_1, \dots, \omega_N$ and applying Propositions 1.2 and 1.3.

By definition, K is a convex cone defined by finitely many linear inequalities, and since $H \cap H^{1,1}_{\mathrm{nef}}$ is compact, we can choose finitely many elements $\eta_1, \dots, \eta_m \in K$ whose convex hull contains $H \cap H^{1,1}_{\mathrm{nef}}$. The expression $M(\eta)$ from Lemma 2.2 is a convex function of η, so we conclude that

$$C := \max_{1 \leq j \leq m} M(\eta_j) \geq \sup_{\eta \in H \cap H^{1,1}_{\mathrm{nef}}} M(\eta)$$

gives the upper bound for $\gamma^+\eta(x)$ when η is nef. Together with (13), the fact that $\left\|f^{j*}\eta\right\| \leq C'\rho^j \|\eta\|$ allows us to extend the bound on $\gamma^+\eta$ to an upper bound for $g_n^+\eta$ that does not depend on n. $\square$

Proposition 2.4. *Given $t > 1$, there exists a constant C such that for any form $\omega \in \Omega$ and any $n \in \mathbf{N}$,*

$$\int |\gamma^+(\omega)| \circ f^j \, dV \leq Ct^j \|\omega\|.$$

Proof. From [DF, §6], we have that for any $t > 1$, there exist constants $C_1, C_2 > 0$ such that

$$\operatorname{Vol} f^{-n}(B_{I(f)}(r)) \leq C_1 r^{C_2/t^n}$$

for all $n \in \mathbf{N}$ and all $r > 0$.

From Theorem 2.3, we have

$$\int |\gamma^+\omega| \circ f^j \, dV \leq \|\omega\| \left(A + B \int |\log \operatorname{dist}(f^j(x), I(f))| \, dV(x)\right).$$

Now the volume estimate above gives

$$\begin{aligned} \int |\log \operatorname{dist}(f^j(x), I(f))| \, dV(x) &\leq A + \int_0^\infty \operatorname{Vol} f^{-j}(B_{I(f)}(e^{-s})) \, ds \\ &\leq A + \int_0^\infty C_1 e^{-C_2 s/t^j} \, ds \leq Ct^j, \end{aligned}$$

which combines with the first estimate to finish the proof. $\square$

Let us define $\gamma^+ := \gamma^+\omega^+ = \gamma^+\theta^+$ so we have

$$dd^c\gamma^+ = f^*\omega^+ - \rho\omega^+.$$

This form is smooth away from $I(f)$, so γ^+ is smooth away from $I(f)$. And since the class of ω^+ is nef, Theorem 2.3 tells us that γ^+ is bounded above. Adjusting the value of γ^+ at points in $I(f)$ if necessary, we may therefore assume that γ^+ is upper semicontinuous. Thus the infinite sum

$$g^+ := \sum_{j=0}^{\infty} \frac{\gamma^+ \circ f^j}{\rho^j} \tag{15}$$

is essentially decreasing and defines an upper semicontinuous function (which is possibly $-\infty$ at some points).

Theorem 2.5. *The function g^+ in (15) belongs to $L^1(X)$. Further, for any smooth, closed $(1,1)$ form η, we have*

$$\lim_{n \to \infty} g_n^+\eta = c \cdot g^+$$

where $c = \langle \eta, \theta^- \rangle$, and the convergence takes place in $L^1(X)$.

Proof. Let us first consider the case $\eta = \omega^+$. Recall that $\{f^{n*}\omega^+\} = \rho^n\theta^+$ and that γ^+ depends only on the cohomology class. If we set $\gamma^+ := \gamma^+\omega^+ = \gamma^+\theta^+$, then $\gamma^+ f^{j*}\omega^+ = \rho^j\gamma^+$. Further, since $\omega(\omega^+) = \omega^+$, we have $p(\omega^+) = 0$. Thus

$$g_n^+\omega^+ = \sum_{j=0}^{n-1} \frac{\gamma^+ \circ f^j}{\rho^j}.$$

If we take $1 < t < \rho$, then by Proposition 2.4 we have

$$\int |\gamma^+ \circ f^j| dV \le C' t^j.$$

Thus the sequence $\{g_n^+\}$ converges in $L^1(X)$ to

$$g^+ := \sum_{j=0}^{\infty} \frac{\gamma^+ \circ f^j}{\rho^j}.$$

Since $\gamma^+\eta$ depends only on the cohomology class $\{\eta\}$, we may assume $\eta \in \Omega$. We use the decomposition (10): $\eta = c\omega^+ + \eta^\perp$. Thus

$$g_n^+\eta^\perp = \rho^{-n} f^{n*} p\eta^\perp + \rho^{-n} \sum_{j=0}^{n-1} f^{(n-j-1)*}\gamma^+(f^{j*}\eta^\perp).$$

Since $\eta^\perp$ is smooth, so is $p\eta^\perp$, and so we have $|f^{n*}p\eta^\perp| \le C$ on X. Thus $\rho^{-n}f^{n*}p\eta^\perp$ converges uniformly to zero. By Proposition 2.4 again and the fact that $\gamma^+ = \gamma^+ \circ \omega$, we have

$$\int |g_n^+\eta^\perp| dV \le C\frac{\mathrm{Vol}(X)}{\rho^n} + \frac{1}{\rho^n}\sum_{j=0}^{n-1} C t^{n-j-1} ||\omega f^{j*}\eta^\perp||.$$

By Theorem 2.1, there is a constant C' such that $||\omega f^{j*}\eta^\perp|| \le C't^j$. Thus

$$\int |g_n^+\eta^\perp| dV \le C\frac{\mathrm{Vol}(X)}{\rho^n} + CC'n\frac{t^{n-1}}{\rho^n}.$$

This tends to zero as $n \to \infty$, and $g_n^+\eta$ is linear in η, so the Theorem follows. □

Theorem 2.6. *The current $\mu^+ := \omega^+ + dd^c g^+$ has the following properties:*

- *for every smooth closed $(1,1)$ form η on X, we have*

$$\lim_{n\to\infty} \frac{f^{n*}\eta}{\rho^n} = \mu^+ \cdot \langle \eta, \theta^- \rangle.$$

- *μ^+ is positive;*
- *$f^*\mu^+ = \rho\mu^+$.*

Proof. By Theorem 2.5,

$$\lim_{n\to\infty} \frac{f^{n*}\eta}{\rho^n} = \lim_{n\to\infty} \frac{\omega f^{n*}\eta}{\rho^n} + dd^c \lim_{n\to\infty} g_n^+\eta = \langle \eta, \theta^- \rangle (\omega^+ + dd^c g^+).$$

Taking $\eta = \beta$ to be the Kähler form on X, we have $\langle \beta, \theta^- \rangle = 1$. Thus $\mu^+ = \lim_{n\to\infty} \rho^{-n} f^{n*}\beta$ is a limit of positive currents and therefore positive. Since f^* acts continuously on positive closed $(1,1)$ currents, we also get that $f^*\mu^+ = \lim_{n\to\infty} \rho^{-n} f^{(n+1)*}\beta = \rho\mu^+$. □

The following observation of Favre [Fav1, Theorem 1] will be useful for us later.

Corollary 2.7. *The Lelong number $\nu(\mu^+, x)$ vanishes for $x \in X - \bigcup_{n\ge1} I(f^n)$.*

We close this section by discussing the extent to which μ^+ is invariant under bimeromorphic conjugacy.

Proposition 2.8. *Let $h : Y \to X$ be a proper modification of X. Suppose that both $f : X \circlearrowleft$ and $f_Y := h^{-1} \circ f \circ h : Y \circlearrowleft$ satisfy (1) and (8). Let μ^+ and μ_Y^+ denote the invariant currents associated to f and f_Y. Then $h_*\mu_Y^+$ is a positive multiple of μ^+.*

Proof. By hypothesis $I(h) = \mathcal{C}(h) = \emptyset$. Thus $h^*\beta$ is a smooth, positive and closed $(1,1)$ form on X, and we compute

$$ch_*\mu_Y^+ = h_* \lim_{n\to\infty} \frac{f_Y^{n*}(h^*\beta)}{\rho^n} = \lim_{n\to\infty} \frac{h_* f_Y^{n*} h^*\beta}{\rho^n} = \lim_{n\to\infty} \frac{f^{n*}\beta}{\rho^n} = \mu^+$$

The first and last equalities follow from Theorem 2.6. The second inequality follows from continuity of h_* acting on positive closed $(1,1)$ currents, and the third equality is a consequence of the proof of Proposition 1.13 in [DF]. Since μ^+ and $h_*\mu_Y^+$ are positive, and μ^+ is non-trivial, it follows that $c > 0$. □

Proposition 2.9. *Let $h : Y \to X$ be a bimeromorphic map. Suppose that both f and $f_Y := h^{-1} \circ f \circ h : Y \circlearrowleft$ satisfy (1) and (8), and let μ^+ and μ_Y^+ be the associated invariant currents. Then $h^*\mu^+ = c\mu_Y^+ + [V]$, where $c > 0$ and V is an effective divisor supported on $\mathcal{C}(h)$.*

Proof. Let G be the desingularized graph of h and $\pi_Y : G \to Y$, $\pi_X : G \to X$ be the projections onto first and second factors. That is, $h = \pi_X \circ \pi_Y^{-1}$. After blowing up points in G if necessary, we can assume that the common lift $F : G \to G$ of f and f_Y to G satisfies (1). Let ν^+ denote the invariant current associated to F. Then by the previous lemma, we see that

$$\pi_{X*}\nu^+ = c_1\mu^+ \qquad \pi_{Y*}\nu^+ = c_2\mu_Y^+$$

for constants $c_1, c_2 > 0$. Hence, $\pi_X^*\mu^+ - [V'] = c\pi_Y^*\mu_Y^+ - [V'']$ where $c > 0$ and V' and V'' are effective divisors (with possibly non-integer coefficients) supported on $\mathcal{C}(\pi_X)$ and $\mathcal{C}(\pi_Y)$, respectively. We apply the 'pushpull formula' [DF, Theorem 3.3] to π_Y and conclude

$$c\mu_Y^+ = c\pi_{Y*}\pi_Y^*\mu_Y^+ = \pi_{Y*}(\pi_X^*\mu^+ + [V''] - [V']) = h^*\mu^+ - [\pi_{Y*}V'].$$

Since $V := \pi_{Y*}V'$ is an effective divisor supported on $\mathcal{C}(h)$, we are done. □

3. Energy

Let T be a positive, closed $(1,1)$ current on X. Then T defines an inner product on the space of smooth, real functions on X via the formula

$$\mathcal{E}_T(\varphi, \psi) := \int d\varphi \wedge d^c\psi \wedge T.$$

We denote the seminorm associated with this inner product by

$$|\varphi|_T = (\mathcal{E}(\varphi,\varphi))^{1/2} = \left(\int d\varphi \wedge d^c\varphi \wedge T\right)^{1/2}.$$

We will say that functions $u_j \in C^\infty(X)$, $j \geq 0$ form a *regularizing sequence* for a function u if u_j decreases pointwise to u and $dd^cu_j \geq -c\beta$ for some $c > 0$ and all j. The limit u necessarily belongs to P, and indeed any $u \in P$ admits a regularizing sequence (see [Dem1, Theorem 1.1]). We will use the following property of a function $u \in P$:

$$\text{Every regularizing sequence } \{u_j\} \text{ for } u \text{ is Cauchy in } |\cdot|_T. \tag{16}$$

The union of two regularizing sequences is (essentially) a regularizing sequence. Thus if u satisfies (16) then all regularizing sequences define the same element of the completion with respect to $|\cdot|_T$. In particular, if u and v satisfy (16), then we may define $\mathcal{E}_T(u,v)$ by taking the limit along any regularizing sequences.

The special case $T = \beta$ is classical: condition (16) for $T = \beta$ is equivalent to the condition that $\nabla u \in L^2$.

Our principal use of condition (16) is to define $(dd^cu) \wedge T$. If u satisfies (16), then we may define $(dd^cu) \wedge T$ as a distribution via the pairing

$$\psi \mapsto \langle dd^cu \wedge T, \psi\rangle := -\mathcal{E}_T(\psi, u).$$

It is evident that $dd^cu \wedge T + c\beta \wedge T \geq 0$, so $dd^cu \wedge T$ is represented by a (signed) Borel measure. Further, since $dd^cu_j \geq -c\beta$, it also follows from (16) that $dd^cu_j \wedge T$ converges to $dd^cu \wedge T$ in the weak* topology on the space of measures.

Proposition 3.1. *If $u \in L^1(T \wedge \beta)$, and if u satisfies (16), then $(dd^c u) \wedge T = dd^c(uT)$.*

Proof. Let us remark first that if $u \in L^1(T \wedge \beta)$, then uT is a well-defined (1,1) current, and thus $dd^c(uT)$ is a well-defined current. If $\{u_j\}$ is a sequence satisfying (16), then $u_j T$ converges to uT weakly as currents. Thus $dd^c(u_j T)$ converges to $dd^c(uT)$. Finally, $(dd^c u_j) \wedge T = dd^c(u_j T)$ when u_j is smooth, and we have observed above that $\lim_{j\to\infty}(dd^c u_j) \wedge T = (dd^c u) \wedge T$. □

Proposition 3.2. *If $u, v \in P$ both satisfy (16), and if $v \in L^1(T \wedge \beta)$, then $v \in L^1(dd^c u \wedge T)$.*

Proof. Let $\{u_j\}$ and $\{v_k\}$ denote regularizing sequences for u and v. For fixed j and k, an integration by parts gives $\int v_k\, dd^c u_j \wedge T = -\int dv_k \wedge d^c u_j \wedge T$. Now $dd^c u_j + c\beta \geq 0$, so $(dd^c u + c\beta) \wedge T$ defines (positive) Borel measure. Letting $j \to \infty$, we have

$$\int |v_k|\, (dd^c u + c\beta) \wedge T = -\mathcal{E}_T(v_k, u) + c \int |v_k|\, \beta \wedge T.$$

If we let $k \to \infty$, then the right hand side stays bounded since $v \in L^1(T \wedge \beta)$, and thus $v \in L^1(dd^c u \wedge T)$ by monotone convergence. □

The motivation for our work in the following sections is as follows. We will show that $g^+ \in L^1(T \wedge \beta)$, and g^+ satisfies (16) for the current $T = \mu^- + \beta$. It will then follow that $dd^c g^+ \wedge \mu^-$ is well defined, so the wedge product defines a (signed) measure

$$\mu = \mu^+ \wedge \mu^- = \omega^+ \wedge \mu^- + dd^c g^+ \wedge \mu^-, \tag{17}$$

and $g^+ \in L^1(\mu)$. Since $\mu^\pm \geq 0$, it follows that μ is positive. The total mass of μ is $\int \omega^+ \wedge \omega^- = \langle \theta^+, \theta^- \rangle = 1$, so μ is a probability measure.

Lemma 3.3. *Let $u, v \in C^\infty(X)$ satisfy $dd^c u, dd^c v \geq -c\beta$ and $v \geq u$. Then for any positive, closed $(1,1)$ current T,*

$$\begin{aligned} \mathcal{E}_T(u,v) - \mathcal{E}_T(v,v) &\geq -c \int (v-u)\, \beta \wedge T \\ \mathcal{E}_T(u,u) - \mathcal{E}_T(u,v) &\geq -c \int (v-u)\, \beta \wedge T. \end{aligned}$$

Proof. It is sufficient to prove the first inequality.

$$\begin{aligned} \mathcal{E}_T(u,v) - \mathcal{E}_T(v,v) &= \int d(u-v) \wedge d^c v \wedge T \\ &= \int (v-u) dd^c v \wedge T \geq -c \int (v-u) \beta \wedge T \end{aligned}$$

Here we used Stokes' Theorem to pass from the first line to the second line, and the inequality is obtained because $v - u \geq 0$ and $dd^c v \geq -c\beta$. □

Theorem 3.4. *Suppose that T is a positive closed $(1,1)$ current and $u \in L^1(\beta \wedge T)$. If there exists a regularizing sequence $\{u_j\}$ for which $\{|u_j|_T\}$ is bounded, then u satisfies (16).*

Proof. By hypothesis there exists $c > 0$ such that $dd^c u_j + c\beta \geq 0$ for all j. Now $\beta \wedge T$ is a positive, finite Borel measure, and by the monotonicity of the sequence $\{u_j\}$, we have

$$\lim_{j\to\infty} \int |u_j - u|\, \beta \wedge T = \lim_{j,k\to\infty} \int |u_j - u_k|\, \beta \wedge T = 0.$$

It follows from the Lemma 3.3 that for $k \geq j$, we have

$$\begin{aligned} \mathcal{E}_T(u_k,u_k) &- \mathcal{E}_T(u_j,u_j) \\ &\geq \mathcal{E}_T(u_k,u_k) - \mathcal{E}_T(u_k,u_j) + \mathcal{E}_T(u_k,u_j) - \mathcal{E}_T(u_j,u_j) \\ &\geq -2c \int |u_k - u_j|\, \beta \wedge T \end{aligned}$$

Thus the sequence $|u_k|_T = \mathcal{E}_T(u_k,u_k)^{1/2}$ is essentially increasing. Since we have assumed that it is also bounded, we conclude $\lim_{k\to\infty} |u_k|_T$ exists and is finite.

Now we observe that

$$|u_j - u_k|_T^2 = \mathcal{E}_T(u_k,u_k) - 2\mathcal{E}_T(u_j,u_k) + \mathcal{E}_T(u_j,u_j)$$

If $k \geq j$, then $u_k \leq u_j$, so by the Lemma, we have $\mathcal{E}_T(u_j,u_j) \leq \mathcal{E}_T(u_j,u_k) \leq \mathcal{E}_T(u_k,u_k)$, modulo an error of size $2c\int |u_k - u_j|\beta \wedge T$. Thus

$$\lim_{j\to\infty} \mathcal{E}_T(u_j,u_j) = \lim_{j,k\to\infty} \mathcal{E}_T(u_j,u_k) = \lim_{k\to\infty} \mathcal{E}_T(u_k,u_k)$$

and so $\lim_{j,k\to\infty} |u_j - u_k|_T = 0$.

Now we show that (16) holds. Let $\{v_j\}$ be any regularizing sequence for u. Since v_j is smooth, there exists $k = k_j$ such that $u_{k_j} \leq v_j$. Thus by Lemma 3.3, $|v_j|_T$ is essentially bounded by $|u_{k_j}|_T$. From the first part of the proof, then, it follows that $\{v_j\}$ is Cauchy. □

Proposition 3.5. *Suppose that $u \in \tilde{P}(X,S)$ and that T is a positive closed $(1,1)$ current. Then $u \in L^1(T \wedge \beta)$.*

Proof. First recall from Jensen's inequality that $[\beta \wedge T](B_S(r)) \leq Cr^2$ for all $r \geq 0$. Since $u(x) \geq A \log \operatorname{dist}(x,S) + B$, we have that

$$[\beta \wedge T]\{|u| \geq t\} \leq [\beta \wedge T](B_S(e^{(-t+B)/A})) \leq C' e^{-2t/A}$$

for constants $C, C' > 0$ and all $t \geq 0$. Therefore,

$$\int |u|\, \beta \wedge T = \int_0^\infty [\beta \wedge T]\{|u| \geq t\}\, dt \leq C' \int_0^\infty r^{-2t/A}\, dt < \infty.$$

□

Theorem 3.6. *Suppose that $u \in \tilde{P}(X,S)$ and that T is a positive closed $(1,1)$ current with local potentials that are finite at each point in S. Then u satisfies (16), and so $|u|_T := \lim_{j\to\infty} |u_j|_T$ is well defined and finite; and*

$$|u|_T^2 = \int_{X-S} du \wedge d^c u \wedge T.$$

The expression $du \wedge T$ defines a current on $X - S$ which has finite mass. The (trivial) extension of $du \wedge T$ to X is equal to the current $d(uT)$, i.e., for all smooth 1-forms η we have

$$\langle d(uT), \eta \rangle = \int_{X \setminus S} \eta \wedge du \wedge T.$$

Proof. First we show that u satisfies (16). Choose a function $m \in C^\infty(\mathbf{R})$ that is convex, increasing and equal to $\max\{0,x\}$ outside a small neighborhood of 0. For all $j \geq 0$ let $u_j(p) = m(u(p)+j) - j \approx \max\{u, -j\}$. Clearly u_j is smooth and decreases to u pointwise on X. If $dd^c u \geq -c\beta$, then a quick computation using the fact that $m' \leq 1$ verifies that $dd^c u_j \geq -c\beta$, as well, with the same constant c. That is, u_j regularizes u. By Theorem 3.4 it suffices to show that $|u_j|_T$ is bounded.

Let $L \in C^\infty(X \setminus S)$ be a function satisfying $L \leq u$ and $L(z) = C(q) \log \|z\|$ with respect to local coordinates centered at each point $q \in S$. For each $j \in \mathbf{N}$, let $L_j(z) = m(L(p) + j) - j$. Then the above argument applied to L, L_j instead of u, u_j shows that $L \in L^1(\beta \wedge T)$ and that L_j regularizes L. Since $u_j \geq L_j$, we have from Lemma 3.3 that

$$|u_j|_T \leq |L_j|_T + C \int (u_j - L_j)\, \beta \wedge T \leq |L_j|_T + C \int (\|u\|_\infty - L)\, \beta \wedge T \leq |L_j|_T + C$$

for every $j \in \mathbf{N}$. Hence our problem reduces to showing that $\{|L_j|_T\}$ is bounded.

To do this, we can restrict attention to a coordinate neighborhood $B_0(2)$ centered at $q \in X$ and assume that $L(z) = C \log \|z\|$ in these coordinates. We choose a smooth, radially symmetric and compactly supported function $\chi : B_0(2) \to [0,1]$ such that $\chi \equiv 1$ on $B_0(1)$. We choose a local potential v for T on $B_0(2)$ and estimate

$$\int_{B_0(1)} dL_j(z) \wedge d^c L_j(z) \wedge T \leq \int \chi\, dL_j(z) \wedge d^c L_j(z) \wedge dd^c v.$$

Integrating by parts, we see that the right side is dominated by

$$\begin{aligned} &\left|\int v\, dd^c\chi \wedge dL_j \wedge d^cL_j\right| + 2\left|\int v\, d\chi \wedge d^cL_j \wedge dd^cL_j\right| + \left|\int \chi v\, dd^cL_j \wedge dd^cL_j\right| \\ = &\left|\int v\, dd^c\chi \wedge dL \wedge d^cL\right| + 2\left|\int v\, d\chi \wedge d^cL \wedge dd^cL\right| + \left|\int \chi v\, dd^cL_j \wedge dd^cL_j\right| \\ &= C + \int (-v)\chi\, dd^cL_j \wedge dd^cL_j \end{aligned}$$

for j large enough that $L_j = L$ on $\operatorname{supp} d\chi$. The measures $\chi(dd^cL_j)^2$ are radially symmetric and converging to a point mass at the origin as $j \to \infty$. Since v is subharmonic, we obtain that

$$-\lim_{j\to\infty}\int \chi v\, dd^cL_j \wedge dd^cL_j = -v(0) < \infty.$$

Thus $\{|L_j|_T\}$ is bounded, and (16) holds by Theorem 3.4.

The sequence $\{du_j\}$ is bounded in $L^2_T(X-S)$ and thus has a weak limit in $L^2_T(X-S)$, which must be du. Further, we must have $\int_{X-S} du \wedge d^cu \wedge T \le \lim_{j\to\infty} |u_j|_T = |u|_T$. It follows that $du \wedge T$ defines a current on $X-S$ of finite total mass, and the trivial extension of this current to X is $d(uT)$. This proves the second equation in the statement of the Theorem.

Finally, in order to prove the first equation in the statement of the Theorem it suffices to show that for $x \in S$ we may choose r small so that the integral $\int_{B_x(r)} du_j \wedge d^cu_j \wedge T$ is arbitrarily small, uniformly in j. Arguing as above, we may assume that $x = 0$ and replace u_j by the logarithm $L(z)$, so it suffices to show that $\int_{B_x(r)} du_j \wedge d^cu_j \wedge T$ is uniformly small in j. In local coordinates near $x = 0$, we have $dL \wedge d^cL \le c||z||^{-2}\beta$. We may identify the Laplacian of the local potential v for T as the measure $\Delta v = T \wedge \beta$. Since $-||z||^{-2}$ is the Newtonian potential on $\mathbf{R}^4$, we may assume that v is given by convolution: $v(0) = -v \star ||z||^{-2}(0) = -\int ||z||^{-2}\Delta v$. Since $v(0)$ is finite, it follows that $||z||^{-2}$ is integrable with respect to the measure Δv. Thus the integral over $B_x(r)$ may be taken uniformly small in j. □

Corollary 3.7. *Let $u \in \tilde{P}(X,S)$ where $S \cap I(f^{-1}) = \emptyset$, and T be a positive closed $(1,1)$ current with local potentials that are finite at each point in $I(f) \cup f^{-1}(S)$. Then $|f^*u|_T$ is well-defined, and*

- $\int_{X-S} du \wedge d^cu \wedge f_*T < \infty$
- $|f^*u|^2_T = \int_{X-(I(f)\cup f^{-1}S)} d(f^*u) \wedge d^c(f^*u) \wedge T < \infty$
- $|u|_{f_*T} = |f^*u|_T$.

Proof. The hypotheses imply that f_*T does not charge $\mathcal{C}(f^{-1}) = f(I(f))$ and that f_*T has local potentials that are finite at each point $p \in S \subset f(f^{-1}(S))$. The function f^*u is a difference of elements of $\tilde{P}(X, f^{-1}(S) \cup I(f))$ by Proposition 1.3. So by Theorem 3.6, the integrals defining $|u|_{f_*T}$ and $|f^*u|_T$ are finite. We compute

$$\begin{aligned} |u|^2_{f_*T} &= \int_{X-I(f^{-1})-\mathcal{C}(f^{-1})} du \wedge d^cu \wedge f_*T \\ &= \int_{X-I(f)-\mathcal{C}(f)} d(u\circ f) \wedge d^c(u\circ f) \wedge T \\ &= \int_{X-I(f)} d(u\circ f) \wedge d^c(u\circ f) \wedge T = |f^*u|^2_T. \end{aligned}$$

The first equality holds because f_*T charges neither points nor $\mathcal{C}(f^{-1})$. The second equality follows by the change of variables formula because $f : X - I(f) - \mathcal{C}(f) \to X - I(f^{-1}) - \mathcal{C}(f^{-1})$ is a biholomorphism. The third equality is a consequence of the fact that $u \circ f$ is constant on $\mathcal{C}(f) - I(f)$. Finally, the fourth equality holds because T does not charge points. □

4. Invariant Measure

Up to this point, we have required that conditions (1) and (8) hold. We will now impose two further conditions. The first of these is:

$$\langle\theta^+, f(x)\rangle > 0 \text{ for every } x \in I(f), \text{ and } \langle\theta^-, f^{-1}(y)\rangle > 0 \text{ for every } y \in I(f^{-1}). \tag{18}$$

Like condition (1), condition (18) may be thought of as a property of the underlying space X used to represent the map f; it will be shown in Proposition 4.1 that (18) may always be assumed to hold. Next we consider condition (3) more carefully (Theorems 4.3 and 4.6). After this, we will assume for the rest of the paper that (3) holds, by which we mean implicitly that (1), (3), (8) and (18) all hold. The main results of this section are that if (3) holds, then the expression μ in (17) is well defined (Theorem 4.7) and invariant (Theorem 4.11).

Proposition 4.1. *If $f : X \to X$ satisfies (1), we may blow down curves in X if necessary so that both (1) and (18) hold.*

Proof. Suppose to the contrary that $\langle\theta^+, f(x)\rangle \le 0$ for some $x \in I$. Then since θ^+ is nef, we have $\langle\theta^+, V\rangle = 0$ for every component $V \subset f(x)$. From this and the Hodge index theorem on surfaces, we see that either $\langle\theta^+, \theta^+\rangle = 0$ or the intersection form is negative definite on $f(x)$. In the first case [DF, Theorem 0.4] guarantees that after blowing down an appropriate curve, f conjugates to an automorphism and satisfies the conclusion of the proposition vacuously.

In the second case, we note that $\pi_1^{-1}(x)$ constitutes a single connected component of $\mathcal{C}(\pi_1)$. We can therefore apply the argument of [DF, Proposition 1.7] to obtain a smooth rational curve of self-intersection -1 in $f(x)$. After blowing this curve down, (1) still holds. However the dimension of $H^{1,1}(X)$ drops by one. If on the new surface we still have $\langle\theta^\pm, f(x)\rangle = 0$ for some $x \in I(f^{\pm 1})$, then we can repeat this process. This cannot happen more than $\dim H^{1,1,}(X)$ times, so eventually we will descend to a surface on which f satisfies both (1) and (18). □

The following is a companion to Corollary 2.7.

Corollary 4.2. *If f satisfies (18), then $\nu(\mu^+, x) > 0$ for $x \in I(f^n)$, $n \ge 1$.*

Proof. The proof uses the fact that $\mu^+ = \rho^{-n} f^{n*}\mu^+$, but it is otherwise identical to the proof of the second conclusion in Proposition 1.2. □

Theorem 4.3. *Suppose that (1), (8), and (18) hold. Then (3) holds if and only if the function g^+ defined in (15) is finite at each point of $I(f^{-1})$. More generally,* (3) *implies that*

$$\lim_{n\to\infty} (g_n^+\eta)(p) = \left\langle \eta, \theta^- \right\rangle g^+(p) \tag{19}$$

for every smooth $(1,1)$ *form η and every $p \in I(f^{-1})$.*

Proof. Remarking that $f^*\omega^+ = \rho\omega^+ + dd^c\gamma^+$, we see from Proposition 1.3 and Corollary 4.2 that

$$A \log \operatorname{dist}(x, I(f)) - B \le \gamma^+(x) \le A' \log \operatorname{dist}(x, I(f)) + B'.$$

Replacing x by $f^j x$ and summing we see that g^+ is bounded above and below by infinite sums of the form $S := A \sum \rho^{-n} \log \operatorname{dist}(f^n x, I(f)) + B$. Thus $g^+(x) > -\infty$ if and only if S is finite. Since $I(f^{-1})$ is a finite set, we have

$$\operatorname{dist}(f^n I(f^{-1}), I(f)) = \min_{x\in I(f^{-1})} \operatorname{dist}(f^j x, I(f)),$$

and we see that (3) holds if and only if S is finite for all $x \in I(f^{-1})$.

Equation (19) is established using the same argument together with Theorem 2.3 and equations (10) and (11). □

The following Theorem says that condition (3) is symmetric in f and f^{-1}; we refer the reader to [Dil1, Theorem 5.2] for a proof.

Theorem 4.4. *Suppose that (1), (8), and (18) hold. Then (3) holds if and only if*

$$\sum_{j=0}^{\infty} \frac{\log \operatorname{dist}(f^{-j}(I(f)), I(f^{-1}))}{\rho^j} < \infty. \tag{20}$$

Another important thing about condition (3) is that it tends to be satisfied by "most" birational maps. The referee pointed out the following result in this direction.

Proposition 4.5. *Let $f : \mathbf{P}^2 \to \mathbf{P}^2$ be a birational map satisfying* (1) *and* (3). *Then for all $A \in PGL(3, \mathbf{C})$ outside a pluripolar set, the map $A \circ f$ satisfies* (1) *and* (3).

Proof. $H^{1,1}(\mathbf{P}^2)$ is one dimensional, so the spectral radius ρ of f^* must be a positive integer wherever (1) is satisfied, and we may suppose that $\omega^+ = \beta$ is the Fubini-Study Kähler form. For each $n \in \mathbf{N}$, the condition $f^n I(f^{-1}) \cap f^{-n} I(f) \neq \emptyset$ is algebraic, so (1) is satisfied outside a countable union of subvarieties of $PGL(3, \mathbf{C})$. Since the complement of these subvarieties is connected and ρ varies continuously, it follows that ρ is constant wherever (1) holds.

Let $\pi : \mathbf{C}^3 - \{0\} \to \mathbf{P}^2$ be the holomorphic map sending complex lines through the origin onto points in $\mathbf{P}^2$. The map f lifts to a homogeneous map $F : \mathbf{C}^3 \to \mathbf{C}^3$, and $\pi^* \omega^+ = dd^c \log \|z\|$. Hence

$$\pi^*(A \circ f)^{n*} \omega^+ = dd^c \log \|(A \circ F)^n(z)\|$$

for all $n \geq 0$. The functions $\log \|(A \circ F)^n(z)\|$ are plurisubharmonic in both A and z, and on each compact subset of $PGL(3, \mathbf{C})$, they are essentially decreasing in n. We let G denote their limit. For each fixed A such that $A \circ f$ satisfies (1), we have that $G(\cdot, A)$ is a potential for the lift $\pi^* \mu^+$ of the invariant current μ^+ corresponding to $A \circ f$. In particular $G = \lim G_n$ is not identically $-\infty$ on $\mathbf{C}^3 \times PGL(3, \mathbf{C})$ and must rather be plurisubharmonic.

The map $A \circ f$ satisfies condition (3) if and only if $G(z) \neq -\infty$ for $\pi(z) \in I(f^{-1})$. On the other hand,

$$\{(z, A) \in \mathbf{C}^3 \times PGL(3, \mathbf{C}) : \pi(z) \in I((A \circ f)^{-1})\},$$

is an analytic variety, and on each of its irreducible components, G must be finite outside a pluripolar set or identically $-\infty$. Each component contains a point of the form (z, id) where $z \neq 0$, so our hypothesis on f implies that the latter possibility does not occur. That is, (3) is satisfied by $A \circ f$ for all A outside a pluripolar set. □

Proposition 4.6. *If (3) holds, then there exists a constant $C > 0$ such that for each $\eta \in \Omega$ and every $n \in \mathbf{N}$*

$$|f^{n*} \gamma^+ \eta|_{\beta + \mu^-} \leq C \rho^{n/2} \|\eta\| .$$

Proof. The function $\gamma^+ \eta \circ f^n$ is a difference of functions in $\tilde{P}(X, I(f^n))$ by Theorem 2.3, and $f^n_* \beta$ is smooth away from $I(f^{-n})$, so by Corollary 3.7 $f^{n*} \gamma^+ \eta$ is a difference of functions which satisfy condition (16) for $T = f^m_* \beta$. The condition (3) together with Proposition 4.3 imply that $f^{n*} \gamma^+ \eta$ is also a difference of functions satisfying (16) for $T = \mu^-$.

We have

$$|f^{n*} \gamma^+ \eta|_{\mu^-} = |\gamma^+ \eta|_{f^n_* \mu^-} = \rho^{n/2} |\gamma^+ \eta|_{\mu^-} \leq C \rho^{n/2} \|\eta\| .$$

The first equality uses Corollary 3.7, the second proceeds from invariance of μ, and the inequality follows from writing η as a linear combination of the basis elements $\omega_1, \dots, \omega_n \in \Omega$ and noting that $|\gamma^+ \omega_j|_{\mu^-} < \infty$ for each j.

Corollary 3.7 also gives

$$|f^{n*} \gamma^+ \eta|_\beta = \rho^{n/2} |\gamma^+ \eta|_{\rho^{-n} f^n_* \beta} \leq \rho^{n/2} ||\eta|| \max_k |\gamma^+ \omega_k|_{\rho^{-n} f^n_* \beta}.$$

It will suffice to show that $|\gamma^+ \omega_k|_{\rho^{-n} f^n_* \beta}$ is bounded. Using the notation from §2, we write

$$\rho^{-n} f^n_* \beta = \rho^{-n} \omega f^n_* \beta + dd^c g^-_n \beta.$$

Hence

$$|\gamma^+ \omega_k|^2_{\rho^{-n} f^n_* \beta} \leq |\gamma^+ \omega_k|^2_{\rho^{-n} \omega f^n_* \beta} + |\gamma^+ \omega_k|^2_{dd^c g^-_n \beta}. \tag{21}$$

The sequence $\{\rho^{-n} \omega f^n_* \beta : n = 1, 2, 3, \dots\} \subset \Omega$ converges uniformly to $c\omega^-$, so the first term on the right hand side is bounded.

Before we analyze the second term, we make some observations. Since $\gamma^+ \omega_k \in P$ is smooth except for logarithmic singularities at $I(f)$, the wedge product yields a well-defined measure (see [Dem2])

$$(dd^c \gamma^+ \omega_k)^2 = \sum_{x \in I(f)} c_x \delta_x + (dd^c \gamma^+ \omega_k)^2|_{X - I(f)}.$$

We claim that the integral $\int(g_n^-\beta)(dd^c\gamma^+\omega_k)^2$ is uniformly bounded in n if (3) holds. To see this we use the projections $\pi_1,\pi_2:\Gamma\to X$ and the formula $f^*\omega_k=\pi_{1*}\pi_2^*\omega_k$. Since $\pi_1:\Gamma-\mathcal{C}(\pi_1)\to X-I(f)$ is biholomorphic, we may pull back by to Γ, obtaining

$$\int_{X-I(f)}(g_n^-\beta)(dd^c\gamma^+\omega_k)^2=\int_{X-I(f)}(g_n^-\beta)(f^*\omega-\omega f^*\omega_k)^2=\int_\Gamma\pi_1^*(g_n^-\beta)(\pi_2^*\omega_k-\pi_1^*\omega f^*\omega_k)^2.$$

Now $\pi_2^*\omega_k-\pi_1^*\omega f^*\omega_k$ is a smooth (1,1)-form on Γ. By Theorems 2.3 and 2.5, $\{g_n^-\beta\}_{n\in\mathbf{N}}$ is uniformly bounded above and converges in L^1 to g^-. In particular, no subsequence of $\{\pi^*(g_n^-\beta)\}$ converges uniformly to $-\infty$. Also, since β is positive, $dd^c\pi^*(g_n^-\beta)\geq -C\pi^*\beta$ for some $C>0$ and all $n\in\mathbf{N}$. Therefore, a standard result (see [Hör], Theorem 4.1.9) concerning sequences of subharmonic functions shows that the integral on the right side of the previous display is bounded independent of n. On the indeterminacy set itself, we have

$$\int_{I(f)}(g_n^-\beta)(dd^c\gamma^+\omega_k)^2=\sum_{x\in I(f)}c_x(g_n^-\beta)(x)$$

is uniformly bounded by Theorems 4.3 and 4.4.

We now claim that

$$|\gamma^+\omega_k|^2_{dd^cg_n^-\beta}=\int d(\gamma^+\omega_k)\wedge d^c(\gamma^+\omega_k)\wedge dd^cg_n^-\beta=-\int(g_n^-\beta)(dd^c\gamma^+\omega_k)^2.$$

In light of the previous paragraph this implies that the second term in (21) is bounded. To see that the claim is true, replace $\gamma^+\omega_k$ by a regularizing sequence γ_j, which coincides with $\gamma^+\omega_k$ outside neighborhood of $I(f)$. Performing two integrations by parts, we have

$$|\gamma_j|_{dd^cg_n^-\beta}=-\int(g_n^-\beta)(dd^c\gamma_j)^2.$$

As $j\to\infty$, the measures $(dd^c\gamma_j)^2$ converge weakly to $(dd^c\gamma^+\omega_k)^2$. Further, since $g_n^-\beta$ is continuous in a neighborhood of $I(f)$, we may assume that it is continuous on the set where $\gamma_j\neq\gamma^+\omega_k$. Letting $j\to\infty$ therefore establishes our claim. □

Now we sharpen Theorem 2.5.

Theorem 4.7. *If (3) holds, then $g^+\in L^1(\mu^-\wedge\beta)$, and g^+ satisfies (16) for $T=\beta+\mu^-$. In fact, if η is any smooth closed (1,1) form on X, then*

$$\lim_{n\to\infty}|g_n^+\eta-\langle\eta,\theta^-\rangle\,g^+|_{\beta+\mu^-}=0.$$

Proof. First we will show that for every smooth, closed (1,1) form η, the sequence $\{g_n^+\eta\}$ is Cauchy with respect to $|\cdot|_{\beta+\mu^-}$. We consider increasingly general forms η. If $\eta=\omega^+$, it follows from (15) and Proposition 4.5 that $\{g_n^+\omega^+\}$ is Cauchy. Now for $\eta\in\Omega$, we write $\eta=c\omega^++\eta^\perp$ as in (10), where $c=\langle\eta,\theta^-\rangle$. By Theorem 2.1, we have $||\omega f^{j*}\eta^\perp||\leq C_tt^j||\eta^\perp||$ for any t greater than 1. By Proposition 4.5 again, we have

$$|\rho^{-n}f^{(n-j-1)*}\gamma^+f^{j*}\eta^\perp|_{\beta+\mu^-}\leq C\rho^{-n}\rho^{(n-j-1)/2}t^j.$$

Choosing $1<t<\sqrt{\rho}$, we see that from this estimate and (13) that $|g_n^+\eta^\perp|_{\beta+\mu^-}$ converges to zero. Since $g_n^+\eta$ is linear in η, we see that $g_n^+\eta$ satisfies (16) with $T=\beta+\mu^-$. Finally, for a general, smooth, closed (1,1) form η, we see that $g_n^+\eta$ differs from $g_n^+\omega\eta$ by $\rho^{-n}f^{n*}p\eta$. The seminorm of this additional term is

$$|\rho^{-n}f^{n*}p\eta|^2_T=\rho^{-2n}|p\eta|^2_{f^n_*T}=\rho^{-n}|p\eta|^2_{\rho^{-n}f^n_*T}.$$

Since $p\eta$ is smooth, and since $\rho^{-n}f^n_*T$ converges to $c\mu^-$, it follows that $|p\eta|_{\rho^{-n}f^n_*T}$ converges to $|p\eta|_{c\mu^-}$. Thus $\rho^{-n}f^{n*}p\eta$ converges to zero, and we conclude that $\{g_n^+\eta\}$ is Cauchy.

Next we show that g^+ satisfies (16) with respect to the current $T=\beta+\mu^-$. We have from Theorems 2.4 and 2.6 that $g_n^+\to g^+$ in $L^1(X)$ and that $g^+\in P$. The sequence $\{g_n^+\beta\}$ is uniformly bounded above everywhere on X by Theorem 2.6. Since $dd^cg_n^+\beta=\rho^{-n}f^{n*}\beta-\rho^{-n}\omega f^{n*}\beta$, and $\rho^{-n}\omega f^{n*}\beta$ converges uniformly to ω^+, we also have that $dd^cg_n^+\beta\geq -c\beta$ for some constant c and all $n\in\mathbf{N}$. It follows (see [Hör], Theorem 4.1.9) that $\overline{\lim}g_n^+\beta\leq g^+$ everywhere on X. Indeed, if $\{u_n\}$ is a regularizing sequence for g^+, then we can pass to a subsequence and assume that $u_n \geq g_n^+$ for all $n \in \mathbf{N}$ large enough. As in the proof of Theorem 3.4, we can conclude that

$$|u_n|_{\beta+\mu^-} \leq |g_n^+|_{\beta+\mu^-} + C$$

for some constant C independent of n. By the first part of the proof the sequence of energies $\{|g_n^+|_{\beta+\mu^-}\}$ converges, so we conclude that $\{|u_n|_{\beta+\mu^-}\}$ is bounded. Theorem 3.4 therefore implies that the limiting function g^+ satisfies (16).

Theorem 4.4 and the paragraph above tell us that g^- satisfies (16) with $T = \mu^+ + \beta$. So if we set $v = g^+$, $u = g^-$, and $T = \beta$ in Proposition 3.2, then it follows that $g^+ \in L^1(\mu^- \wedge \beta)$. □

Corollary 4.8. *If (3) holds, then the expression (17) defines a probability measure μ such that $g^\pm \in L^1(\mu)$.*

Proof. This follows from Proposition 3.2 with $v = u = g^+$ and $T = \mu^-$. □

Together with Theorem 4.4, the next corollary shows among other things that the roles of g^- and g^+ can be reversed without changing the measure μ. It is also the principle place in which we use the fact that $g^\pm$ satisfies (16) with $T = \beta$.

Corollary 4.9. *Suppose that* (3) *holds. Then for any smooth closed* $(1,1)$ *forms η_1, η_2 on X, we have*

$$\lim_{n,m\to\infty} \frac{f^{n*}\eta_1}{\rho^n} \wedge \frac{f_*^m \eta_2}{\rho^m} = \mu \cdot \left\langle \eta_1, \theta^- \right\rangle \left\langle \eta_2, \theta^+ \right\rangle,$$

in the weak topology on Borel measures.*

Proof. Without loss of generality, we may assume that $\langle \eta_1, \theta^- \rangle = \langle \eta_2, \theta^+ \rangle = 1$. We write

$$\mu_n^+ := \rho^{-n} f^{n*}\eta_1 = \omega_n^+ + dd^c \tilde{g}_n^+$$

where $\omega_n^+ = \omega\mu_n^+$ and $dd^c\tilde{g}_n^+ = \mu_n^+ - \omega_n^+$. We similarly define μ_m^-, $\tilde{g}_m^-$ and ω_n^-. Let $\psi : X \to \mathbf{R}$ be a smooth test function. We will that show that

$$\int \psi\, \mu_n^+ \wedge \mu_m^- - \mu = \int \psi\, \mu_n^+ \wedge (\mu_m^- - \mu^-) + \int \psi\, (\mu_n^+ - \mu^+) \wedge \mu^-$$

is small for n, m large by dealing with each term on the right side separately.

The second term is easier, because it depends only on n. Since $\langle \eta_1, \theta^- \rangle = 1$, it follows that ω_n^+ converges uniformly to ω^+. Theorem 2.5 tells us, moreover, that $\tilde{g}_n^+$ converges in L^1 to g^+, and Theorem 4.7 that $\tilde{g}_n^+ \to g^+$ with respect to the energy semi-norm $|\cdot|_{\beta+\mu^-}$. Thus,

$$\int \psi\, (\mu_n^+ - \mu^+) \wedge \mu^- = \int \psi\, (\omega_n^+ - \omega^+) \wedge \mu^- - \int d\psi \wedge d^c(\tilde{g}_n^+ - g^+) \wedge \mu^- \to 0$$

as $n \to \infty$.

Turning to the first term, we have

$$\left| \int \psi\, \mu_n^+ \wedge (\mu_m^- - \mu^-) \right| \leq \left| \int \psi\, \mu_n^+ \wedge (\omega_m^- - \omega^-) \right| + \left| \int \psi\, \omega_n^+ \wedge (\mu_m^- - \mu^-) \right| + \left| \int \psi\, dd^c\tilde{g}_n^+ \wedge dd^c(\tilde{g}_m^- - g^-) \right|.$$

The first two terms on the right side of the inequality tend to zero as n and m increase, because one factor in each integrand converges uniformly whereas the other tends weakly to zero. As for the third term on the right, we have

$$\left| \int \psi\, dd^c\tilde{g}_n^+ \wedge dd^c(\tilde{g}_m^- - g^-) \right| = \left| \int dd^c\psi \wedge d\tilde{g}_n^+ \wedge d^c(\tilde{g}_m^- - g^-) \right| \leq |\psi|_{C^2} |\tilde{g}_n^+|_\beta |\tilde{g}_m^- - g^-|_\beta \to 0$$

as $m \to \infty$, because $|\tilde{g}_n^+|_\beta$ is bounded uniformly in n and $|\tilde{g}_m^- - g^-|_\beta \to 0$. □

Theorem 4.10. *μ does not charge points.*

Proof. By Corollary 2.7 there is no point in X at which both μ^+ and μ^- have positive Lelong number. So given $x \in X$ we may assume that $\nu(\mu^-, x) = 0$. Choose a local coordinate system such that $x = 0$ and U is the unit ball, and choose a smooth, compactly supported function $\chi : U \to [0, 1]$ such that $\chi(z) = 1$ for $\|z\|$ small enough. Set $\chi_r(z) = \chi(r^{-1}z)$ for all $0 < r < 1$. We have

$$\begin{aligned}\mu(x) &\leq \int \chi_r\,\mu \leq \left|\int \chi_r\,\omega^+ \wedge \mu^-\right| + \left|\langle d\chi_r, d^c(g^+ \wedge \mu^-)\rangle\right| \\ &\leq C_1 r^2 + C_2|\chi_r|_{\mu^-} \leq C_1 r^2 + \frac{C_2}{r^2}\int_{B_x(r)} \beta \wedge \mu^-.\end{aligned}$$

The last expression on the right converges to a multiple of $\nu(\mu^-, x) = 0$ as $r \to 0$. □

Since μ does not charge $I(f)$, its pushforward $f_*\mu$ is well-defined.

Theorem 4.11. *μ is f-invariant.*

Proof. Let $\psi : X \to \mathbf{R}$ be a continuous function. Then by definition and Corollary 4.9

$$\int \psi\, f_*\mu = \int (\psi \circ f)\,\mu = \lim_{n\to\infty} \int (\psi \circ f)\,\frac{f^{n*}\beta}{\rho^n} \wedge \frac{f^n_*\beta}{\rho^n}.$$

The measure $f^{n*}\beta \wedge f^n_*\beta$ does not charge points because there is no point in X at which both factors are singular. Both factors are smooth at points outside a finite set, so this measure does not charge curves either. Thus

$$\begin{aligned}\int_X (\psi \circ f)\,\frac{f^{n*}\beta}{\rho^n} \wedge \frac{f^n_*\beta}{\rho^n} &= \int_{X-(\mathcal{C}(f)\cup I(f))} \psi \circ f\,\frac{f^{n*}\beta}{\rho^n} \wedge \frac{f^n_*\beta}{\rho^n} \\ &= \int_{X-(\mathcal{C}(f^{-1})\cup I(f^{-1}))} \psi\,\frac{f^{(n-1)*}\beta}{\rho^{n-1}} \wedge \frac{f^{n+1}_*\beta}{\rho^{n+1}} \\ &= \int_X \psi\,\frac{f^{(n-1)*}\beta}{\rho^{n-1}} \wedge \frac{f^{n+1}_*\beta}{\rho^{n+1}},\end{aligned}$$

where second equality holds because $f : X - \mathcal{C}(f) - I(f) \to X - \mathcal{C}(f^{-1}) - I(f^{-1})$ is a biholomorphism. If we let $n \to \infty$, then the last expression on the right converges to $\int \psi\mu$. We conclude that $f_*\mu = \mu$. □

Corollary 4.12. *μ does not charge $\mathcal{C}(f)$ or $\mathcal{C}(f^{-1})$.*

Proof. We already know that $\mu(I(f)) = 0$. So by invariance,

$$\mu(\mathcal{C}(f)) = \mu(\mathcal{C}(f) - I(f)) = \mu(I(f^{-1})) = 0.$$

Similarly, $\mu(\mathcal{C}(f^{-1})) = 0$. □

5. Mixing

Let T be a positive, closed (1,1) current, and let J denote the operator on 1-forms induced by the complex structure. For a smooth 1-form η, $\eta \wedge J\eta$ is a positive (1,1)-form, and we define the $L^2(T)$ seminorm

$$||\eta||^2_{L^2(T)} := \int \eta \wedge J\eta \wedge T.$$

If $\varphi \in L^1(T \wedge \beta)$, then we may define the quantity

$$|\varphi|^\natural_T := \sup\{\int \varphi \wedge d\eta \wedge T : \eta \text{ a smooth 1-form with } ||\eta||_{L^2(T)} \leq 1\};$$

$|\cdot|^\natural_T$ is a seminorm on the space $\{\varphi \in L^1(T \wedge \beta) : |\varphi|^\natural_T < \infty\}$. If φ is smooth, then we may take $\eta = d^c\varphi$ and integrate by parts in the integral defining $|\varphi|^\natural_T$ and apply Cauchy's inequality to see that $|\varphi|^\natural_T = |\varphi|_T$. Thus $|\cdot|^\natural_T$ extends $|\cdot|_T$ from the space of smooth functions to a larger space.

Lemma 5.1. *Suppose that (3) holds. If $\psi : X \to \mathbf{R}$ is a smooth function, then*

$$|\psi \circ f^n|^\natural_{\mu^+} \leq \rho^{-n/2}|\psi|_{\mu^+}.$$

Proof. Recall that $f^n : X-(\mathcal{C}(f^n)\cup I(f^n)) \to X-(\mathcal{C}(f^{-n})\cup I(f^{-n}))$ is biholomorphic. The function $f^{n*}\psi$ is smooth and bounded on $X-I(f^n)$, and $\beta\wedge\mu^+$ puts no mass on $I(f^n)$, so $f^{n*}\psi \in L^1(\beta\wedge\mu^+)$. Since $\beta\wedge\mu^+$ also puts no mass on $\mathcal{C}(f^n)$, it follows that if η is a smooth 1-form, then

$$-\int (f^{n*}\psi)d\eta\wedge\mu^+ = -\int_{X-(\mathcal{C}(f^n)\cup I(f^n))}(f^{n*}\psi)d\eta\wedge\mu^+ = -\int_{X-(\mathcal{C}(f^n)\cup I(f^n))} f^{n*}(\psi\frac{\mu^+}{\rho^n})\wedge d\eta.$$

Now let Γ denote the (desingularized) graph of f^n, and let $\pi_1,\pi_2:\Gamma\to X$ be the associated projections. Thus $f^n=\pi_2\pi_1^{-1}$, and $f^{n*}=\pi_{1*}\pi_2^*$. Now $\pi_1:\Gamma-\mathcal{C}(\pi_1)\to X-I(f^n)$ is a biholomorphism, so we may pull this last integral back under π_1^* to obtain an integral over Γ:

$$= -\int_{\Gamma-\mathcal{C}(\pi_1)}\pi_2^*(\psi\wedge\rho^{-n}\mu^+)\wedge\pi_1^*(d\eta) = -\int_{\Gamma-\mathcal{C}(\pi_1)}\pi_2^*(\psi)\wedge\pi_1^*(d\eta)\wedge\pi_2^*(\rho^{-n}\mu^+).$$

Now $\pi_2^*\psi$ and $\pi_1^*(d\eta)$ are smooth on Γ, and $\pi_2^*\psi$ and $\pi_1^*(d\eta)$ are smooth on Γ, and $\pi_2^*(\rho^{-n}\mu^+)$ puts no mass on $\mathcal{C}(\pi_1)=\pi_1^{-1}(I(f^n))$, so we obtain

$$= -\int_\Gamma \pi_2^*(\psi)\wedge\pi_1^*(d\eta)\wedge\pi_2^*\left(\rho^{-n}\mu^+\right) = \int_\Gamma \pi_2^*(d\psi)\wedge\pi_1^*(\eta)\wedge\pi_2^*\left(\rho^{-n}\mu^+\right).$$

Applying the Schwarz inequality to this last term, we have

$$\left|\int (f^{n*}\psi)\wedge d\eta\wedge\mu^+\right| \le \left(\int_\Gamma \pi_2^*(d\psi)\wedge\pi_2^*(d^c\psi)\wedge\pi_2^*\left(\rho^{-n}\mu^+\right)\right)^{\frac12}\left(\int_\Gamma \pi_1^*(\eta)\wedge J\pi_1^*(\eta)\wedge\pi_2^*\left(\rho^{-n}\mu^+\right)\right)^{\frac12}.$$

Again, since $\pi_2^*(d\psi)$ and $\pi_1^*(\eta)$ are smooth, there is no change if we integrate over just $\Gamma-\mathcal{C}(\pi_1)$ or $\Gamma-\mathcal{C}(\pi_2)$. Pushing the last integral forward under π_{1*} and the first integral forward under π_{2*}, we obtain

$$\left|\int (f^{n*}\psi)\wedge d\eta\wedge\mu^+\right| \le \left(\int_{X-I(f^{-n})} d\psi\wedge d^c\psi\wedge\rho^{-n}\mu^+\right)^{\frac12}\left(\int \eta\wedge J\eta\wedge f^{n*}\left(\rho^{-n}\mu^+\right)\right)^{\frac12} =$$

$$= \rho^{-n/2}|\psi|_{\mu^+}||\eta||_{L^2(\mu^+)}.$$

Now we take the supremum over η with $||\eta||_{L^2(\mu^+)}\le 1$, and the Lemma follows. □

Theorem 5.2. *Let $\psi : X\to\mathbf{R}$ be a smooth function. Then*

$$\lim_{n\to\infty}(\psi\circ f^n)\mu^+ = \mu^+\cdot\int\psi\,\mu.$$

Proof. We assume without loss of generality that $0\le\psi\le 1$ so that $0\le(\psi\circ f^n)\mu^+\le\mu^+$. Thus every subsequence of $\{(\psi\circ f^n)\mu^+\}$ has a sub-subsequence converging to a positive $(1,1)$ current S dominated by μ^+. To see that S is closed, let η be any smooth 1-form,

$$|\langle\eta, dS\rangle| = \lim_{n_j\to\infty}|\langle -d\eta,(\psi\circ f^{n_j})\mu^+\rangle| \le ||\eta||_{L^2(\mu^+)}\,|\psi\circ f^n|^\natural_{\mu^+}.$$

The right hand side tends to zero by Lemma 5.1. The current μ^+ is extremal in its cohomology class [Gue, Theorem 5.1], so $S=c\mu^+$. We will conclude the proof by showing that $c=\int\psi\mu$.

Let $\pi_1,\pi_2:\Gamma\to X$ be as in the proof of Lemma 5.1, so $f^{n*}=\pi_{1*}\pi_2^*$. Since β is smooth and μ^+ puts no mass on $\mathcal{C}(f^{-1})\cup\mathcal{C}(f)$, we may pull integrals up from X to Γ and push them back down as before:

$$\begin{aligned}\int_X(\psi\circ f^n)\,\mu^+\wedge\beta &= \int_X (f^{n*}\psi)\wedge\left(\rho^{-n}f^{n*}\mu^+\right)\wedge\beta = \int_\Gamma(\pi_1^*\psi)\wedge\left(\rho^{-n}\pi_1^*\mu^+\right)\wedge\pi_2^*\beta\\ &= \int_X\psi\wedge\pi_{1*}\left(\pi_2^*\beta\wedge\frac{\pi_1^*\mu^+}{\rho^n}\right) = \int_X\psi\,\frac{f^n_*\beta}{\rho^n}\wedge\mu^+.\end{aligned}$$

Note that the second and fourth equalities depend on the fact that none of the measures involved charge curves. Finally,

$$c\int\mu^+\wedge\beta = \int S\wedge\beta = \lim_{n\to\infty}\psi\circ f^n\,\mu^+\wedge\beta = \left(\int\psi\,\mu\right)\langle\beta,\theta^+\rangle = \left(\int\psi\,\mu\right)\int\mu^+\wedge\beta.$$

The third equality follows from the previous display and Corollary 4.9; the fourth follows from the fact that θ^+ is the cohomology class of μ^+. □

Theorem 5.3. *The measure μ is mixing for f.*

Proof. We show that for any smooth functions φ and ψ we have

$$\lim_{n\to\infty}\int(\psi\circ f^n)\cdot\varphi\,\mu=\int\psi\,\mu\;\cdot\int\varphi\,\mu.$$

Let $\{g_j^-\}$ be a regularizing sequence for g^-. By Theorem 5.2, we have

$$\lim_{n\to\infty}\int(\psi\circ f^n)\mu^+\wedge\varphi\wedge(\omega^-+dd^cg_j^-)=\int\varphi\mu^+\wedge(\omega^-+dd^cg_j^-)\cdot\int\psi\,\mu. \tag{22}$$

By Theorem 4.7, $\mu^+\wedge(\omega^-+dd^cg_j^-)$ converges to μ as $j\to\infty$. Thus it suffices to show that we may interchange the limits $n\to\infty$ and $j\to\infty$ in the left hand integral in (22). For this, it suffices to show that

$$\lim_{j,k\to\infty}\int(\psi\circ f^n)\varphi\mu^+\wedge dd^c(g_j^--g_k^-)=0 \tag{23}$$

holds uniformly in n. Integration by parts gives

$$\int(\psi\circ f^n)\varphi\mu^+\wedge dd^c(g_j^--g_k^-)=$$

$$=-\int\varphi\wedge d(\psi\circ f^n)\wedge d^c(g_j^--g_k^-)\wedge\mu^+-\int(\psi\circ f^n)d\varphi\wedge d^c(g_j^--g_k^-)\wedge\mu^+=I+II.$$

The first integral is estimated by

$$|I|\le|\psi\circ f^n|^{\natural}_{\mu^+}\,||\varphi d^c(g_j^--g_k^-)||_{L^2(\mu^+)}\le|\psi\circ f^n|^{\natural}_{\mu^+}\,||\varphi||_{L^\infty}\,|g_j^--g_k^-|_{\mu^+}.$$

This term converges to zero as $j,k\to\infty$ because $|\psi\circ f^n|^{\natural}_{\mu^+}$ tends to zero by Lemma 5.1 and because $\{g_j^-\}$ is Cauchy for $|\cdot|_{\mu^+}$ by Theorem 4.7. The second term is estimated by

$$|II|\le\left(\int(\psi\circ f^n)^2d\varphi\wedge d^c\varphi\wedge\mu^+\right)^{1/2}|g_j^--g_k^-|_{\mu^+}\le||\psi||_{L^\infty}\,|\varphi|_{\mu^+}\,|g_j^--g_k^-|_{\mu^+},$$

which converges to zero because $\{g_j^-\}$ is Cauchy. □

We can now strengthen Corollary 4.12.

Corollary 5.4. *Every compact curve in X has zero μ-measure.*

Proof. Let V be a compact irreducible curve. By Corollary 4.12, we can assume that V is not critical for any iterate of f. If V is not fixed by any iterate of f, then $f^n(V)\cap f^m(V)$ is finite for any $n\neq m$. Hence,

$$\infty>\mu(X)\ge\mu(\bigcup_{n=0}^{\infty}f^n(V))=\sum_{n=0}^{\infty}\mu(f^n(V))=\sum_{n=0}^{\infty}\mu(V)$$

which can happen only if $\mu(V)=0$.

Now suppose that $V=f^n(V)$ is fixed by an iterate of f. Since μ is mixing, we see that V has either full or zero μ-measure. In the former case, it would follow that $f^n|_V$ lifts to an automorphism of a compact Riemann surface (the desingularization of V) with a non-trivial mixing invariant measure. No such automorphism exists in dimension one, so we must have $\mu(V)=0$. □

Theorem 5.5. *Any bimeromorphic map $h:X\to Y$ is defined at μ almost every point. The pushforward $h_*\mu$ is a probability measure, does not charge compact curves in Y, and is invariant and mixing for $g:=h^{-1}\circ f\circ h:Y\to Y$. If g satisfies conditions (3), and μ_g^+ and μ_g^- denote the associated currents, then $h_*\mu=\mu_g^+\wedge\mu_g^-$.*

Proof. Let $\tilde{h}$ denote the restriction of h to $X - (I(h) \cup \mathcal{C}(h))$. All but the last conclusion are immediate from Corollary 5.4 and the fact that $\tilde{h} : X - (I(h) \cup \mathcal{C}(h)) \to Y - (I(h^{-1}) \cup \mathcal{C}(h^{-1}))$ is a biholomorphism. Let us use the notation $\tilde{\cdot}$ denote the restriction of a current or measure on Y to $Y - (I(h^{-1}) \cup \mathcal{C}(h^{-1}))$. With this notation, it follows from Corollary 2.9 that $\tilde{h}^* \tilde{\mu}_g^+$ is equal to the restriction of μ^+ to $X - (I(h) \cup \mathcal{C}(h))$. Thus, since $h_* = (\tilde{h}^*)^{-1}$ on $X - (I(h) \cup \mathcal{C}(h))$, we have $h_*\mu = \widetilde{h_*\mu} = \widetilde{h_*\mu^+} \wedge \widetilde{h_*\mu^-} = \mu_g^+ \wedge \mu_g^-$. □

6. Lyapunov Exponents

In this Section we will show μ is hyperbolic (Theorem 6.4). In order to do this, we first show that the Lyapunov exponents are finite μ a.e. Next we give an estimate on μ of a ball with a certain mapping property (Proposition 6.3). Then, using the machinery of Pesin Theory, we convert this estimate into a proof of Theorem 6.4.

Proposition 6.1. *The quantities* $\log^+ \|Df\|$, $\log^+ \left\|(Df)^{-1}\right\|$, *and* $\log^+ \left\|D^2 f\right\|$ *are μ integrable.*

Proof. It suffices to consider a neighborhood of a point $x \in I(f)$; let z be a local coordinate system such that x corresponds to $z = 0$. By Proposition 1.1, we have $\log^+ ||Df|| \le C\,|\log ||z||\,|$. Now if $v = \log ||z||$, $u = g^+$, and $T = \mu^-$, we may apply Theorems 4.7 and 3.6 and Proposition 3.2 to conclude that v, and thus $\log^+ ||Df||$, is μ integrable. Likewise, $\log^+ \left\|D(f^{-1})\right\|$ is integrable with respect to μ. Since μ is f-invariant, this gives the μ-integrability of $\log^+ \left\|(Df)^{-1}\right\|$.

By the Cauchy estimates applied to the entries of Df we have $\log ||D^2 f|| \le C'\,|\log ||z||\,|$, so $\log ||D^2 f||$ is μ-integrable. □

Proposition 6.1 allows us to invoke Oseledec's Theorem (see [KH, Theorem S.2.9]) to conclude:

Proposition 6.2. *The limits*

$$\chi^+ = \lim_{n\to\infty} \frac{1}{n} \log ||Df_x^n||, \qquad \chi^- = -\lim_{n\to\infty} \frac{1}{n} \log ||Df_x^{-n}||$$

exist and are finite at μ almost every point $x \in X$. Since μ is ergodic, these limits are a.e. independent of x.

For $v \in T_x X - \{0\}$, the Lyapunov exponent of f at x in the direction v is defined as the limit

$$\chi(x, v) = \lim_{n\to\infty} \frac{1}{n} \log |Df_x^n v|,$$

provided that this limit exists. If $\chi^+ \ne \chi^-$, it is a consequence of the Oseledec Theorem is that there is an f-invariant splitting $T_x X = E_x^s \oplus E_x^u$ for μ a.e. x, and

$$\chi^\pm = \chi(x, v) = \lim_{n\to\infty} \frac{1}{n} \log |Df_x^n v|,$$

for $v \in E_x^{u/s} - \{0\}$.

Proposition 6.3. *There exist $C < \infty$ and $R > 0$ such that if $f^n(B_x(2r)) \subset B_y(R)$ for some $n \ge 0$, then*

$$\mu(B_x(r)) \le C\rho^{-n/2}.$$

Proof. From invariance of μ and Lemma 5.1, we obtain

$$\begin{aligned}
\left|\int \varphi\,\mu\right| &= \left|\int (\varphi\circ f^{-n})\,\mu\right| \le \left|\int (\varphi\circ f^{-n})\,\omega^+ \wedge \mu^-\right| + \left|\int d(\varphi\circ f^{-n}) \wedge d^c g^+ \wedge \mu^-\right| \\
&\le \frac{1}{\rho^n}\left|\int \varphi \wedge (f^{n*}\omega^+) \wedge \mu^-\right| + |g^+|_{\mu^-} |\varphi\circ f^{-n}|_{\mu^-} \\
&\le \frac{1}{\rho^n}\left|\int \varphi \wedge (f^{n*}\omega^+) \wedge \mu^-\right| + \frac{C|\varphi|_{\mu^-}}{\rho^{n/2}}
\end{aligned}$$

for any $\varphi \in C^\infty(X)$. Now let us take φ to be bounded above and below by the characteristic functions for $B_x(2r)$ and $B_x(r)$, respectively. We may choose φ such that $\|\varphi\|_{C^1}^2, \|\varphi\|_{C^2} \le Cr^{-2}$ for some constant C independent of x and r.

Let us choose $R > 0$ small enough that there is a local potential u for ω^+ on $B_y(R)$. Since ω^+ is smooth, we may assume that the L^∞ norm of u is bounded above independently of y. We use the local potential $dd^c u = \omega^+$ on the first right hand term in the inequality above and integrate by parts twice to find:

$$\begin{aligned}
\left|\int \varphi\,\mu\right| &\leq \frac{1}{\rho^n}\left|\int_{B_p(2r)} (u\circ f^n)\, dd^c\varphi\wedge\mu^-\right| + \frac{C}{\rho^{n/2}}\left(\int_{B_p(2r)} d\varphi\wedge d^c\varphi\wedge\mu^-\right)^{1/2} \\
&\leq \frac{C_1}{r^2\rho^n}\int_{B_p(2r)}\beta\wedge\mu^- + \frac{C_2}{r\rho^{n/2}}\left(\int_{B_p(2r)}\beta\wedge\mu^-\right)^{1/2} \\
&\leq \frac{C_1}{\rho^n} + \frac{C_2}{\rho^{n/2}} \leq \frac{C}{\rho^{n/2}},
\end{aligned}$$

where C does not depend on x, y, or r. ☐

Theorem 6.4. *The Lyapunov exponents satisfy:*

$$\chi^- \leq -\frac{\log\rho}{8} < 0 < \frac{\log\rho}{8} \leq \chi^+.$$

In particular, μ is hyperbolic of saddle type.

Proof. Pesin Theory provides us with the following setup (see [KH, Theorem S.3.1]). For any $\epsilon > 0$, there exist measurable radius and distortion functions $r, A : X \to \mathbf{R}^+$, and a constant $c > 0$ with the following properties. For μ almost every $x \in X$ there is an embedding $\psi_x : B_{\mathbf{0}}(r(x)) \to X$ such that:

- $\psi_x(\mathbf{0}) = x$;
- both r and A are 'ϵ-slowly varying,' i.e., $-\epsilon < \log A(fx)/A(x), \log r(fx)/r(x) < \epsilon$;
- $c\,\mathrm{dist}(\psi_x(a),\psi_x(b)) \leq \mathrm{dist}(a,b) \leq A(x)\mathrm{dist}(\psi_x(a),\psi_x(b))$ for $a, b \in B_x(r(x))$;
- if $f_x = \psi_{f(x)}^{-1}\circ f\circ\psi_x$, then $D_{\mathbf{0}}f_x$ is a diagonal matrix with diagonal entries e^{χ_1}, e^{χ_2} satisfying
$$|\mathrm{Re}\,\chi_1 - \chi^+|, |\mathrm{Re}\,\chi_2 - \chi^-| < \epsilon;$$
- $\|f_x - D_{\mathbf{0}}f_x\|_{C^1} \leq \epsilon$ on the domain of f_x.

With this notation, we set $r_n(x) = e^{-n(\chi^+ + 3\epsilon)}cr(x)/A(x)$. By the properties above, it follows that for $1 \leq j \leq n$ we have

$$f^j|_{B_x(r_n(x))} = \psi_{f^j(x)}\circ f_{f^{j-1}(x)}\circ\cdots\circ f_x\circ\psi_x^{-1}.$$

That is, by keeping track of the diameters of the successive images of $B_x(r_n(x))$ one sees that each stage of the composition on the right side is well-defined and that, moreover,

$$f^n B_x(r_n(x)) \subset B_{f^n(x)}(r(f^n(x)))$$

Therefore Lemma 6.3 gives us the bound

$$\mu(B_x(r_n(x))) \leq C\rho^{-n/2}.$$

Now Lusin's Theorem provides us with a compact subset $K \subset \operatorname{supp}\mu$ such that $\mu(K) > 1/2$ and on which r and A vary continuously. Thus for all $x \in K$ the radius of $B_n(x)$ is bounded below by $Ce^{-n(\chi^+ + 3\epsilon)}$. We can therefore choose $m \leq Ce^{4n(\chi^+ + 3\epsilon)}$ points $x_1, \ldots, x_m \in K$ such that $K \subset \bigcup_{j=1}^m B_{x_j}(r_n(x_j))$. Using this cover, we estimate

$$1/2 < \mu(K) \leq \sum_{j=1}^m \mu(B_{x_j}(r_n(x_j))) \leq Ce^{4n(\chi^+ + 3\epsilon)}\rho^{-n/2}.$$

Letting n tend to ∞ and then ϵ to 0 yields

$$1 < e^{4\chi^+}\rho^{-1/2},$$

which yields the estimate of Theorem 6.4. ☐

Using Theorem 6.4 and the fact that μ is mixing, one can apply a contraction mapping argument, (see, for example [Dil2, §8]), to obtain:

Corollary 6.5. $\operatorname{supp}\mu$ *is contained in the closure of the saddle periodic points of* μ.

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