ON A QUESTION OF EREMENKO CONCERNING ESCAPING SETS OF ENTIRE FUNCTIONS (DRAFT)

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ABSTRACT. Let f be an entire function of finite order whose set of singular values is bounded or, more generally, a finite composition of such functions. We show that every escaping point of f can be connected to ∞ by a curve in I(f). This provides a positive answer to a question of Eremenko for a large class of entire functions.

1. INTRODUCTION

The dynamical study of transcendental entire functions was initiated by Fatou in 1926 [F]. In this *mémoire*, Fatou observed that the Julia sets of several explicit entire functions contain (analytic) curves of points which escape to infinity under iteration. He then remarks

Il serait intéressant de rechercher si cette propriété n'appartiendrait pas à des substitutions beaucoup plus générales.¹

Sixty years later, Eremenko [E] made a precise study of the escaping set

$$I(f) := \{ z \in \mathbb{C} : |f^n(z)| \to \infty \}$$

of an entire transcendental function. In particular, he showed that every component of $\overline{I(f)}$ is unbounded, and asks whether in fact each component of I(f) is unbounded (we will call this problem *Eremenko's conjecture*). He also states that

It is plausible that the set I(f) always has the following property: every point $z \in I(f)$ can be joined with ∞ by a curve in I(f).

This can be seen as making Fatou's original question more precise, and will be referred to in the following as the *strong form* of Eremenko's conjecture.

These problems are of particular importance since the existence of such curves can be used to study entire functions using combinatorial methods. This is analogous to the notion of "dynamic rays" of polynomials introduced by Douady and Hubbard [DH], which has proved to be one of the fundamental tools for the successful study of polynomial dynamics. (The idea that curves in I(f) may be seen as the limits of dynamic rays of approximating polynomials, and have similar combinatorial properties, was first championed in [DGH].)

There has been some recent progress in the study of these questions [RS, RRS2] (compare the remarks below). Nonetheless, even Fatou's problem — whether there are *any* curves of escaping points — has remained open even for function-theoretically well-behaved classes,

¹"It would be interesting to study whether this property holds for much more general functions."



FIGURE 1. The Julia set of the linearizing coordinate Ψ of a postcritically finite polynomial. Such functions have finite order and have finitely many singular values, so $I(\Psi)$ consists entirely of curves to ∞ by our theorem. The picture on the right is a magnification of the picture near the origin.

such as that of finite-order transcendental entire functions with a finite set of singular values; compare Figure 1. (Recall that f is of *finite order* if

$$\limsup_{r \to \infty} \frac{\log \log \max_{|z|=r} |f(z)|}{\log r} < \infty.$$

In this article, we will prove Eremenko's conjecture in its strong form for such functions, and in fact for a much larger subset of the class \mathcal{B} of entire transcendental functions with bounded singular sets.

1.1. Theorem (I(f) consists of curves). Suppose that $f : \mathbb{C} \to \mathbb{C}$ can be written as a finite composition $f = f_1 \circ \cdots \circ f_n$, where each $f_i \in \mathcal{B}$ is of finite order. Then $I(f) \cup \{\infty\}$ is path-connected.

Some history. It has long been known [DK, DGH] that for exponential maps $f(z) = \exp(z) + \kappa$, the set I(f) contains certain curves to ∞ . A similar construction was carried out by Devaney and Tangerman [DT] for a class of functions with finitely many singular values whose tracts (see Section 2) satisfy some explicit geometric and growth conditions.

For the class of exponential maps, the strong form of Eremenko's conjecture was first proved by Schleicher and Zimmer [SZ]; this result was transferred to the space of cosine maps in [RoS]. Recently, Rückert, Schleicher and the second author proved the strong Eremenko conjecture for a subclass of \mathcal{B} consisting of functions whose tracts satisfy certain geometric conditions [RRS2] (including the type of functions treated in [DT]). On the other hand, there is some evidence [RRS1] that Eremenko's conjecture does *not* hold in its strong form for all functions in \mathcal{B} .

Idea and structure of proof. Our work consists of three parts.

- (a) We introduce a suitable subclass $\mathcal{H} \subset \mathcal{B}$ of entire functions, which is closed both under composition and under quasiconformal equivalence in the sense of [EL]. Rather than using a geometric definition as in [RRS2], this class is defined primarily by a growth property which we call a *head-start condition*.
- (b) We give a straightforward proof of the strong form of Eremenko's conjecture for functions in class \mathcal{B} .
- (c) We show that all finite-order functions $f \in \mathcal{B}$ belong to class \mathcal{H} .

Our class \mathcal{H} can be shown to contain the class of functions treated in [RRS2]. (Conversely, functions in our class \mathcal{H} have some nice geometric properties — compare Proposition 2.5 — which indicates that the methods of [RRS2] could also be extended to give a, somewhat less direct, proof of (b).

Further results. In [DT], it was shown that for certain functions there exist "Cantor *n*-bouquets" in the Julia set. We show the following stronger statement.

1.2. Theorem (Existence of Absorbing Brush).

Let $f \in \mathcal{H}$. Then there exists a closed unbounded set $X \subset J(f)$ with the following properties.

- (a) $f(X) \subset X$,
- (b) every component of X is an injective curve $\gamma : [0, \infty) \to J(f)$ with $\gamma(t) \to \infty$ and $\gamma(t) \in I(f)$ for t > 0, and
- (c) if $z \in I(f)$, then $f^n(z) \in X$ for some $n \ge 0$.

In [RR], we use this result to show that every path-connected component of

$$I(f) \setminus \bigcup_{n \ge 0; z: f'(z) = 0} f^{-n}(z)$$

is a curve.

We should mention that Eremenko's conjecture (albeit not in its strong form) was recently proved by Rippon and Stallard for a rather different class of entire functions, namely those with a multiply-connected periodic Fatou component [RS]. More precisely, Rippon and Stallard show that for *any* entire function, every component of the set A(f) introduced by Bergweiler and Hinkkanen [BH] is unbounded. In the case where F(f) has a multiply connected periodic Fatou component, A(f) is connected, and I(f) = A(f).

2. Functions satisfying a head-start condition

Let $f \in \mathcal{B}$ and set

$$R_0 := 1 + |f(0)| + \max_{s \in S(f)} |s|.$$

The components of

$$f^{-1}(\{z \in \mathbb{C} : |z| > R_0))$$

are called the *tracts* of f. Each tract T maps to the complement of the singular disk as a universal covering.

If T is a tract of f, then we can define branches of $\arg z$ and $\arg f(z)$ on T, which are unique up to an additive constant in $2\pi\mathbb{Z}$. The same remark applies to the functions $\log z$ and $\log f(z)$.

In particular, the functions

wind
$$(z, w) := \frac{|\arg z - \arg w|}{\max\{\log |w|, \log |z|\}}$$
 and
wind_f $(z, w) := \frac{|\arg f(z) - \arg f(w)|}{\max\{\log |f(w)|, \log |f(z)|\}}$

are well-defined for $z, w \in T$.

2.1. Definition (Head-Start Condition). Let $f \in \mathcal{B}$.

(a) We say that f has at most logarithmically spiralling tracts (of order $\theta > 0$) if there exists $M > R_0$ such that, for every tract T and all $w_0, z_0 \in T \cap \{|z| \ge M\}$,

wind
$$(z_0, w_0) \leq \theta$$
.

(b) We say that f satisfies a head-start condition (of exponent K > 1) if for every $\theta > 0$ there exists a constant $M > R_0$ with the following property. Suppose that T is a tract of f and $w_0, z_0 \in T$ satisfy $|f(z_0)|, |f(w_0)| \ge M$ and $\operatorname{wind}_f(z_0, w_0) \le \theta$. If $|z_0| \ge |w_0|^K$, then

(1)
$$|f(z_0)| > |f(w_0)|^K$$
.

We denote by \mathcal{H} the class of all functions $f \in \mathcal{B}$ with at most logarithmically spiralling tracts which satisfy a head-start condition.

REMARK 1. The idea behind this definition is that, if two orbits escape in the same directions, and the first has a "head start" over the other one, then this orbit will escape faster. This will allow us to introduce an order on the set of points with such orbits, showing that they are actually curves.

REMARK 2. Saying that f has at most logarithmically spiralling tracts is the same as saying that, if γ is an asymptotic path for the asymptotic value at ∞ — i.e., $\gamma(t) \rightarrow \infty$ and $f(\gamma(t)) \rightarrow \infty$ — then γ does not spiral more than logarithmically.

REMARK 3. It would be possible to replace the explicit bounds in this definition by a more flexible condition. We have chosen the above version since it yields a large class with many nice properties (see Theorem 2.4 below) and is very convenient for our purposes.

A study of functions satisfying more general "head-start conditions" by combinatorial means will be contained in [RR].

The fundamental tool for establishing the important properties of class \mathcal{H} is the following expansion statement for class \mathcal{B} . The proof is not difficult, but is more conveniently carried out in *logarithmic coordinates*, which is why we postpone this argument to Section 5.

2.2. Lemma (Growth of Orbits).

Let $f \in \mathcal{B}$, and let $\theta > 0$. Then there exist constants $M > R_0$ and $\eta, \delta > 0$ with the following properties. Let T be a tract of f, and suppose that $z, w \in T$ such that $|f(z)| \ge |f(w)| \ge M$.

(a) Then

$$\left|\log z - \log w\right| < \frac{1}{2} \left|\log f(z) - \log f(w)\right|.$$

(b) Suppose furthermore that wind $f(z, w) \leq \theta$ and that $|\log z - \log w| \leq \delta$. Then

$$|f(z)| > |f(w)|^{\exp(\eta |\log z - \log w|)} \ge |f(w)|^{\max(|z|/|w|,|w|/|z|)^{\eta}}$$

2.3. Corollary. (a) Replacing (1) by

(2)
$$|f(z_0)| \ge |f(w_0)|^{1/K}$$

does not change the class of functions satisfying a head-start condition of exponent K.

(b) Suppose that f satisfies a head-start condition of exponent $K_0 > 1$. Then f satisfies a head-start condition of exponent K for every $K \ge K_0$.

PROOF. Clearly (b) is a consequence of (a). So let $\theta > 0$ and suppose that there is $M_1 > 0$ such that (2) holds whenever z_0 and w_0 , satisfying $|f(z_0)|, |f(w_0)| \ge M_1, |z_0| \ge |w_0|^K$ and wind $f(z_0, w_0) \le \theta$, belong to a common tract.

Let M, δ and η be the constants from Lemma 2.2. Since f is continuous, we can choose $M_2 \geq M$ so large that

$$|w| > \max\left(K^{\frac{1}{\eta(K-1)}}, \exp(\frac{\delta}{K-1})\right)$$

whenever $|f(w)| \ge M_2$.

If z_0, w_0 are as above with $|f(z_0)|, |f(w_0)| \ge \max(M_1, M_2)$, then

$$|log z_0 - \log w_0| \ge \log \frac{|z_0|}{|w_0|} \ge (K-1) \log |w_0| > \delta$$
 and
 $\left(\frac{|z_0|}{|w_0|}\right)^{\eta} \ge |w_0|^{\eta(K-1)} > K.$

By Lemma 2.2, either

$$|f(z_0)| > |f(w_0)|^K$$
 or $|f(w_0)| > |f(z_0)|^K$.

By (2), the second possibility is excluded, so (1) holds, as required.

2.4. Theorem (Properties of \mathcal{H}). (a) \mathcal{H} is closed under composition.

- (b) \mathcal{H} is closed under quasiconformal equivalence. That is, if $f \in \mathcal{H}$ and $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ are quasiconformal such that $g := \varphi \circ f \circ \psi$ is holomorphic, then $g \in \mathcal{H}$.
- (c) If $f \in \mathcal{B}$ has finite order, then $f \in \mathcal{H}$.

PROOF. If f is an entire function and g has at most logarithmically spiralling tracts, then the tracts of $f \circ g$ also spiral at most logarithmically, since every asymptotic path of ∞ for $f \circ g$ is also an asymptotic path for g.

Now suppose that f and g satisfy a head-start condition and that f has at most logarithmically spiralling tracts; then it is easy to see that the composition $f \circ g$ also satisfies a head-start condition.

To prove (b), we will use the following fact. If φ is quasiconformal in a neighborhood of ∞ with $\varphi(\infty) = \infty$, and we fix any branch of $\arg(\varphi(z)) - \arg(z)$, then there exists $C_{\varphi} > 1$ such that

$$|z|^{1/C_{\varphi}} \le |\varphi(z)||z|^{C_{\varphi}}$$

and

(3)
$$|\arg\varphi(z) - \arg z| \le C_{\varphi} \log|z|$$

for all sufficiently large z; compare [EL, Lemma 4]².

Now suppose that φ and ψ are quasiconformal such that $g = \varphi \circ f \circ \psi$ is holomorphic, and set $C := \max(C_{\varphi}, C_{\psi})$. Let γ be an asymptotic path for g. Then $\psi \circ \gamma$ is an asymptotic path for f, and thus does not spiral more than logarithmically. Thus γ does not spiral more than logarithmically by (3), and g also has at most logarithmically spiralling tracts.

Furthermore, let $\theta > 0$, K > 1 and choose M sufficiently large (to be fixed below). Suppose that z and w belong to a common tract of g with $|g(z)|, |g(w)| \ge M, |z| \ge |w|^K$ and wind_g $(z, w) \le \theta$. Provided M is large enough, the points $\tilde{z} := \psi(z)$ and $\tilde{w} := \psi(w)$ belong to a common tract of f, with $|f(\tilde{z})|, |f(\tilde{w})| \ge M^{1/C}$ and

$$|\widetilde{z}| \ge |z|^{1/C} \ge |w|^{K/C} \ge |\widetilde{w}|^{K/C^2}.$$

Furthermore,

$$\begin{aligned} |\arg f(\widetilde{z}) - \arg f(\widetilde{w})| &\leq |\arg g(z) - \arg g(w)| + C(\log |\widetilde{z}| + \log |\widetilde{w}|) \\ &\leq \theta \max(\log |z|, \log |w|) + 2C \max(\log |\widetilde{z}|, \log |\widetilde{w}|) \\ &\leq (\theta + 2) \cdot C \cdot \max(\log |\widetilde{z}|, \log |\widetilde{w}|). \end{aligned}$$

Thus wind $f(\tilde{z}, \tilde{w}) \leq (\theta + 2)C =: \tilde{\theta}$.

So if $K > C^2$ was large enough such that f satisfies a head-start condition of exponent $\widetilde{K} := K/C^2$ for $\widetilde{\theta}$, then we can let \widetilde{M} be the corresponding constant for f and set $M := \widetilde{M}^C$. Then

$$|g(z)| \ge |f(\tilde{z})|^{1/C} \ge |f(\tilde{w})|^{\tilde{K}/C} \ge |g(w)|^{\tilde{K}/C^2} > |g(w)|^{1/K}.$$

By Corollary 2.3, g satisfies a head-start condition for K.

To prove (c), first recall that the tracts of any finite order function $f \in \mathcal{B}$ spiral at most logarithmically by the Ahlfors spiral theorem [H].

So it is sufficient to show that an entire function $f \in \mathcal{B}$ which does not satisfy a headstart condition has infinite order. Let $\theta > 0$ and suppose that for every K > 1, there exist

²Eremenko and Lyubich refer to [LV], but we did not find a proof of the — surely classical — estimate (3) there. A short proof can be found in the appendix of [vS].

z and w which satisfy the hypotheses of both the head-start condition and of Lemma 2.2 (b), but

$$|f(z)| < |f(w)|.$$
 Then $|z|/|w| > |w|^{K-1}$ and, by Lemma 2.2 (b),

$$|f(w)| > |f(z)|^{|w|^{\eta(K-1)}},$$

or in other words,

$$\log \log |f(w)| > \eta(K-1) \log |w| + \log \log |f(z)| > \eta(K-1) \log |w|.$$

Since K is arbitrary, this means that f has infinite order.

Finally, let us observe that functions in our class \mathcal{H} have geometrically "well-behaved" tracts. In particular, this is the case for finite-order functions in \mathcal{B} , which may be of independent interest.

2.5. Proposition (Geometry of tracts).

Suppose that f satisfies a head-start condition of exponent K. Then there exists $M > R_0$ with the following property: if $z \in \mathbb{C}$ with $|f(z)| \geq M$, there exists a curve $\gamma : [0, \infty) \to \mathbb{C}$ connecting z and ∞ with $|\gamma(t)| > |z|^{1/K}$ and $|f(\gamma(t))| > M$ for all t > 0.

REMARK. This means that the boundaries of the components of $f^{-1}(\{|z| > M\})$ cannot "wiggle" too much.

PROOF. Let M be the constant from the head-start condition, and let α be the radial ray connecting f(z) to ∞ . Define γ to be the preimage of α containing z. Then wind_f $(z, \gamma(t)) = 0$ and $|f(\gamma(t))| > |f(z)| \ge M$, and thus $|\gamma(t)| > |z|^{1/K}$ by the head-start condition.

3. UNBOUNDEDNESS OF ESCAPING COMPONENTS

In this section, we prove Eremenko's conjecture for $f \in \mathcal{H}$.

3.1. Theorem (Connected components of I(f)). Let $f \in \mathcal{H}$.

- (a) Every connected component of I(f) is unbounded.
- (b) More precisely, let $R > R_0$. Then there exists $R' \ge R$ with the following property. If $z \in \mathbb{C}$ with $|f^n(z)| \ge R'$ for all $n \ge 0$, then there is an unbounded closed connected set $C \subset J(f)$ with $z \in C$ and

 $|f^n(w)| \ge R$

for all $w \in C$ and for all $n \ge 0$. If $z \in I(f)$, then C can be chosen to lie in I(f) as well.

PROOF. First note that (b) implies (a). Indeed, let $z \in I(f)$. Then we can find n_0 such that (a) applies to $f^n(z)$. If C is the unbounded closed connected set from (a), then the component of $f^{-n_0}(C)$ containing z is unbounded (because f is continuous and open).

Thus, we need to prove (b). Let M, θ and K be the constants from Definition 2.1. We may suppose that $R \ge M$. We set $R' := R^K$.

Let z be a point as in (b), and set $z_n := f^n(z)$ for $n \ge 0$. Let T_n be the tract of f containing z_n . Consider the sequence of sets

$$K_n := \{ w \in T_n : \text{wind}_f(z_n, w) \le \theta \text{ and } |f(w)| \ge |z_{n+1}|^{1/K} \}$$

Since f maps T_n to $\{|z| > R_0\}$ by a universal covering, the set K_n is connected.

CLAIM. For every $n \ge 0$, $K_{n+1} \subset f(K_n)$.

Proof. Let $w \in T_{n+1} \setminus f(K_n)$. Since f has at most logarithmically spiralling tracts of order θ ,

wind
$$(z_{n+1}, w) \le \theta$$
.

Our assumption of $w \notin f(K_n)$ thus implies $|w| < |z_{n+1}|^{1/K}$. By the head-start condition, this implies

$$|f(w)| < |z_{n+2}|^{1/K}$$

and thus $w \notin K_{n+1}$.

Now let C_n be the connected component of $f^{-n}(K_n)$ containing z_0 . It follows from our claim that

$$K_0 = C_0 \supset C_1 \supset C_2 \cdots \ni z_0.$$

Furthermore, each C_j is closed, connected and unbounded. Thus the set

$$\mathcal{C} := \bigcap \widehat{C}_j$$

is compact, nonempty and connected, with $\infty, z_0 \in \mathcal{C}$. Let C be the component of $\mathcal{C} \setminus \{\infty\}$ containing z_0 . Since \mathcal{C} is compact, C is unbounded.

Then, by definition,

$$|f^n(w)| \ge |z_n|^{1/K} \ge R$$

for all $w \in C_n$. In particular, if $z \in I(f)$, then every point in w also escapes to ∞ .

4. Curves in I(f)

We will now prove the strong form Eremenko's conjecture for functions in classs \mathcal{H} . More precisely, we show the following.

4.1. Theorem.

Let $f \in \mathcal{H}$. Then there exists a number R with the following property. Suppose that $C \subset \mathbb{C}$ is a closed connected set, consisting of more than one point, with

$$f^n(C) \subset \{|z| \ge R\}$$

for all $n \geq 0$.

Then the closure \hat{C} of C in $\hat{\mathbb{C}}$ is homeomorphic to [0,1]. If C is unbounded, then ∞ is an endpoint of \hat{C} . Every point of C, with the possible exception of one finite endpoint, belongs to I(f).

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REMARK. This result, together with Theorem 3.1, establishes Theorem 1.1.

PROOF. Recall that f has at most logarithmically spiralling tracts of some order $\theta > 0$, and that f satisfies a head-start condition of some order K (with respect to θ). Let R be larger than both the constant from this condition, and the constant M from Lemma 2.2.

We will use the fact that C as in the statement of the theorem is naturally ordered in terms of growth rates. More precisely, let $n \ge 0$, and let T_n be the tract containing $f^n(C)$. Since f has tracts spiralling at most logarithmically of order θ ,

wind
$$(f(z), f(w)) \le \theta$$

for all $z, w \in f^n(C)$. Since $f^n(C)$ is connected, it follows that also

wind_f $(z, w) \le \theta$

for all $z, w \in f^n(C)$. Thus, if

(4)
$$|f^{n_0}(z)| < |f^{n_0}(w)|^K$$

for some $n_0 \ge 0$, we can apply the head-start condition to $f^{n_0}(z)$ and $f^{n_0}(w)$, finding that

$$|f^n(z)| < |f^n(w)|$$

for all $n \ge n_0$. We will say that $z \prec w$ if (4) holds for some $n_0 \ge 0$. Note that, by what we have just said, the relation \prec is antisymmetric and transitive.

CLAIM 1. The order \prec is total; that is, for every $z, w \in C$, either $z \prec w, w \prec z$ or z = w. *Proof.* Let δ, η be the constants from Lemma 2.2. If neither $z \prec w$ nor $w \prec z$, then

$$\left|\log f^{n}(z) - \log f^{n}(w)\right| \le \delta'$$

for all n, where $\delta' = \max(\delta, K^{1/\eta})$.

By Lemma 2.2 (a), it follows that

$$\left|\log z - \log w\right| < 2^{-n} \delta'$$

for all n. Thus z = w.

CLAIM 2. If $z \prec w$, then $w \in I(f)$.

Proof. This follows easily from Lemma 2.2 (a).

If C is unbounded, then the order \prec is easily extended to \hat{C} by setting $z \prec \infty$ for all $z \in C$.

It is clear from the definition of \prec that, for every $w \in \hat{C}$, the sets $U_w^- := \{z \in \hat{C} : z \prec w\}$ and $U_w^+\{z \in \hat{C} : w \prec z\}$ are open in \hat{C} . Thus the usual topology of \hat{C} (induced from the Riemann sphere) is at least as large as the order topology of \hat{C} with respect to \prec . Since \hat{C} is compact and the order topology is Hausdorff, the two topologies coincide.

That is, \hat{C} is a compact connected metric space with a compatible total order. Thus \hat{C} is homeomorphic to the interval [0, 1], and the endpoints of C are the largest and the smallest point of C with respect to \prec . In particular, if C is unbounded, then ∞ is an endpoint. By Claim 2, C can contain at most one nonescaping point, which is necessarily the \prec -smallest point in C.

PROOF OF THEOREM 1.2. Let R be as in Theorem 4.1, and pick R' as in Theorem 3.1 (b). We set

$$Y := Y_R := \{ z \in \mathbb{C} : |f^n(z)| \ge R \text{ for all } n \ge 0 \}.$$

Then Y is clearly closed. Let \widehat{X} be the connected component of \widehat{Y} containing ∞ , and let $X := \widehat{X} \setminus \{\infty\}.$

Then X is closed. Every component C of X satisfies the assumptions of Theorem 4.1, and thus is a curve to ∞ with at most one nonescaping point.

By choice of R',

$$Y_{R'} \subset X,$$

and every point in I(f) will eventually iterate into $Y_{R'}$. This completes the proof.

5. EXPANSION AND FUNCTIONS OF FINITE ORDER

Throughout this section, $f : \mathbb{C} \to \mathbb{C}$ will be a transcendental function of class \mathcal{B} . We will work exclusively in *logarithmic coordinates* (compare [EL]). More precisely, set $\rho_0 := \log R_0$,

$$H_{\rho_0} := \{ \zeta \in \mathbb{C} : \operatorname{Re} \zeta > \rho_0 \} \quad \text{and} \\ \mathcal{T} := \exp^{-1} \big(f(\exp(H_{\rho_0})) \big).$$

Then every component T of \mathcal{T} is mapped conformally to a tract of f by exp. Thus we can define a map

$$F: \mathcal{T} \to H_{\rho_0}$$

with $\exp \circ F = f \circ \exp$. We call the components of \mathcal{T} the *tracts* of F; note that F maps each such tract conformally to the half plane H_{ρ_0} . The key property of class \mathcal{B} which we will employ is that the map F is strongly expanding on tracts.

5.1. Lemma ([EL, Lemma 1]).

$$|F'(\zeta)| \ge \frac{1}{4\pi} (\operatorname{Re} F(\zeta) - \rho_0) \text{ for all } \zeta \in \mathcal{T}.$$

We will now prove Lemma 2.2 using this expansion estimate. To translate its statement into logarithmic coordinates, let us define, for $\mu \ge \rho_0$, $\theta > 0$ and $m \in \mathbb{Z}$, define

$$W_{\mu,\theta,m} := \{ \zeta \in \mathbb{C} : \operatorname{Re}(\zeta) \ge \mu \text{ and } |\operatorname{Im}(\zeta) - 2\pi m| \le \theta \operatorname{Re}(\zeta) \}.$$

5.2. Lemma (Growth of Orbits).

For every $\theta > 0$, there exist constants $\mu > rho_0$ and $\lambda, \eta, \delta > 0$ with the following properties. Let ζ, ω belong to the same component T of \mathcal{T} with $\operatorname{Re} F(\zeta) \geq \operatorname{Re} F(\omega) \geq \mu$, and let $m \in \mathbb{Z}$. Then

(a)
$$|\zeta - \omega| < \frac{1}{2} |F(z) - F(\omega)|;$$

(b) if $F(\zeta), F(\omega) \in W_{\mu,\theta,m}$, then
 $\operatorname{Re}(F(\zeta)) > \lambda |\zeta - \omega| \operatorname{Re}(F(\omega));$

(c) if
$$F(\zeta), F(\omega) \in W_{\mu,\theta,m}$$
 and furthermore $|\zeta - \omega| \ge \delta$, then
 $\log \operatorname{Re} F(\zeta) > \log \operatorname{Re} F(\omega) + \eta |\zeta - \omega|.$

PROOF. Set $\mu := \max(2\rho_0, \rho_0 + 8\pi)$. Let us connect $F(\omega)$ and $F(\zeta)$ by a straight line segment γ , and set $\gamma' := (F|_T)^{-1} \circ \gamma$. Then γ' is a curve connecting ω and ζ . By Lemma 5.1,

(5)
$$|F(\zeta) - F(\omega)| = \ell(\gamma) \ge \frac{\ell(\gamma')(\operatorname{Re}(F(\omega)) - \rho_0)}{4\pi} \ge \frac{|\zeta - \omega|(\operatorname{Re}F(\omega) - \rho_0)}{4\pi}$$

(where ℓ denotes euclidean length). Since Re $F(\omega) - \rho_0 > 8\pi$, this proves (a).

To prove (b), recall that, by assumption, $\operatorname{Re}(F(\zeta)) \ge \operatorname{Re}(F(\omega))$. Therefore,

$$|F(\zeta) - F(\omega)| \le (2\theta + 1) \operatorname{Re}(F(\zeta))$$

by definition of $W_{2\rho_0,\theta,m}$. Using $\operatorname{Re} F(\omega) \geq 2\rho_0$, we obtain from (5) that

$$\operatorname{Re}(F(\omega)) \cdot \frac{|\zeta - \omega|}{8\pi} \leq \frac{|\zeta - \omega|(\operatorname{Re}F(\omega) - \rho_0)}{4\pi} \leq |F(\zeta) - F(\omega)|$$
$$\leq (2\theta + 1) \operatorname{Re}(F(\zeta)) = \operatorname{Re}F(\omega) \cdot (2\theta + 1) \cdot \frac{\operatorname{Re}(F(\zeta))}{\operatorname{Re}(F(\omega))}$$

Thus

$$\frac{\operatorname{Re}(F(\zeta))}{\operatorname{Re}(F(\omega))} \ge \frac{|\zeta - \omega|}{8\pi(2\theta + 1)} =: \lambda |\zeta - \omega|.$$

This proves (b) if we choose $\lambda > 8\pi(2\theta + 1)$.

To prove (c), we assume furthermore that $|\zeta - \omega| \geq \delta$. By (b), if we choose δ sufficiently large, then $F(\zeta) > 2F(\omega)$.

In this case, the slope of the line segment γ is bounded by 3θ :

$$\frac{\operatorname{Im} F(\zeta) - \operatorname{Im} F(\omega)}{\operatorname{Re} F(\zeta) - \operatorname{Re} F(\omega)} \le \theta \frac{\operatorname{Re} F(\zeta) + \operatorname{Re} F(\omega)}{\operatorname{Re} F(\zeta) - \operatorname{Re} F(\omega)} \le \theta \left(1 + \frac{2 \operatorname{Re} F(\omega)}{\operatorname{Re} F(\zeta) - \operatorname{Re} F(\omega)}\right) < 3\theta.$$

So if we parametrize γ by real parts as a curve $\gamma : [\operatorname{Re} F(\omega), \operatorname{Re} F(\zeta)] \to \mathbb{C}$, then

$$\begin{aligned} |\zeta - \omega| &\leq \ell(\gamma') = \int_{\operatorname{Re} F(\omega)}^{\operatorname{Re} F(\zeta)} \left| \frac{\gamma'}{F'(\gamma(t))} \right| dt \\ &< 3\theta \int_{\operatorname{Re} F(\omega)}^{\operatorname{Re} F(\zeta)} \frac{dt}{t - \rho_0} = 3\theta (\log(\operatorname{Re} F(\zeta) - \rho_0) - \log(\operatorname{Re} F(\omega) - \rho_0)). \end{aligned}$$

 So

 $\log \operatorname{Re} F(\zeta) \ge \log(\operatorname{Re} F(\zeta) - \rho_0) > \log \operatorname{Re} F(\omega) - \log \rho_0 + 3\theta |\zeta - \omega| > \log \operatorname{Re} F(\omega) + 2\theta |\zeta - \omega|$ if δ was chosen large enough. This completes the proof.

PROOF OF LEMMA 2.2. Let μ, ν, λ, η be the constants from the previous lemma, and set $M := \exp(\mu)$. Choose ζ, ω with $\exp(\zeta) = z$ and $\exp(\omega) = w$ belonging to the same component of \mathcal{T} . Part (a) of Lemma 2.2 follows directly from (a) in the previous lemma. Note that wind_f(z, w) $\leq \theta$ if and only if $F(\zeta), F(\omega) \in W_{\rho_0,\theta,m}$ for some $m \in \mathbb{Z}$. So if the

hypotheses of (b) in Lemma 2.2 hold for z and w, then $|\zeta - \omega| = |\log z - \log w| \ge \delta$.

Now by (c) in the previous Lemma, we have

$$\log \operatorname{Re} F(\zeta) \ge \log \operatorname{Re} F(\omega) + \eta |\zeta - \omega|.$$

Since $\operatorname{Re} F(\zeta) = \log |f(z)|$, we get, by exponentiating twice, that,

$$|f(z)| \ge |f(w)|^{\exp(\eta |\log z - \log w|)} = |f(w)|^{\max\left(\frac{|z|}{|w|}, \frac{|w|}{|z|}\right)^{\eta}}.$$

This completes the proof.

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