1 Fun with symbols

$$(S,\sigma) \overset{f}{\underset{e}{\rightrightarrows}} (T,\tau)$$

$$W(\sigma) \in \mathcal{WAD}(S)$$
$$W(\tau) \in \mathcal{WAD}(T)$$

$$W(\sigma) = f^*(W(\tau))$$

$$W(\sigma) + B \multimap W(\tau)$$
$$\|B\|_\infty \leq 2$$

2 The Degeneration Theorem

1 Theorem

$$(S,\sigma_i) \overset{f}{\underset{e}{\rightarrow}} (T,\tau_i)$$

If $[\tau_i] \to \infty$ in $\mathrm{Teich}^*(T)$ then, letting $W_i = W(T,\tau_i)$, then $\|W_i\|_\infty \to \infty$, and, passing to a subsequence,

$$\frac{W_i}{\|W_i\|_\infty} \to W_\infty,$$

with

$$f^*(W_\infty) \overset{e}{\multimap} W_\infty.$$

3 B-invariant arc diagram, $B \in \mathbb{Z}^+$

$X \in \mathcal{AD}(T)$ is B-invariant if $\forall [\alpha] \in X$, $\exists [\alpha_1], \ldots, [\alpha_n] \in f^*(X)(\in \mathcal{AD}(S))$ such that $\alpha_1, \ldots, \alpha_n \stackrel{e}{\to} \alpha$.

$$f^*W \multimap \frac{1}{2}W \Rightarrow \text{supp}(W) \text{ is } B\text{-invariant}$$

4 The Arc Lemma

2 Lemma
Given

$$S \underset{[e]}{\overset{[f]}{\rightrightarrows}} T$$

and B, there exist finitely many B-invariant arc diagrams for $(S, T, [e], [f])$.

5 Cone schemes and additive relations

Let C be a (possibly infinite) simplicial complex. We define $W(C)$ to be all finite formal sums of vertices of C, with positive coefficients, such that if $X \in W(C)$, then the support of X comprises the vertices of a simplex in C. We can then multiply an element of $W(C)$ by a non-negative real, and add elements of $W(C)$ if the union of their supports is again the set of vertices of a simplex in C.

6 Weighted arc diagrams

Let S be a compact surface with (non-trivial) boundary. An *arc* on S is an embedding $(I, \partial I) \to (S, \partial S)$, up to isotopy (that is not isotopic to a constant map). We denote the set of arcs on S by $\mathcal{A}(S)$. Any two arcs $\alpha, \beta \in \mathcal{A}(S)$ have an intersection number $\langle \alpha, \beta \rangle$ that is equal to the minimum number of times that representatives of α and β intersect. We say that two arcs are *disjoint* if they have zero intersection number.

An *arc-diagram* is a (necessarily finite) set of arcs in S that are pairwise disjoint. We denote the set of arc-diagrams by $\mathcal{AD}(S)$.

A *weighted arc-diagram* is a arc-diagram with positive real weights assigned to each arc. We denote the set of weighted arc-diagrams by $\mathcal{WAD}(S)$. If $X \in \mathcal{WAD}(S)$, then the *support* of X, denoted $\mathrm{supp}(X)$, is the underlying arc-diagram. If $X, Y \in \mathcal{WAD}(S)$, and $\mathrm{supp}\, X \cup \mathrm{supp}\, Y \in \mathcal{AD}(X)$ (i.e. no arc for X intersects one for Y) then we can form the weighted arc-diagram $X+Y$ by adding the weights of arcs that appear in both. We say that $X \geq Y$ if $\exists Z \in \mathcal{WAD}(S)$ such that $Y + Z = X$.

Given $X \in \mathcal{WAD}(S)$, we can write

$$X = \sum_i w_i \alpha_i$$

where the α_i are distinct. Then we write $X|_\alpha = w_i$ if $\alpha = \alpha_i$, and $X|_\alpha = 0$ if $\alpha \notin \mathrm{supp}\, X$. (Note that $X \geq Y$ if and only if $X|_\alpha \geq Y|_\alpha$ for all $\alpha \in \mathcal{A}(S)$). Also, we write

$$\|X\|_\infty = \sup_{\alpha \in \mathrm{supp}\, X} X|_\alpha,$$

and

$$\|X\|_1 = \sum_{\alpha \in \mathrm{supp}\, X} X|_\alpha.$$

If $f : S \to T$ is a covering map, and $Y \in \mathcal{WAD}(T)$, we define $f^*(Y) \in \mathcal{WAD}(S)$ by

$$f^*(Y)|_\alpha = Y|_{f_*\alpha}$$

(of course, if $f_*\alpha$ is not an embedded arc, then $Y|_{f_*\alpha} = 0$).

7 Lollypop

Given

$$S \xrightarrow{e} T$$

we say

$$\alpha_1, \ldots, \alpha_n \to \alpha$$

if we can find embedded paths a_i,a such that $\alpha_i = [a_i]$, $\alpha = [a]$, and

$$e^{-1}(a) = \bigcup_{i=1}^{n} a_i$$

(here we use the functions a,a_i as shorthand for their images). We then define the relation $\multimap$ between $\mathcal{WAD}(S)$ and $\mathcal{WAD}(T)$ to be the least relation such that

1. $$\sum_{i=1}^{n} w_i \cdot \alpha_i \multimap w \cdot \alpha$$

   whenever

   $$\alpha_1, \ldots, \alpha_n \to \alpha$$

   and

   $$\sum_{i=1}^{n} \frac{1}{w_i} \leq \frac{1}{w},$$

2. $A \multimap 0$ for all $A \in \mathcal{WAD}(S)$,

3. $$X \multimap Y \quad \& \quad X' \multimap Y' \qquad \Longrightarrow \qquad X + X' \multimap Y + Y'$$

   whenever $X + X'$ and $Y + Y'$ are well-defined.

8 Neat path-families and weight

Let S now be a bordered Riemann surface. A *path* in S is a piecewise smooth map $a : (I, \partial I) \to (S, \partial S)$ (*not* up to isotopy). A *path family* on S is a set of paths. A path family is *neat* if

- every path is an embedding,
- every two paths have disjoint images, and
- no path is homotopic (through paths) to a constant map.

Given any neat path family $\mathcal{F}$ and arc $\alpha \in \mathcal{A}(S)$ we define $\mathcal{F}|_\alpha$ to be the set of paths in $\mathcal{F}$ that are representatives of α, and we let $\operatorname{supp} \mathcal{F} \in \mathcal{AD}(S)$ be those α for which $\mathcal{F}|_\alpha$ is non-empty. We then define $W(\mathcal{F}) \in \mathcal{WAD}(S)$ by setting $W(\mathcal{F})|_\alpha$ to be the reciprocal of the *extremal length* of $F|_\alpha$; we then have $\operatorname{supp} W(\mathcal{F}) \subset \operatorname{supp} F$. We say that $X \in \mathcal{WAD}(S)$ is *valid* for a given Riemann surface structure on S if there exists a neat path-family $\mathcal{F}$ such that $W(\mathcal{F}) = X$.

9 The Canonical neat path-family

Now consider the set of all bordered Riemann surfaces. We can assign to each surface S a neat path-family, called the *canonical* neat path-family $\mathcal{F}^N(S)$ in such a way that

- if $\mathcal{F}'$ is any neat path-family on S, then

  $$W(\mathcal{F}') \leq W(\mathcal{F}^N(S)) + B$$

  for some $B \in \mathcal{WAD}(S)$ with $\|B\|_\infty \leq 2$;

- if $f : S \to T$ is a conformal covering map, then $\mathcal{F}^N(S) = f^*(\mathcal{F}^N(T))$, in the sense that $a \in \mathcal{F}^N(S)$ iff $f \circ a \in \mathcal{F}^N(T)$.

We define $W(S) = W(\mathcal{F}^N(S)) \in \mathcal{WAD}(S)$.

10 The inclusion lemma

3 Lemma

If $Y \in \mathcal{WAD}(T)$ is valid for τ, and $e : (S, \sigma) \to (T, \tau)$ is a conformal embedding, then $\exists X \in \mathcal{WAD}(S)$ valid for σ such that

$$X \overset{e}{\multimap} Y.$$

11 Almost maximality

4 Lemma
If $X \in \mathcal{WAD}(S)$ is valid for σ, then $\exists B, \|B\|_\infty \leq 2$ such that

$$W_S(\sigma) + B \geq X.$$

If M is an $n \times n$ matrix with non-negative entries, then either $Mx \geq x$ for some $x > 0$, or $\forall x(\langle Mx, u\rangle < \langle x, u\rangle)$ for some $u > 0$. We can generalize this statement to postive linear relations:

5 Theorem
Suppose that $T \subset \mathbb{R}^n \times \mathbb{R}^n$ is a positive linear relation. Then either $T(x, x)$ for some $x \geq 0$, or there exists $u > 0$ such that $T(x, y) \Rightarrow \langle u, x\rangle > \langle u, y\rangle$.

We can generalize the multiplication of matrices to positive linear relations as follows: Given postive linear relations $R \subset^n \times^m$ and $S \subset^m \times^l$, we define $RS \subset^n \times^l$ by

$$RS(x, z) \iff \exists y : R(x, y) \,\&\, S(y, z).$$

Given any set S, let

$$\Delta(S) = \{(x, x) : x \in S\}$$

be the diagonal. Here's a cool theorem:

6 Theorem
A positive linear relation $R \subset^n \times^n$ satisfies $R^2 = R$ and $R \supset I$ if and only if there exists $V \subset^n$ such that $R = \Delta(V^\perp)$.

12 Domination revisited

Given a surface S, let γ be a weighted sum of simple closed curves on S. Then we define $V_\gamma(\alpha) = \langle \gamma, \alpha \rangle^2$, and we extend V_γ linearly to $\mathcal{WAD}(S)$.

7 Theorem
Given $e : S \to T$ an embedding, we have

$$X \overset{e}{\multimap} Y$$

if and only if for all weighted sums of simple closed curves γ,

$$V_\gamma(X) \geq V_{e_*\gamma}(Y).$$

8 Theorem
Let (S, T, e, f) be a covering system. Consider $w \in \mathcal{AD}(T)$. Then either there exists $W \in \mathcal{WAD}(T)$ with $\operatorname{supp} W \subset w$ such that

$$f^*W \overset{e}{\multimap} W$$

or there exists a weighted sum γ of disjoint simple closed curves in S such that

$$V_\gamma(f^*W) < V_{e_*\gamma}(W)$$

whenever $\operatorname{supp} W \subset w$.

13 The finite basis theorem

Let $\mathbb{N}$ denote the natural numbers $0, 1, 2, \ldots$.

9 **Theorem**
Let $S \subset \mathbb{N}^k$. Then there exists a finite subset $S' \subset S$, such that for all $x \in S$, there exists $x' \in S'$ such that $x' \leq x$.

14 Exact Fit

We say that $g : (S, [\sigma]) \to (T, [\tau])$ is holomorphic if there exists $\sigma' \in [\sigma], \tau' \in [\tau]$ such that $g : (S, \sigma') \to (T, \tau')$ is holomorphic. The point here is that you isotope the structures, not the maps. Now, given $e : S \to T$ an embedding, we again define

$$\alpha_1, \ldots, \alpha_n \xrightarrow{e} \alpha$$

to mean that there exist a_i,a such that $\alpha_i = [a_i]$, $\alpha = [a]$, and

$$e^{-1}(a) = \bigcup_{i=1}^{n} a_i.$$

We then define $\multimap$ as before. Given a degenerating sequence

$$(S, [\sigma_i]) \underset{f}{\overset{e}{\rightrightarrows}} (T, [\tau_i])$$

we have, letting $W_i = W_T^N(\tau_i)$ as before, and passing to a subsequence,

$$\frac{W_i}{\|W_i\|_\infty} \to \infty,$$

with

$$f^* W_\infty \overset{e}{\multimap} W_\infty.$$

The statement looks exactly like the one before, but it's been made more precise: we do not isotope e in either the maps $(S, [\sigma_i]) \xrightarrow{e} (T, [\tau_i])$ or the definition of $\multimap$.

15 Exact fit and Hubbard trees

Suppose we have an embedding $e : S \to T$. We introduce the *topological arrow*. Let $A \in \mathcal{AD}(S)$ be an arc-diagram. For $\alpha \in \mathcal{A}(T)$, we say that

$$A \overset{e}{\rightsquigarrow} \alpha$$

to mean that there exist $\alpha_1, \dots \alpha_n$ such that

$$\alpha_1, \dots \alpha_n \overset{e}{\to} \alpha$$

and $\alpha_i \in A$ for all i.

Now, suppose that $f : \mathbb{C} \to \mathbb{C}$ is a critically periodic quadratic polynomial. We can define $H \in \mathcal{AD}(\mathbb{C} - P_f)$ to be those arcs which can be homotoped to lie in the Hubbard tree of f. We let $H' \supset H$ be H augmented by those arcs from finite points of P_f to ∞ that do not cross any arc in H. We have the following remarkable theorem:

10 Theorem
Suppose $A \in \mathcal{AD}(\mathbb{C} - P_f)$ satifies

$$f^*A \overset{e}{\rightsquigarrow} A$$

where $e : \mathbb{C} - f^{-1}P_f \to \mathbb{C} - P_f$ is the inclusion. Then $A \subseteq H'$. Moreover, $A \supseteq H$ or $A \cap H = \emptyset$ unless f has a non-trivial renormalization, and then $A \cap H$ is either empty or the forward saturation of the H' of some renormalization of f.

This theorem depends very much on the more precise definition of $\to$ (and hence $\rightsquigarrow$).

Now, let $f : U \to V$ be a polynomial-like restriction of f, and let n be the period of 0 for f. We have the following corollary:

11 Corollary
Suppose $A \in \mathcal{AD}(V - P_f)$ satisfies

$$f^*A \overset{e}{\rightsquigarrow} A$$

where $e : U - f^{-1}P_f \to V - P_f$ is the inclusion. Then $(f^{3n})^*(A)$ satisfies the conclusion of the previous theorem.

16 Hubbard Trees and Sums of Weights

Continuing with the notation of the previous section, suppose now that $W \in \mathcal{WAD}(V - P_f)$ satisfies

$$f^*W \multimap W.$$

and that $\operatorname{supp} W \subset H'$. Suppose also that f is m-renormalizable with $n > m \geq n/B$.

12 Theorem

$$\sum_{\alpha \in H} W|_\alpha < K \sum_{\alpha \in H' - H} W|_\alpha$$

where K depends only on B.

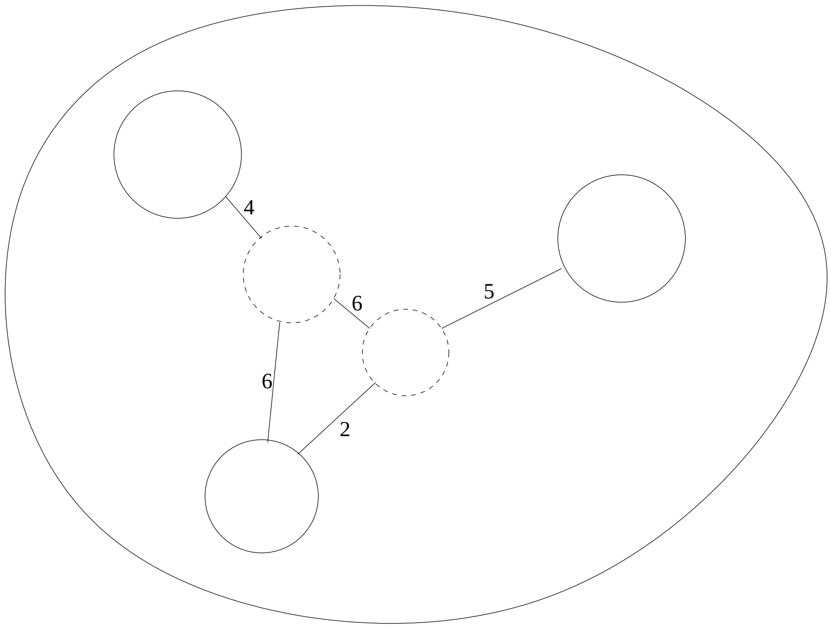

Figure 1: The weighted arc-diagram on U

17 Contracting by a finite union of disjoint chords

Suppose that $U \subset V \subset \hat{\mathbb{C}}$, and $V - U$ is a finite union of disks. Let $X \in \mathcal{WAD}(U)$ be given, as in figure 1. Then split X into a weighted sum of paths, and join those endpoints of the paths that lie in $\partial(V - U)$ by disjoint chords in $V - U$, as in figure 2. Finally, join the paths along the chords, to form chains of weighted paths, and replace each chain with a single path in V with weight equal to the harmonic sum of the weights of the paths in the chain, as in figure 3. The resulting weighted arc-diagram Y satisfies

$$X \multimap Y,$$

and indeed any maximal Y satisfying that equation can be realized in the above manner.

Now let $f : S \to T$ be a smooth branched cover, and let $B \subset T$ be a disjoint union of Jordan domains, such that all of the branch values $f(C)$ of f lie in B. Let $A \subset S$ be a union of components of $B' = f^{-1}(B)$. Now, suppose we are given a weighted arc-diagram $Y \in \mathcal{WAD}(T - B)$. Again,

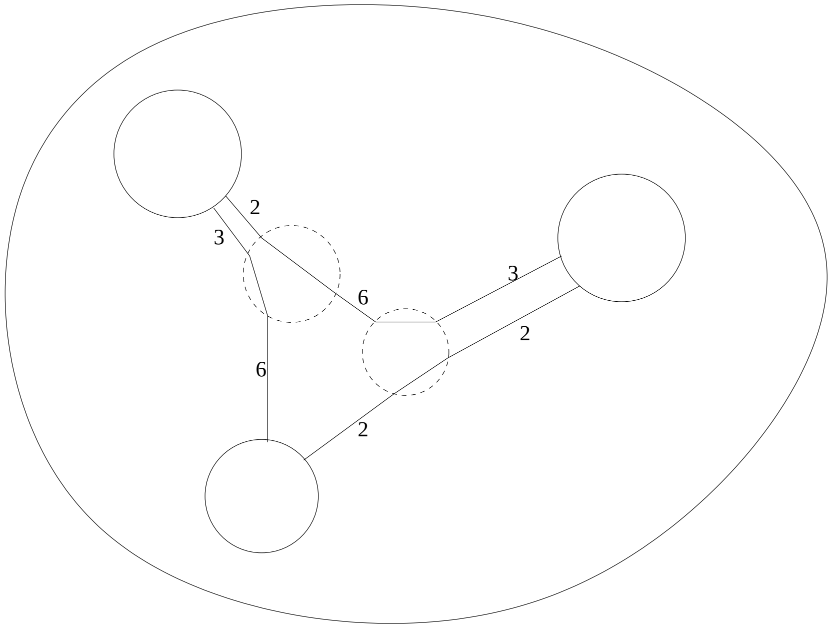

Figure 2: The weighted arc-diagram on U, about to be contracted by a lamination

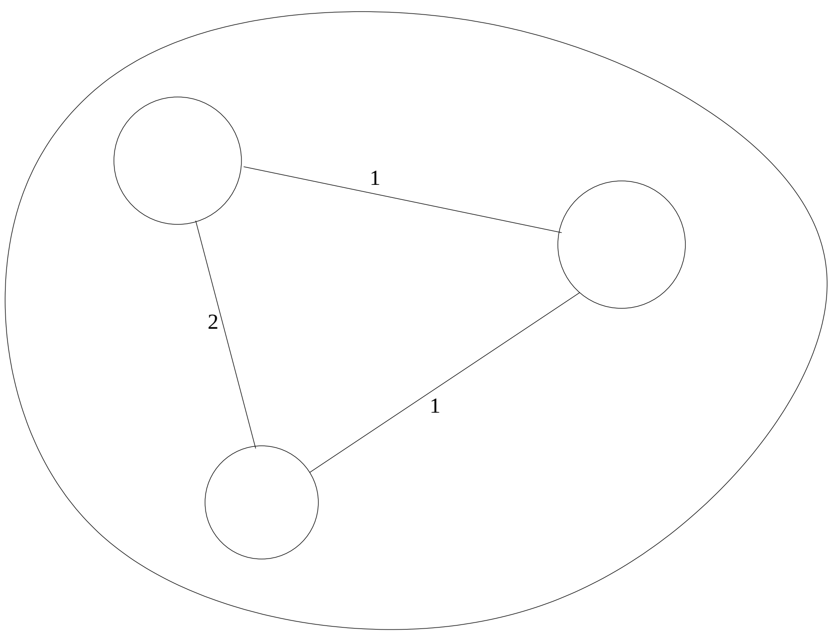

Figure 3: The weighted arc-diagram on V, after being contracted

split it into weighted paths, and connect those endpoints of the paths that lie in ∂B by chords in $B - f(C)$. Then lift the whole picture, via f, to S, and form chains by connecting paths by those lifts of the above chords that lie in $B' - A$. Again replace the chains by paths in $S - A$, to obtain in the end a weighted arc-diagram $X \in \mathcal{WAD}(S - A)$. If X can be obtained from Y in this way, we say that

$$X \overset{f}{\multimap} Y.$$

We can now state our theorem:

13 Theorem
Suppose we have S, T, A, B as above. Then for any Riemann surface structures σ, τ on S, T respectively that make $f : S \to T$ conformal, we have

$$W_{S-A}(\sigma) \overset{f}{\multimap} (1 + \epsilon) W_{T-B}(\tau)$$

with $\epsilon \to 0$ as $\|W_{T-B}(\tau)\| \to \infty$.

18 Contraction Applied

Let $f : U \to V$ be a quadratic-like map, and suppose that f is n-renormalizable. Let K_n be the small Julia set (around 0) of the renormalization, and let $\mathcal{K}_n$ be the union of small Julia sets. Let $k \geq 3$, and suppose $X \in \mathcal{WAD}(f^{-kn}(V) - \mathcal{K}_n)$ and $Y \in \mathcal{WAD}(V - \mathcal{K}_n)$.

We say that the *horizontal* part of Y, or Y_h, is the part supported on arcs that do not touch ∂V, and the *vertical* part of Y, or Y_v, is the rest. Likewise for X.

14 Theorem
Suppose that

1. $\|Y_h\|_1 < C\|X_v\|_1$
2. $X \overset{f^{kn}}{\circ\!\!-} Y$

Then

$$\|X_v\|_1 < 100\, C^2\, 2^k \|Y_v\|_1.$$

We will apply the above Theorem in the case where we know that

$$\|X_h\|_1 < C\|X_v\|_1.$$

Because

$$X -\!\!\circ\, Y$$

via the inclusion of $f^{-kn}(V) - \mathcal{K}_n$ into $V - \mathcal{K}_n$, we have

$$\|Y_h\|_1 < \|X\|_1 = \|X_h\|_1 + \|X_v\|_1,$$

and hence condition 1 above.

19 Controlled Immersion

20 We say ... if ...

We introduce the following notational convention:
"We say

$$Q(f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n))$$

if

$$P(a_1, \ldots, a_n)''$$

means
We define $Q(b_1, \ldots, b_m)$ by

$$Q(b_1, \ldots, b_m) \Leftrightarrow$$
$$\exists x_1, \ldots, x_n : P(x_1, \ldots, x_n) \& f_1(x_1, \ldots, x_n) = b_m, \ldots, f_m(x_1, \ldots, x_n) = b_m.$$