1 What's on my whiteboard

1.1 left panel

$$(S,\sigma) \overset{f}{\underset{e}{\to}} (T,\tau)$$

$$W(\sigma) \in \mathcal{WAD}(S)$$
$$W(\tau) \in \mathcal{WAD}(T)$$

$$W(\sigma) = f^*(W(\tau))$$

$$W(\sigma) + B \multimap W(\tau)$$
$$\|B\|_\infty \leq 2$$

1.2 middle

- $\mathcal{WAD}(S)$

$$f^*, \langle , \rangle , +, \| \, \|_\infty$$

- Path-family
  - neat
  - canonical
  - valid

1.3 left panel

1 Lemma
If $X \in \mathcal{WAD}(S)$ is valid for σ, then $\exists B, \|B\|_\infty \leq 2$ such that

$$W_S(\sigma) + B \geq X.$$

2 Lemma

If $Y \in \mathcal{WAD}(T)$ is valid for τ, and $e : (S, \sigma) \to (T, \tau)$ is a conformal embedding, then $\exists X \in \mathcal{WAD}(S)$ valid for σ such that

$$X \overset{e}{\multimap} Y.$$

2 What's on Misha's blackboard

2.1 The Degeneration Theorem

3 Theorem

$$(S, \sigma_i) \underset{\stackrel{e}{\to}}{\stackrel{f}{\to}} (T, \tau_i)$$

If $[\tau_i] \to \infty$ in $\mathrm{Teich}^*(T)$ then, letting $W_i = W(T, \tau_i)$, then $\|W_i\|_\infty \to \infty$, and, passing to a subsequence,

$$\frac{W_i}{\|W_i\|_\infty} \to \infty,$$

with

$$f^*(W_\infty) \stackrel{e}{\multimap} W_\infty.$$

2.2 B-invariant arc diagram, $B \in \mathbb{Z}^+$

$X \in \mathcal{AD}(T)$ is B-invariant if $\forall [\alpha] \in X$, $\exists [\alpha_1], \ldots, [\alpha_n] \in f^*(X) (\in \mathcal{AD}(S))$ such that $\alpha_1, \ldots, \alpha_n \stackrel{e}{\to} \alpha$.

$$f^* W \multimap \frac{1}{2} W \Rightarrow \mathrm{supp}(W) \text{ is } B\text{-invariant}$$

2.3 The Arc Lemma

4 Lemma
Given

$$S \underset{\stackrel{[e]}{\to}}{\stackrel{[f]}{\to}} T$$

and B, there exist finitely many B-invariant arc diagrams for $(S, T, [e], [f])$.

3 Lollypop

Given

$$S \xrightarrow{e} T$$

we say

$$\alpha_1, \ldots, \alpha_n \to \alpha$$

if

$$e^{-1}(\alpha) = \bigcup_{i=1}^{n} \alpha_1 \cup \{\text{homotopically trivial arcs}\}$$

(more precisely, if $e^{-1}(\alpha)$ is $\alpha_1, \ldots, \alpha_n$ *in sequence*). Then

$$\sum_{i=1}^{n} w_i \cdot \alpha_i \multimap w \cdot \alpha$$

if

$$\alpha_1, \ldots, \alpha_n \to \alpha$$

and

$$\sum_{i=1}^{n} \frac{1}{w_i} \leq \frac{1}{w}.$$

And

$$A + \sum A_j \multimap \sum B_j$$

if $A \geq 0$ and $A_j \multimap B_j$.

4 Weighted arc diagrams

Let S be a compact surface with (non-trivial) boundary. An *arc* on S is an embedding $(I, \partial I) \to (S, \partial S)$, up to isotopy (that is not isotopic to a constant map). We denote the set of arcs on S by $\mathcal{A}(S)$. Any two arcs $\alpha, \beta \in \mathcal{A}(S)$ have an intersection number $\langle \alpha, \beta \rangle$ that is equal to the minimum number of times that representatives of α and β intersect. We say that two arcs are *disjoint* if they have zero intersection number.

An *arc-diagram* is a (necessarily finite) set of arcs in S that are pairwise disjoint. We denote the set of arc-diagrams by $\mathcal{AD}(S)$.

A *weighted arc-diagram* is a arc-diagram with positive real weights assigned to each arc. We denote the set of weighted arc-diagrams by $\mathcal{WAD}(S)$. If $X \in \mathcal{WAD}(S)$, then the *support* of X, denoted $\text{supp}(X)$, is the underlying arc-diagram. If $X, Y \in \mathcal{WAD}(S)$, and $\operatorname{supp} X \cup \operatorname{supp} Y \in \mathcal{AD}(X)$ (i.e. no arc for X intersects one for Y) then we can form the weighted arc-diagram $X+Y$ by adding the weights of arcs that appear in both. We say that $X \geq Y$ if $\exists Z \in \mathcal{WAD}(S)$ such that $Y + Z = X$.

Given $X \in \mathcal{WAD}(S)$, we can write

$$X = \sum_i w_i \alpha_i$$

where the α_i are distinct. Then we write $X|_\alpha = w_i$ if $\alpha = \alpha_i$, and $X|_\alpha = 0$ if $\alpha \notin \operatorname{supp} X$. (Note that $X \geq Y$ if and only if $X|_\alpha \geq Y|_\alpha$ for all $\alpha \in \mathcal{A}(S)$). Also, we write

$$\|X\|_\infty = \sup_{\alpha \in \operatorname{supp} X} X|_\alpha,$$

and

$$\|X\|_1 = \sum_{\alpha \in \operatorname{supp} X} X|_\alpha.$$

If $f : S \to T$ is a covering map, and $Y \in \mathcal{WAD}(T)$, we define $f^*(Y) \in \mathcal{WAD}(S)$ by

$$f^*(Y)|_\alpha = Y|_{f_* \alpha}$$

(of course, if $f_* \alpha$ is not an embedded arc, then $Y|_{f_* \alpha} = 0$).

5 Neat path-families and weight

Let S now be a bordered Riemann surface. A *path* in S is a piecewise smooth map $a : (I, \partial I) \to (S, \partial S)$ (*not* up to isotopy). A *path family* on S is a set of paths. A path family is *neat* if

- every path is an embedding,
- every two paths have disjoint images, and
- no path is homotopic (through paths) to a constant map.

Given any neat path family $\mathcal{F}$ and arc $\alpha \in \mathcal{A}(S)$ we define $\mathcal{F}|_\alpha$ to be the set of paths in $\mathcal{F}$ that are representatives of α, and we let $\operatorname{supp} \mathcal{F} \in \mathcal{AD}(S)$ be those α for which $\mathcal{F}|_\alpha$ is non-empty. We then define $W(\mathcal{F}) \in \mathcal{WAD}(S)$ by setting $W(\mathcal{F})|_\alpha$ to be the reciprocal of the *extremal length* of $F|_\alpha$; we then have $\operatorname{supp} W(\mathcal{F}) = \operatorname{supp} F$. We say that $X \in \mathcal{WAD}(S)$ is *valid* for a given Riemann surface structure on S if there exists a neat path-family $\mathcal{F}$ such that $W(\mathcal{F}) = X$.

6 The Canonical neat path-family

Now consider the set of all bordered Riemann surfaces. We can assign to each surface S a neat path-family, called the *canonical* neat path-family $\mathcal{F}^N(S)$ in such a way that

- if $\mathcal{F}'$ is any neat path-family on S, then

  $$W(\mathcal{F}') \leq W(\mathcal{F}^N(S)) + B$$

  for some $B \in \mathcal{WAD}(S)$ with $\|B\|_\infty \leq 2$;

- if $f : S \to T$ is a conformal covering map, then $\mathcal{F}^N(S) = f^*(\mathcal{F}^N(T))$, in the sense that $a \in \mathcal{F}^N(S)$ iff $f \circ a \in \mathcal{F}^N(T)$.

We define $W(S) = W(\mathcal{F}^N(S)) \in \mathcal{WAD}(S)$.

7 The inclusion lemma

5 Lemma
If $Y \in \mathcal{WAD}(T)$ is valid for τ, and $e : (S,\sigma) \to (T,\tau)$ is a conformal embedding, then $\exists X \in \mathcal{WAD}(S)$ valid for σ such that

$$X \stackrel{e}{\multimap} Y.$$

8 Almost maximality

6 Lemma
If $X \in \mathcal{WAD}(S)$ is valid for σ, then $\exists B, \|B\|_\infty \leq 2$ such that

$$W_S(\sigma) + B \geq X.$$