WANDERING TRIANGLES EXIST

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Abstract. W. P. Thurston introduced closed σ_d-invariant laminations (where $\sigma_d = z^d : S^1 \to S^1$, $d \geq 2$) as a tool in complex dynamics. He defined *wandering triangles* as triples $T \subset S^1$ such that $\sigma_d^n(T)$ consists of three distinct points for all $n \geq 0$ and the convex hulls of all the sets $\sigma_d^n(T)$ in the plane are pairwise disjoint, and proved that σ_2 admits no wandering triangles. We show that for every $d \geq 3$ there exist uncountably many σ_d-invariant closed laminations with wandering triangles and pairwise non-conjugate factor maps of σ_d on the corresponding quotient spaces.

1. Introduction

Laminations were introduced by Thurston [10] as a tool for studying both individual complex polynomials and the space of all of them. In the case of degree d the latter reduces to studying the parameter space of degree $d \geq 2$ monic centered polynomials of the form $z \mapsto z^d + a_{d-2}z^{d-2} + \cdots + a_0$ [3]. The set of parameters for which the corresponding Julia set is connected is called the *connectedness locus* (if $d = 2$ the connectedness locus is called the *Mandelbrot set* and denoted by $\mathcal{M}$).

Let $P : \mathbb{C}^* \to \mathbb{C}^*$ be a degree d polynomial with a connected Julia set J_P acting on the complex sphere $\mathbb{C}^*$. Denote by K_P the corresponding filled-in Julia set. Let $\theta = z^d : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ ($\mathbb{D} \subset \mathbb{C}$ is the unit disk). There exists a conformal isomorphism $\Psi : \mathbb{D} \to \mathbb{C}^* \setminus K_P$ with $\Psi \circ \theta = P \circ \Psi$ [4, 5]. If J_P is locally connected, then Ψ extends to a continuous function $\overline{\Psi} : \overline{\mathbb{D}} \to \overline{\mathbb{C}^* \setminus K_P}$. Let $\sigma_d = \theta|_{\partial\mathbb{D}}$, $\psi = \overline{\Psi}|_{\partial\mathbb{D}}$, for each $y \in J_P$ let $C(y)$ be the convex hull of the set $\psi^{-1}(y)$ in the unit disk, and let $\mathcal{L}_P$ be the collection of all chords of S^1 contained in the boundary of all the sets $C(y)$, $y \in J_P$ (if $C(y)$ is a point then this point is included in $\mathcal{L}_P$ too). Then $\mathcal{L}_P$ is an example of a d-invariant lamination. Such

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a lamination gives a combinatorial description of J_P. By Kiwi [7] a similar construction is possible for all polynomials with connected Julia sets and no irrational neutral cycles.

In the case $d = 2$ Thurston [10] proved that the space of all 2-invariant (*quadratic*) laminations can be interpreted through a "meta-lamination" called QML (*quadratic minor lamination*). The exact relationship between QML and $\mathcal{M}$ is unknown (Thurston conjectured that the boundary of $\mathcal{M}$ is essentially QML). A major ingredient of Thurston's theory is the non-existence of wandering triangles for quadratic laminations. From the standpoint of the dynamics in the Julia set this is equivalent to the non-existence of non-preperiodic non-precritical branch points in J, and can be viewed as a natural extension of the same result for maps of finite graphs.

Extensions of Thurston's results beyond $d = 2$ have been hampered by the lack of information about the existence of wandering triangles for $d > 2$. Here we sketch the construction of invariant laminations with wandering triangles.

A *lamination* $\mathcal{L}$ is a closed set of chords and points in $\overline{\mathbb{D}} \subset \mathbb{C}$ such that any two distinct chords in $\mathcal{L}$ (called *leaves*) intersect at most at a common endpoint; leaves may be degenerate. A leaf with endpoints $p, q \in S^1$ is denoted by $\ell = \overline{pq}$. Denote the union of all leaves in $\mathcal{L}$ by $\mathcal{L}^*$. A *gap* G of $\mathcal{L}$ is the closure of a complementary domain of $\mathcal{L}^*$ in $\overline{\mathbb{D}}$. For each chord $\ell = \overline{pq}$ let $\sigma_d(\ell)$ be the chord joining the points $\sigma_d(p)$ and $\sigma_d(q)$. The lamination $\mathcal{L}$ is *d-invariant* if for each $\ell \in \mathcal{L}$ we have $\sigma_d(\ell) \in \mathcal{L}$, there exist d pairwise disjoint leaves $\ell_i \in \mathcal{L}$ $(i = 1, \dots, d)$ with $\sigma_d(\ell_i) = \ell$, and for each gap G either $|\sigma_d(G \cap S^1)| \leq 2$, or there exists a gap H of $\mathcal{L}$ such that $\sigma_d|_{G\cap S^1}$ maps $G \cap S^1$ onto $H \cap S^1$ as a covering map with positive orientation; in this case we write $\sigma_d(G) = H$.

Given a lamination $\mathcal{L}$ there exists the finest closed equivalence relation $\approx_{\mathcal{L}}$ (or simply $\approx$) on S^1 with the property that if $\overline{pq} \in \mathcal{L}$ then $p \approx q$ (for some laminations $\mathcal{L}$ all of S^1 is a single class and $S^1/\approx$ is a point). If $\mathcal{L}$ is d-invariant then $\approx$ is σ_d-invariant, and σ_d induces a branched covering map $f_{\mathcal{L}} : J_{\mathcal{L}} \to J_{\mathcal{L}}$, where $J_{\mathcal{L}}$ is the quotient space $S^1/\approx$. For a polynomial P with locally connected Julia set J_P the above defined lamination $\mathcal{L}_P$ gives rise to the equivalence $\approx_P$ with equivalence classes being the sets $C(y) \cap S^1, y \in J_P$ so that $P|_{J_P}$ and $f_{\mathcal{L}_P}|_{J_{\mathcal{L}_P}}$ are topologically conjugate. In particular $J_{\mathcal{L}_P}$ is non-degenerate. To avoid ambiguity we from now on consider only *q-laminations*, i.e. closed d-invariant laminations $\mathcal{L}$ such that the convex hull of each non-degenerate equivalence class of $\approx$, is either a leaf or a gap of $\mathcal{L}$.

Let us introduce some notions. Assume that $\mathcal{L}$ is a d-invariant q-lamination, $\approx$ is its equivalence, and $X \subset S^1$ is an equivalence class of $\approx$. Call X *critical* iff $\sigma_d|_X$ is not 1-to-1 and *precritical* iff $\sigma_d^j(X)$ is critical for some $j \geq 0$. Call X *preperiodic* if $\sigma_d^i(X) = \sigma_d^j(X)$ for some $0 \leq i < j$. A gap G is a *wandering n-gon* if $|G \cap S^1| = n \geq 3$ and $G \cap S^1$ is neither preperiodic nor precritical. A wandering 3-gon is a *wandering triangle*.

Now we list some known facts. J. Kiwi [6] extended Thurston's theorem by showing that every non-preperiodic non-precritical gap in a d-invariant lamination is at most a d-gon. In [8] G. Levin showed that laminations with one critical class do not have wandering n-gons. Another result was obtained in [1] (see Theorem 1.1). Let $k_{\mathcal{L}}$ be the maximal number of critical classes X of $\approx_{\mathcal{L}}$ with pairwise disjoint infinite σ_d-orbits such that $\sigma_d(X)$ is a singleton.

Theorem 1.1. *Let $\mathcal{L}$ be a d-invariant q-lamination and let Γ be a non-empty collection of wandering d_j-gons ($j = 1, 2, \dots$) with distinct grand orbits. Then $\sum_j (d_j - 2) \leq k_{\mathcal{L}} - 1 \leq d - 2$.*

Until now, it has not been known if wandering triangles exist; our main result shows that they do.

Theorem 1.2. *For each $d \geq 3$ there exists an uncountable collection of d-invariant q-laminations $\mathcal{L}(\alpha)$ with a wandering triangle such that the induced maps $f_{\mathcal{L}(\alpha)}|_{J_{\mathcal{L}(\alpha)}}$ are pairwise non-conjugate.*

2. Construction

For several obvious reasons we call laminations with wandering triangles *WT-laminations*. Here we outline the construction of one 3-invariant (*cubic*) WT-lamination. The example was inspired by ideas of [1] and [9]. In a later paper we will use the freedom of the construction to prove the full version of Theorem 1.2.

The circle S^1 is identified with the factor space $\mathbb{R}/\mathbb{Z}$; points of S^1 are denoted by real numbers $x \in [0, 1]$ with the induced circular order. By an arc (p, q) in the circle we mean the positively oriented arc from p to q. A few necessary conditions for a cubic lamination $\mathcal{L}$ to be a WT-lamination follow from [1] (or from [6]). Indeed, by Theorem 1.1 if $\mathcal{L}$ is a cubic WT-lamination then $k_{\mathcal{L}} = 2$. This implies that the two critical classes are leaves of $\mathcal{L}$ and $J_{\mathcal{L}}$ is a dendrite. Other more dynamical facts about cubic WT-laminations follow from [2].

Let $X \subset S^1$. A map $g : X \to g(X) \subset S^1$ is said to be *σ-extendable* if $X \cup g(X)$ can be embedded into S^1 by means of an order-preserving (not necessarily continuous) map φ so that the induced map $g' : \varphi(X) \to$

$g'(\varphi(X)) = \varphi(g(X)) \subset S^1$ (defined as $g' = \varphi \circ g \circ \varphi^{-1}$) coincides with the map $\sigma_d|_{\varphi(X)}$ for some d. The minimal such d is said to be the *pseudo-degree* of g.

The idea is to construct sets $A \subset A' \subset S^1$ and a σ-extendable map $g : A \to A'$ of pseudo-degree 3 so that A contains the g-orbit of a triple T_0 and T_0 is a wandering triangle of g. The construction is flexible and can be implemented in uncountably many ways. By the definition we can then embed A' into S^1 by means of an order-preserving map φ so that the induced map on $\varphi(A)$ coincides with σ_3. The set $\varphi(T_0)$ is a wandering triangle for σ_3. The σ_3-forward invariant lamination $\mathcal{L}'$ consisting of the sides of all triangles $\varphi(T_i)$ can be extended to a non-degenerate 3-invariant (*cubic*) lamination $\mathcal{L}$, and the uncountably many implementations of the construction give rise to uncountably many essentially distinct laminations $\mathcal{L}$. The extension onto higher degrees relies upon the techniques of "inserting an extra wrap" and completes the proof of Theorem 1.2.

Set $B = \{0 < c' < s_0 < u_0 < \frac{1}{2} < v_0 < d' < t_0 < 1\}$ and denote by $\bar{c}_0$ the chord with the endpoints u_0 and v_0 and by $\bar{d}_0$ the chord with the endpoints s_0 and t_0. Let the point u_{-k} be the only point such that $u_{-k} \in (u_0, v_0)$, $\sigma_3(u_{-k}) \in (u_0, v_0), \dots, \sigma_3^k(u_{-k}) = u_0$. Similarly we define points v_{-k}, s_{-k}, t_{-k}. Observe that $\lim_{n\to\infty} u_{-n} = 1/2$ and $\sigma_3(u_{-i}) = u_{-i+1}$; similar facts hold for v_{-n}, s_{-n}, and t_{-n}. All these points together with the set B form the set B'. The chord connecting u_{-k}, v_{-k} is denoted by $\bar{c}_{-k}$ and the chord connecting s_{-k}, t_{-k} is denoted by $\bar{d}_{-k}$.

The set B' is an initial part of A used to determine the location of other points of A on the circle. Below we will define the triple $T_0 = \{\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0\}$ and the set $X_0 = B' \cup T_0$. On each step a new triple $T_n = \{\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n\}$ is added and the set $X_n = X_{n-1} \cup \{\mathbf{x}_n, \mathbf{y}_n, \mathbf{z}_n\}$ is defined. We denote new points added on each step by boldface letters. This explains the notation in the next phrase: the map g on points $x_{n-1}, y_{n-1}, z_{n-1}$ is defined as $g(x_{n-1}) = \mathbf{x}_n$, $g(y_{n-1}) = \mathbf{y}_n$, $g(z_{n-1}) = \mathbf{z}_n$. Below by a "triple" we mean one of the sets T_i and by a "triangle" the convex hull of a triple.

We suggest the following system of notation and rules which are enforced throughout. Suppose X_{i-1} has been defined. The location of the i-th triple T_i is determined by points $p, q, r \in X_{i-1}$ with $p < x_i < q < y_i < r < z_i$ and $[(p, x_i) \cup (q, y_i) \cup (r, z_i)] \cap X_{i-1} = \emptyset$. In this case we write $T_i = T(p, \mathbf{x}_i, q, \mathbf{y}_i, r, \mathbf{z}_i)$. If 2 or 3 points of a triple are located between two adjacent points of X_{i-1} then we need less than 6 points to denote T_i - e.g., $T(p, x_i, y_i, q, z_i)$ where $p, q \in X_{i-1}$ means that

$p < x_i < y_i < q < z_i$ and $[(p, y_i) \cup (q, z_i)] \cap X_{i-1} = \emptyset$. Define the map g on all points of $(B' \cap [s_{-1}, t_{-1}]) \cup \{0\}$ as σ_3. Set $g(u_0) = g(v_0) = c'$, $g(s_0) = g(t_0) = d'$. This defines a map $g : B' \setminus \{c', d'\} \to B'$. In what follows the map g is constructed to be order preserving on subsets of A contained in the closures of components of $S^1 \setminus \{0, s_0, u_0, 1/2, t_0, v_0\}$.

Now we introduce locations of some initial triples: $T_0 = T(u_0, \mathbf{x}_0, \mathbf{y}_0, t_{-1}, \mathbf{z}_0)$, $T_1 = T(c', \mathbf{x}_1, \mathbf{y}_1, t_0, \mathbf{z}_1)$, $T_2 = T(v_0, \mathbf{x}_2, \mathbf{y}_2, d', \mathbf{z}_2)$, $T_3 = T(c', \mathbf{x}_3, \mathbf{y}_3, \mathbf{z}_3)$, $T_4 = T(s_{-1}, \mathbf{x}_4, v_{-1}, \mathbf{y}_4, \mathbf{z}_4)$, $T_5 = T(s_0, \mathbf{x}_5, v_0, \mathbf{y}_5, \mathbf{z}_5)$, $T_6 = T(0, \mathbf{x}_6, c', \mathbf{y}_6, \mathbf{z}_6)$, $T_7 = T(x_0, \mathbf{x}_7, \mathbf{y}_7, \mathbf{z}_7)$, $T_8 = (x_1, \mathbf{x}_8, \mathbf{y}_8, \mathbf{z}_8)$, $T_9 = T(x_2, \mathbf{x}_9, \mathbf{y}_9, \mathbf{z}_9)$, $T_{10} = T(x_3, \mathbf{x}_{10}, \mathbf{y}_{10}, \mathbf{z}_{10})$, $T_{11} = (u_{-1}, \mathbf{x}_{11}, \mathbf{y}_{11}, t_{-2}, \mathbf{z}_{11})$. Our rules force the location of some triples, e.g. the fact that $T_7 \subset (x_0, y_0)$ forces the location of T_8, T_9, T_{10}. The segment of triples $\{T_0, \dots, T_{11}\}$ is the basis of induction (see Figure 1).

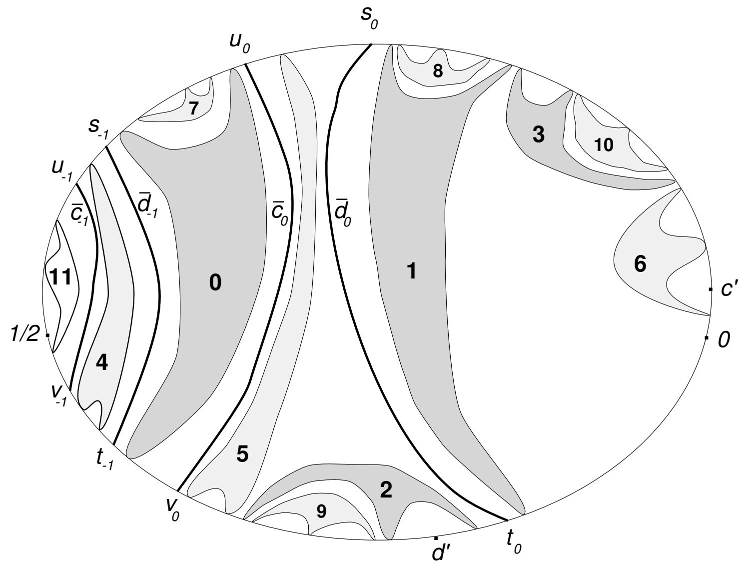

Figure 1. Twelve first triangles

Given two disjoint chords p, q denote by $S(p, q)$ the strip enclosed by p, q and arcs of the circle. Since T_{11} is contained in $S(\bar{d}_{-2}, \bar{c}_{-1})$, its image must be contained in $S(\bar{d}_{-1}, \bar{c}_0)$; set $T_{12} = T(y_0, \mathbf{x}_{12}, \mathbf{y}_{12}, t_{-1}, \mathbf{z}_{12})$. Thus, T_{12} separates the chord $\bar{d}_{-1}$ from T_0 (inside the unit disk). Our rules then imply that $T_{13} = T(y_1, \mathbf{x}_{13}, \mathbf{y}_{13}, t_0, \mathbf{z}_{13})$ separates the chord $\bar{d}_0$ from T_1. Moreover, this fact together with our rules forces the location of forthcoming triples $T_{14}, T_{15}, \dots$ with respect to $X_{13}, X_{14}, \dots$ for some time. The first time when the location of the triple is not forced is when T_1 is mapped onto T_{11} and T_{13} is mapped onto T_{23}. At this moment our rules guarantee that T_{23} must be located in the arc (y_{11}, z_{11}), but otherwise its location is not forced. The freedom of choice of the location of T_{23} at this point, and the similar variety of options which will be available later on at similar moments, is exactly the reason why the construction yields not just one, but uncountably many types of behavior of a wandering triangle.

However here we are only interested in one example, so we choose the location of T_{23} as $T_{23} = T(s_{-2}, \mathbf{x}_{23}, v_{-2}, \mathbf{y}_{23}, \mathbf{z}_{23})$ (in particular $T_{23} \subset S(\bar{c}_{-2}, \bar{d}_{-2})$). This implies that T_{24} must be located inside $S(\bar{c}_{-1}, \bar{d}_{-1})$, and we choose its location so that T_{24} separates $\bar{c}_{-1}$ from T_4 in the disk. This forces the location of T_{25} which separates $\bar{c}_0$ from T_5. As before with $\bar{d}_0, T_1$ and T_{13}, this determines the location of the triples $T_{26}, T_{27}, \dots$ relative to $X_{25}, X_{26}, \dots$ for some time until the choice for the location of the triple is not forced. This happens exactly at the moment when T_5 maps into T_{23} and T_{25} maps into T_{43}. We choose the location for T_{43} as $T_{43} = T(u_{-2}, \mathbf{x}_{43}, \mathbf{y}_{43}, t_{-3}, \mathbf{z}_{43})$ to mimic already existing triples T_0 and T_{11}. Then we choose T_{44} so that it separates $\bar{d}_{-2}$ and T_{11} and proceed with the construction as before. The step from T_{11} to T_{43} is the first inductive step in the construction.

Let us now describe the induction in general. Step n begins at a moment i_n with a triple $T_{i_n} = T(u_{-n}, \mathbf{x}_{i_n}, \mathbf{y}_{i_n}, t_{-n-1}, \mathbf{z}_{i_n}) \subset S(\bar{d}_{-n-1}, \bar{c}_{-n})$. It is followed by a segment of triples which comply with our rules and are contained in strips $S(\bar{d}_{-n}, \bar{c}_{-n+1}), S(\bar{d}_{-n+1}, \bar{c}_{-n+2}), \dots$ closer to chords $\bar{d}_{-n}, \bar{d}_{-n+1}, \dots$ than previously existing triples until the triple T_{j_n} whose triangle is located to the right of $\bar{d}_0$ and separates $\bar{d}_0$ from $T_{j_{n-1}}$ in the disk. From that time on the behavior of T_{j_n} is forced by our rules and behavior of $T_{j_{n-1}}$ until at a later moment the triple $T_{j_{n-1}}$ maps into T_{i_n} and T_{j_n} maps onto a new triple T_{k_n}, closer to $1/2$ than previous triples. We choose T_{k_n} as $T_{k_n} = T(s_{-n-1}, \mathbf{x}_{k_n}, v_{-n-1}, \mathbf{y}_{k_n}, \mathbf{z}_{k_n}) \subset S(\bar{c}_{-n-1}, \bar{d}_{-n-1})$. We now follow this triple by a series of triples contained in the strips $S(\bar{c}_{-n}, \bar{d}_{-n}), S(\bar{c}_{-n+1}, \bar{d}_{-n+1}), \dots$ closer to the chords $\bar{c}_{-n}, \bar{c}_{-n+1}, \dots$ than previously existing triples. This series of triples ends with a triple T_{l_n} whose triangle separates $\bar{c}_0$ from $T_{l_{n-1}}$ and whose future behavior is forced by that of $T_{l_{n-1}}$ until the moment when $T_{l_{n-1}}$ maps onto T_{k_n}. At this moment T_{l_n} maps onto $T_{i_{n+1}}$ and the construction repeats.

This leads to a set $A' = B' \cup (\cup_{i=0}^{\infty} T_i) \subset S^1$ such that all T_i's have pairwise disjoint convex hulls. Moreover, the map g is defined on $A = A' \setminus \{c', d'\}$. We then prove that in fact $g : A \to A'$ is σ-extendable of pseudo-degree 3. Thus, we can embed A' into S^1 by means of an order-preserving map φ so that the induced map on $\varphi(A)$ coincides with σ_3. The set $\varphi(T_0)$ is a wandering triangle for σ_3. This forward invariant non-closed lamination can be completed to a closed cubic invariant lamination. By the construction, there are countably many available choices as to in what strips the triangles T_{i_n} and T_{k_n} can be placed. That leads to uncountably many cubic WT-laminations whose induced maps on the quotient spaces are non-conjugate. The extension onto higher degrees relies upon the techniques of "inserting an extra wrap" and completes the proof of Theorem 1.2.

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