Morse Novikov Theory and Cohomology with Forward Supports

F. Reese Harvey and Giulio Minervini

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Abstract

We extend an approach to Morse theory due to Harvey and Lawson to non compact manifolds and give a new interpretation of the Morse-Novikov complex, via cohomology with supports.

Weakly Proper Morse Smale Flows

Suppose $f : Y \to \mathbb{R}$ is a Morse-Smale function on an oriented (not necessarily compact) Riemannian manifold. Assume that the gradient vector field of f is complete, insuring a flow ϕ on Y. Recall the transversality condition of Smale:

Smale *For any two critical points p, $q \in Cr(f)$, the stable manifold S_p and the unstable manifold U_q intersect transversally.*

To overcome lack of compactness of Y, we will also assume the following:

Weakly Proper *The function f is **weakly proper** if the intersection of each broken flow line with each slab $f^{-1}([a,b])$ is compact.*

This more delicate notion of weakly proper includes the case when f is proper (which is not general enough for our purpose). Next, we briefly recall the Harvey-Lawson technique.

Denote by $\phi = (\phi_t)_{t\in\mathbb{R}}$ the flow of the vector field $V = -grad(f)$ and by Φ the **total graph map**

$$\Phi : \mathbb{R} \times Y \to Y \times Y$$
$$(t,x) \mapsto (\phi_t(x), x)$$

The total graph map Φ is regular near points where $V \neq 0$. In fact, $\Phi|_{\mathbb{R}\times(Y\setminus Cr(f))}$ defines a one-to-one immersed submanifold $T \subset Y \times Y$. The graphs

$$\Phi(t \times Y) = graph(\phi_t)$$

are closed embedded submanifolds for any value of t . These submanifolds are conveniently described as pushforwards, namely

$$T_t = \Phi_*([0,t] \times Y) \quad \text{and} \quad P_t = \Phi_*(t \times Y) = [graph(\phi_t)]$$

define currents of dimension $n+1$ and n, respectively, in $Y \times Y$, in particular the $graph(\phi_0)$ is just the diagonal Δ. Since taking boundary commutes with current pushforward:

$$\partial T_t = \Delta - P_t$$

Finite Volume Flows *If the submanifold T defined by $\Phi|_{\mathbb{R}\times(Y\setminus Cr(f))}$ has locally finite volume in Y then ϕ is called a finite volume flow (this is actually a property of the associated singular foliation).*

Theorem (Minervini) *Each weakly proper Morse Smale flow is a finite volume flow. Moreover, $\overline{T}$ is a stratified space and can be locally described as the image of a compact manifold with corners D under a smooth stratified map σ with the special property that the restriction of σ to each stratum of D is a submersion onto the corresponding stratum of $\overline{T}$.*

A similar result also holds for any stable (unstable) manifold, and in this case the singular strata are other stable (unstable) manifolds. The precise description of the singularities in the statement is needed in full in the theory, because of some intersection issues.

Consequently, T defines a current (which we again denote by T) and the limit

$$\lim_{t\to+\infty} T_t = T$$

holds in the mass norm (i.e. locally the volume of $T - T_t$ decreases to zero). In particular, the currents $P_t = \Delta - \partial T_t$ converge. A simple geometric description is available for $P = \lim_{t\to+\infty} P_t$, namely:

$$P = \textstyle\sum_{p\in Cr(f)} [U_p \times S_p] \qquad (P)$$

The limiting equation

$$\partial T = \Delta - P \qquad (\mathbf{FME})$$

will be referred to as the **Fundamental Morse Equation**. Via the kernel calculus developed by Harvey and Polking in [HP], the FME is equivalent to an operator equation where the operators map test forms on Y to currents on Y (we denote the operators by the same letters we use for the corresponding currents). In particular Δ determines the identity operator and $P_t = [graph\,(\phi_t)]$ determines the pullback operator ϕ_t^*. The operator equation determined by FME:

$$d\circ T + T\circ d = I - P \qquad (\mathbf{MCH})$$

will be referred to as the **Morse Chain Homotopy.** The kernel equation (P) determines an operator equation for $P = \lim_{t\to+\infty} \phi_t^*$, namely:

$$P(\alpha) = \sum_{p\in Cr(f)} \left(\int_{U_p} \alpha\right)[S_p] \qquad (P')$$

for any test form α on Y (the convergence is in the topology of currents, and actually in the "flat topology"). Note that the limit of $\phi_t^*(\alpha)$ where α is a compactly supported form, is generally not a compactly supported current (since this limit is a sum of stable manifolds).

The classical Morse complex is a formal complex of finitely generated free groups having the critical points of the Morse function as generators, and boundary defined in a certain geometric way.

Harvey and Lawson ([HL]) realized this complex (for Y compact and under a certain *tameness* condition), as the subcomplex of the complex of currents on Y consisting of finite sums of (the currents defined by integration over the) stable manifolds (the $\mathcal{S}$**-complex**). In this way the boundary in the "formal" Morse complex can be interpreted as the boundary of stable manifolds as currents, relying on the fact that the boundary of a stable manifold is a finite integer sum of other stable manifolds of lower dimension.

In the non compact case, the critical points are no longer finite in number and, more important, the boundary of a stable manifold (as a current) might be the sum of a family of stable manifolds which might not even be locally finite! Nevertheless one can overcome this problem by requiring the weakly proper condition. Under this assumption:

The boundary of a stable manifold S_p is an integer sum of other stable manifolds which are a finite family on each slab $f^{-1}([a,b])$.

Of course, each stable manifold is *forward directed* in the sense that $f(S_p) \subset [f(p), +\infty)$.

Forward Supports: the Current Morse Complex

The motivation for introducing the family of "forward sets" and considering cohomology with forward support arises by trying to find a setting in which the operators P and T are still defined and the FME holds. The structure and position of the stable and unstable manifolds and the expression for P make it simple to guess that P extends to forms which are supported in $f^{-1}([a,+\infty[)$ for some $a \in \mathbb{R}$ and compactly supported in $f^{-1}([b,c])$ for any constants $b \leq c \in \mathbb{R}$, and it's quite obvious that the range of the operator consists of currents supported in the same way. This justifies the following definition.

Definition (compact/forward set) *A closed set $A \subset Y$ is a compact/forward set (abbreviated c/f set) with respect to the Morse function f if both*

- *$A \cap f^{-1}([b,c])$ is compact for any $b \le c \in \mathbb{R}$ (i.e. A is slab compact)*
- *$A \subset f^{-1}([a,+\infty))$ for some constant $a \in \mathbb{R}$ (i.e. A is forward).*

Equivalently, $A \cap f^{-1}([-\infty,a))$ is compact for all $a \in \mathbb{R}$.

The subscript $c\uparrow$ will denote the family of compact/forward sets. For example, $\mathcal{E}^*_{c\uparrow}(Y) = \Gamma_{c\uparrow}(Y,\mathcal{E}^*)$ denotes the space of smooth forms with c/f support.

Clearly the compact/forward sets are a paracompactifying families for Y in the terminology of [G]. Consequently, either the complex $\mathcal{D}'^*_{c\uparrow}(Y)$ of currents with c/f support, or the complex $\mathcal{E}^*_{c\uparrow}(Y)$ of smooth forms with c/f support can be used to compute cohomology with c/f supports, $H^*_{c\uparrow}(Y,\mathbb{R})$.

Lemma *The operators T and P, a priori defined on $\mathcal{E}^*_{cpt}(Y)$, extend to $\mathcal{E}^*_{c\uparrow}(Y)$ and the Morse Chain Homotopy continues to hold.*

Next we introduce the $\mathcal{S}$-complex and describe cohomology with forward supports in terms of it.

Definition (**the $\mathcal{S}$-complex over** $\mathbb{Z}$) *Let ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)$ denote the subcomplex of $\mathcal{D}'^*_{c\uparrow}(Y)$ (the complex of currents with c/f support) consisting of those currents of the form*

$$\sum_{p\in F} a_p [S_p] \quad \text{where } F \text{ is a c/f set of critical points and } a_p \in \mathbb{Z}$$

The boundary $d : {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \to {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)$ is the current boundary; note that each element of ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)$ is a locally finite sum. We will sometime skip the explicit refence to the function f and just write ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}$. Similarly we define ${}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)$ (the **$\mathcal{S}$-complex over** $\mathbb{R}$).

Theorem *The maps*

$$P : \mathcal{E}^*_{c\uparrow}(Y) \longrightarrow {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f) \quad \text{and} \quad {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f) \hookrightarrow \mathcal{D}'^*_{c\uparrow}(Y)$$

induces isomorphisms in cohomology:

$$H^*_{c\uparrow}(Y,\mathbb{R}) \approx H^*\left({}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)\right)$$

As for integer coefficients, one can use the $\mathcal{S}$-complex over the integers to compute $H^*_{c\uparrow}(Y,\mathbb{Z})$. First, consider the complex of local chain currents. Following deRham ([dR]), we recall that a **local chain current** is a current that can be locally described as a finite integer sum of (currents defined via pushforward by) smooth simplexes. Let's denote by $\mathcal{C}^*(Y)$ the complex of local chain currents and by $\mathcal{C}^*_{c\uparrow}(Y)$ the subcomplex with c/f support. Note that ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow} \subseteq \mathcal{C}^*_{c\uparrow}(Y)$.

It can be proved that the operator P extends to a chain current C if both C and ∂C are transversal to all the unstable manifolds. For the operator T to act on C we need, in addition, to require that $C \times Y$ is transversal to the submanifold $T \subset Y \times Y$. Note that each current in the $\mathcal{S}$-complex fulfils those transversality conditions.

We remark that the unstable manifolds and the submanifold T are not closed sets. Nevertheless, the description of their singularities as "submersed by manifold with corners" gives an additional meaning to requiring transversality with each of them: this allows the intesections needed for extending P and T to act on C.

Observing that P acts as the identity on ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}$, it's not difficult to prove the following.

Theorem *The inclusion of complexes* ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \subseteq \mathcal{C}^*_{c\uparrow}(Y)$ *induces an isomorphism in cohomology*

$$H^*({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)) \approx H^*_{c\uparrow}(Y, \mathbb{Z})$$

That is, the $\mathcal{S}$-complex over the integers computes cohomology with forward supports and integer coefficients. Consequently, if ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}$ is a finitely generated group, then so is $H^*_{c\uparrow}(Y, \mathbb{Z})$ and standard Morse inequalities follow (the strong inequalities over $\mathbb{Z}$).

Remark (stability) *Suppose f_0 and f_1 are two weakly proper functions on Y whose difference is bounded (say by the constant $c \geq 0$). Then f_0 and f_1 determine the same family of compact/forward sets.*

Since $f_1^{-1}([-\infty, a)) \subset f_0^{-1}([-\infty, a+c))$ for any $a \in \mathbb{R}$, if A is c/f with respect to f_0 then A is also c/f with respect to f_1. Actually, either of the notions forward set and slab compact (cf definition of c/f set) are the same for f_0 and f_1.

Forward-Backward duality

We finish this review with some comments on the case of f proper. A first obvious fact is that the assumption "compact" in the definition of a compact/forward set is superfluous. Moreover in this case one can define backward sets in a complete analogue way, and consider the corresponding complexes of forms, currents and local chains currents. The analogous of the $\mathcal{S}$-complex is now the $\mathcal{U}$-complex, defined as the subcomplex of backward supported currents made up of locally finite sums of unstable manifolds. There clearly is a duality between forms with forward support and currents with backward support, and one expects the duality to pass to cohomology. This is true, and a simple proof is provided by showing that the duality between the $\mathcal{S}$-complex and the $\mathcal{U}$-complex pass to cohomology, providing a "forward-bacward duality". In particular, if Y is compact one retrieves Poincare' duality, and if f is "coercive" (i.e. $f^{-1}([-\infty, a))$ are compact) then one retrieves deRham duality between homology (with compact supports) and cohomology (with arbitrary supports).

If f is not proper, a duality might be found between homology with compact forward supports and cohomology with "backward supports". But this has no applications to Novikov theory and will not be pursued here.

Circle Valued Morse Theory

Morse Novikov theory is a variation of the previous theory, governed by the addition of the action of a certain (Novikov) ring Λ on subsets of Y. This action commutes with the flow. In this section we consider the special case of "cyclic coverings". In the general case, there is less compatibility between the algebraic structure and the dynamical system.

Suppose a circle valued Morse function $g : X \longrightarrow \mathbb{R}/\mathbb{Z}$ is given on the compact manifold X, and consider a gradient vector field for g, whose flow ϕ is Smale.

Let now $\sigma : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ be the quotient map and let

$$\begin{array}{ccc} Y & \overset{f}{\longrightarrow} & \mathbb{R} \\ \downarrow \rho & & \downarrow \sigma \\ X & \overset{g}{\longrightarrow} & \mathbb{R}/\mathbb{Z} \end{array}$$

be the pullback covering. The group of deck transformations is the integers $\mathbb{Z} = \langle t \rangle$, where $t : Y \longrightarrow Y$ is a diffeomorphism. The equivariance $f(ty) = f(y) + 1$ relates the covering group and the Morse function f.

Using ρ, the gradient vector field and the flow ϕ can be lifted to a vector field and flow ψ on Y. The flow ψ is the gradient of the Morse function f and is again Smale. The function f is actually proper (not just weakly proper). The critical points upstairs are just the preimages of the critical points downstairs. The main difference between the two dynamical systems is that upstairs there are no closed orbits (nor closed broken flow lines) whereas downstairs there might be some. Actually any flow line that has no finite limit point downstairs lifts to a closed curve (necessarily tending to ∞ in Y).

Consider the **group rings** of the covering

$$\mathbb{R}\left[t, t^{-1}\right] \quad \text{and} \quad \mathbb{Z}\left[t, t^{-1}\right]$$

i.e. the the rings of Laureant polinomials in t. Define the **Novikov rings**

$$\Lambda_{\mathbb{R}} = \mathbb{R}\left[[t], t^{-1}\right] \quad \text{and} \quad \Lambda_{\mathbb{Z}} = \mathbb{Z}\left[[t], t^{-1}\right]$$

to be the ring of formal Laureant series with finite principal parts. In particular $\Lambda_{\mathbb{R}}$ is actually a field. Moreover $\Lambda_{\mathbb{R}}$ is a $\mathbb{R}\left[t, t^{-1}\right]$-module and $\Lambda_{\mathbb{Z}}$ is a $\mathbb{Z}\left[t, t^{-1}\right]$-module.

Compact/forward sets can be defined algebraically, as a simple consequence of the interaction of the deck map t and of f.

Lemma *A closed set $A \subset Y$ is a compact/forward set if and only if there exists a compact set $K \subset Y$ and an integer $N \in \mathbb{Z}$ such that $A \subset \bigcup_{n \geq N} t^n(K)$.*

Let's now reconsider the complexes of forms and currents $\mathcal{E}^*_{c\uparrow}(Y)$, $\mathcal{D}'^*_{c\uparrow}(Y)$, $\mathcal{C}^*_{c\uparrow}(Y)$, and $\mathcal{S}^*_{c\uparrow}(f)$ defined in the previous section. Since t commutes with the flow ψ, the previous lemma implies that the action of t by pushforward is a self map of all the previous complexes. This induces actions of the group rings and Novikov rings. Since the operators

$$T : \mathcal{E}^*_{c\uparrow}(Y) \longrightarrow \mathcal{D}'^*_{c\uparrow}(Y)$$
$$P : \mathcal{E}^*_{c\uparrow}(Y) \longrightarrow {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f) \subset \mathcal{D}'^*_{c\uparrow}(Y)$$

commute with the action of t, they are $\Lambda_{\mathbb{R}}$-linear maps.

Theorem *The map of $\Lambda_{\mathbb{R}}$-complexes*

$$P : \mathcal{E}^*_{c\uparrow}(Y) \longrightarrow {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)$$

induces an isomorphism of $\Lambda_{\mathbb{R}}$-vector spaces

$$H^*_{c\uparrow}(Y, \mathbb{R}) \approx H^*\left({}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)\right)$$

Moreover, $\dim_{\Lambda_{\mathbb{R}}} {}_{\mathbb{R}}\mathcal{S}^k_{c\uparrow} = \#$*(critical points of index k for g) is finite.*

Any choice of a lifting $p \in Cr(g) \longmapsto \bar{p} \in Cr(f)$ for the set of critical points downstairs will provide a $\Lambda_{\mathbb{R}}$ basis for ${}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}(f)$, consisting of the stable manifolds $S_{\bar{p}} \in {}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}$ at those (lifted) points. The inequalities of Morse type between dimensions (over $\Lambda_{\mathbb{R}}$) of ${}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}$ and of $H^k\left({}_{\mathbb{R}}\mathcal{S}^*_{c\uparrow}\right)$ follow in the standard manner.

As for the theory with integer coefficients, observe that the inclusion map ${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \hookrightarrow \mathcal{C}^*_{c\uparrow}(Y)$ commutes with the action (as pushforward) of t and the complexes involved are complexes of $\mathbb{Z}\left[t, t^{-1}\right]$-modules as well as of $\Lambda_{\mathbb{Z}}$-modules.

Theorem *The inclusion map of $\Lambda_{\mathbb{Z}}$-complexes*

$${}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \hookrightarrow \mathcal{C}^*_{\uparrow}(Y)$$

induces an isomorphism of $\Lambda_{\mathbb{Z}}$-modules

$$H^*\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)\right) \approx H^*_{c\uparrow}(Y, \mathbb{Z})$$

Moreover, ${}_{\mathbb{Z}}\mathcal{S}^k_{c\uparrow}$ is finitely generated, with one generator in ${}_{\mathbb{Z}}\mathcal{S}^k_{c\uparrow}$ for every critical point of g of index k (downstairs).

Again the Novikov inequalities are a standard algebraic consequence of this theorem, exactly as in Morse theory.

Next we compare $H^*_{cpt}(Y,\mathbb{Z})$ and $H^*_{c\uparrow}(Y,\mathbb{Z})$, cf [P]. The sheaf cohomology groups $H^*_{cpt}(Y,\mathbb{Z})$ are standard topological invariants of Y (isomorphic to $H_{n-p}(Y,\mathbb{Z})$, i.e. homology). Altough the constructions of T, P and the $\mathcal{S}$-complex depend on f, the cohomology with c/f supports $H^*_{c\uparrow}(Y,\mathbb{Z})$ only depends on the covering translation t. Consequently, to compute the c/f supported cohomology we may replace f by a new f which is the lift to Y of a (real valued!) Morse function $g : X \to \mathbb{R}$. The isomorphism of Λ-modules $H^*\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f)\right) \approx H^*_{c\uparrow}(Y,\mathbb{Z})$ remains valid for the new f.

Now each stable manifold S_p is relatively compact, therefore $[S_p]$ has compact support and its boundary consists of a finite sum of other stable manfiolds. In particular the space ${}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f)$ (made up of finite sums stable manifolds) is closed under taking boundary, i.e. it is a complex. Moreover, the operator P maps $\mathcal{E}^*_{cpt}(Y)$ to ${}_{\mathbb{R}}\mathcal{S}^*_{cpt}(f)$ and the operator T is a chain homotopybetween P and the identity $I : \mathcal{E}^*_{cpt}(Y) \longrightarrow \mathcal{D}'_{cpt}{}^*(Y)$.

Consequently, there are isomorphisms of real vector spaces and abelian groups:

$$H^*_{cpt}(Y,\mathbb{R}) \approx H^*\left(\mathcal{E}^*_{cpt}(Y)\right) \approx H^*\left({}_{\mathbb{R}}\mathcal{S}^*_{cpt}(f)\right)$$

$$H^*_{cpt}(Y,\mathbb{Z}) \approx H^*\left(\mathcal{C}^*_{cpt}(Y)\right) \approx H^*\left({}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f)\right)$$

The (covering) group ring $\mathbb{Z}[\pi] = \mathbb{Z}\left[t,t^{-1}\right]$ of Laureant polynomials acts on ${}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f)$ and $\mathcal{C}^*_{cpt}(Y)$. Therefore ${}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}}$ and $\mathcal{C}^*_{cpt}(f) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}}$ are complexes of $\Lambda_{\mathbb{Z}}$ modules and there are isomorphisms of $\Lambda_{\mathbb{Z}}$ modules:

$${}_{\mathbb{Z}}\mathcal{S}^*_{cpt}(f) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}} = {}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}(f) \quad \text{and} \quad \mathcal{C}^*_{cpt}(f) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}} = \mathcal{C}^*_{c\uparrow}(f)$$

Since $\Lambda_{\mathbb{Z}}$ is flat over $\mathbb{Z}[\pi] = \mathbb{Z}\left[t,t^{-1}\right]$, taking homology of the complexes yelds:

Theorem As finitely generated $\Lambda_{\mathbb{Z}}$-modules:

$$H^*_{c\uparrow}(Y,\mathbb{Z}) \approx H^*_{cpt}(Y,\Lambda_{\mathbb{Z}})$$

where, by definition, $H^*_{cpt}(Y,\Lambda_{\mathbb{Z}}) = H^*_{cpt}(Y,\mathbb{Z}) \underset{\mathbb{Z}[\pi]}{\otimes} \Lambda_{\mathbb{Z}}$.

Modified Novikov Theory

Let ω be a Novikov 1-form on the compact Riemannian manifold X, i.e. a closed form with nondegenerate singularities. Its gradient vector field defines a flow ϕ on X. Using ϕ, one can then define global stable and unstable manifolds. We will assume this flow to be **Smale** (i.e. all stable and unstable manifolds have to intersect transversally: this is known to be a generic condition for this kind of gradient fields, cfr [?2]).

Let $k-1$ be the irrationality index of ω and $\chi = (\chi_1, .., \chi_k)$ denote its periods, which one can assume to be positive numbers. Let $\rho : Y \to X$ be a minimal covering such that ω pulls back to an exact form, say df, with $f : Y \longrightarrow \mathbb{R}$. The group π of deck translations of (Y, ρ) is a free abelian group with k generators, say $t_1, ..., t_k$ (i.e. $\pi \approx \mathbb{Z}^k$) and the **group rings** (over $\mathbb{R}$ and $\mathbb{Z}$) are the Laurent polynomials

$$\mathbb{R}[\pi] = \mathbb{R}\left[t_1, .., t_k, t_1^{-1}, .., t_k^{-1}\right] \quad \text{and} \quad \mathbb{Z}[\pi] = \mathbb{Z}\left[t_1, .., t_k, t_1^{-1}, .., t_k^{-1}\right]$$

The equivariance relations $f(t_i(y)) = f(y) + \chi_i$ hold for any $i = 1, .., k$.

If $k = 1$ the covering is cyclic and the one form ω can be seen as the differential of a circular valued function, which was the case in the previous section.

Using the covering map ρ as it has been done for cyclic coverings, the gradient vector field and the flow ϕ can be lifted to a vector field and flow ψon Y. The flow ψ is the gradient of the (not proper!) Morse function f and is again Smale. The critical points upstairs are just the preimages of the critical points downstairs and "upstairs" there are no closed orbits (nor broken closed orbits).

Lemma *The lifted flow ψ is weakly proper.*

Proof. Suppose $\bar{\gamma} : [0, +\infty[\to Y$ is a forward flow-half line of ψ which is not relatively compact in Y and consider its projection $\gamma = \rho(\bar{\gamma})$ (γ is a forward flow-half line for ϕ): we just need to show that $\int_\gamma \omega = +\infty$ (which is trivially true if γ is a periodic orbit for ϕ). Observe that γ cannot converge to a critical point for ϕ, otherwise $\bar{\gamma}$ would have too, and therefore it has to go around in X, staying often away from critical points. In these parts of the travel $\int_{\gamma[0,t]} \omega$ will increase discretely, being uniformely bounded below by some constant defined using the minimum of the distances between two critical points.

Remark *The previous lemma (and proof) holds for any covering over which ω pullbacks to an exact form, in particular for the universal covering.*

Now we modify the Novikov Theory (for $k > 1$) by introducing a new ring. Let χ also denote the linear functional on $\mathbb{R}^k$ defines by $\chi(v) = \chi{\cdot}v$, where χ is the vector of periods.

Definition (c/f**-set in the lattice**) *A subset $F \subset \mathbb{Z}^k$ is a **compact-forward set in the lattice** (with respect to χ) if F is **slab compact** (i.e. $F \cap \chi^{-1}[a, b]$ is compact for any $a \leq b$) and **forward** (i.e. $F \subset \chi^{-1}([a, +\infty))$ for some $a \in R$).*

Definition (c/f **Ring**) *The compact forward ring $\Lambda_{c/f}$ (or R) consists of all formal Laurent series $\sum_{n \in F} a_n t^n$ whose support F is a c/f-set in the lattice Z^k. The coefficients are taken in Z or R.*

Note that the c/f condition $F \subset \mathbb{Z}^k$ insures that that products $\lambda_1 \cdot \lambda_2$ in $\Lambda_{c/f}$ are well defined, since for any $p \in F(\lambda_1)$, $q \in F(\lambda_2)$, there is just a finite number of solutions to the equation $p+q=n$. The ring ${}_{\mathbb{R}}\Lambda_{c/f}$ is in fact a field.

Exactly as in the cyclic covering case, c/f based of f can be defined algebraically in terms of the covering group $\mathbb{Z}^k$.

Lemma *A closed set $A \subset Y$ is a compact/forward set if and only if there exists a compact set $K \subset Y$ and a c/f set F in the lattice $\mathbb{Z}^k$ such that A is contained in the union of the sets $t^n(K)$ over $n \in F$.*

The three Theorems of the last section and their proof hold in this setting if one just substitutes the ring Λ_{cf} for Λ. The statements will not be repeated, but we will point out an important remark.

Remark (Topological Stability) Any two Novikov forms in the same cohomology class in $H^1(X, \mathbb{R})$ define the same c/f sets on Y. In fact they differ by the differential of a bounded function (since X is compact). Therefore their liftings to Y differ by a bounded function. The last remark in the section on Morse theory applies.

Novikov Theory

Finally, we compare the previous results with Novikov theory. For the sake of conciseness, we will restrict to integer coefficients. It is convenient to define the Novikov ring in terms of supports.

Definition (**N-set in the lattice**) *A subset $F \subset \mathbb{Z}^k$ is a **cone-forward set in the lattice** (with respect to χ) if there exists an $a \in \mathbb{R}$ and $\varepsilon > 0$ s.t. $F \subset \chi^{-1}([a, +\infty))$ and **(stability)** this remains true for all χ^* with $|\chi - \chi^*| < \varepsilon$.*

Definition (**Novikov Ring**) *The Novikov ring Λ consists of all formal Laurent series $\sum_{n \in F} a_n t^n$ whose support F is a cone-forward set in the lattice $\mathbb{Z}^k$.*

Note that any cone-forward set is compact/forward, so that the Novikov ring Λ is a subring of the ring $\Lambda_{c/f}$.

Definition (**Novikov-forward set in** Y) *A closed subset $A \subset Y$ is a **Novikov-forward** set (abbreviated N-set) if there exists a compact set $K \subset Y$ and a cone-forward set F in the lattice $\mathbb{Z}^k$ such that $A \subset \bigcup_{n \in F} t^n(K)$.*

Note that Novikov-forward sets are compact/forward with respect to f, since $f(t^n(y)) = \chi \cdot n + f(y)$.The converse is not always true if $k > 1$ because the lattice contains compact/forward sets which are not cone/forward.

Clearly, each covering translation t_i acts on the various complexes of forms and current with support in N-sets and the different actions commute (since the t_i's do). This allows one to define actions of the group ring and Novikov ring "by linearity" on those complexes; the supports are in fact preserved by the action of the Novikov ring Λ. One can also define an $\mathcal{S}$-complex ${}_{\mathbb{Z}}\mathcal{S}_N^*$ with supports in N-sets and all the previous arguments carry over substituting N-sets for compact/forward sets with respect to f.

The three theorems stated for the cyclic cover case and their proof hold for the ($k > 1$) Novikov case if one substitutes N-supports for compact/forward supports, and will not be repeated here.

We finish with a little degression on the Novikov ring and the invariants involved in Novikov theory.

The Novikov ring is an algebraic device, having two main properties:

- The free group generated by the critical points upstairs is a finitely generated Λ module, with one generator for each critical pointdownstairs;

- Inequalities (like in the Morse case) can be derived by defining a boundary and obtaining a complex of Λ-modules which computes cohomology with Λ-forward supports of Y (the invariant involved are the ranks of the cohomology Λ-modules).

From our point of view, the main result of Novikov theory are the Novikov numbers and the corresponding inequalities. With this spirit, instead of having an algebraically elegant choice for a ring (as Λ is) and obtain cohomology with supports in some family of sets (N-sets), we choosed the supports in a geometric way and defined the ring accordingly. The ring Λ_{cf} satisfies the above properties of the Novikov ring but in addition compact/forward sets for f can be defined algebraically using Λ_{cf}. Moreover, Λ_{cf} coincides with Λ in the case of cyclic coverings. Also, exactly the same Novikov numbers and inequalities come out using Λ_{cf} instead of Λ, even for $k > 1$:

By the way, an extension Π of a ring Λ is said to satisfy the *whatweneed* condition if Π is flat as a module over Λ and the minimal number of generators of any Λ-module A is the same than that of the Π-module $A \bigotimes_\Lambda \Pi$.

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Theorem (we hope) Considering Λ_{cf} as a (not finitely generate!) Λ-module yields isomorphisms of finitely generated Λ_{cf}-modules

$$\mathcal{S}_{c\uparrow} \approx \mathcal{S}_N \otimes_\Lambda \Lambda_{cf}$$

Moreover, Λ_{cf} is a **whatweneed** Λ-module and there are isomorphisms of finitely generated Λ_{cf}-modules :

$$H^*_{c\uparrow}(Y, \mathbb{Z}) \approx H^*\left({}_{\mathbb{Z}}\mathcal{S}^*_{c\uparrow}\right) \approx H^*\left({}_{\mathbb{Z}}\mathcal{S}^*_N\right) \otimes_\Lambda \Lambda_{cf} \approx H^*_N(Y, \mathbb{Z}) \otimes_\Lambda \Lambda_{cf}$$

The novikov numbers and inequalities are the same over Λ or Λ_{cf}.

REFERENCES

[D] deRham,G. Differentiable manifolds. Forms, currents, harmonic forms. *GMW* ***266****. Springer-Verlag, Berlin, 1984*

[F] Federer,H. Geometric measure theory, *GMW* ***153*** *Springer-Verlag, New York 1969*

[G] Godement,R. Topologie algébrique et théorie des faisceaux. *Hermann, Paris, 1973*

[HL] Harvey,F.R. and Lawson, B. Finite volume flows and Morse theory, Ann.of Math. ***153****, 2001, 1-25*

[HP] Harvey,F.R. and Polking, J. Fundamental solutions in complex analysis, Part I, Duke Math J. ***46*** *1979 253-300*

[M] Minervini,G. PhD thesis, Universita' "La Sapienza", Roma.

[N] Novikov,S.P. Topology I, *Encyclopaedia of Math. Sci. 12, Springer 1996*

[P] Pajitinov,A.V. Counting closed orbits of Gradients of Circle Valued Maps, preprint DG/0104273 v2

[S] Smale,S. On gradient dynamical systems. Ann.of Math. *2,* ***74,*** *1961, 199–206;*

[Ranicki?] reference for group rings and cohomology with coefficients in a $Z[\pi]$module.

[Ranicki, Pajitnov?] reference for "gradient of ω is generically smale".