Heegaard splittings and hyperbolic geometry

Hossein Namazi

Abstract

It is well known that every closed 3-manifold has a Heegaard splitting and the combinatorics of the Heegaard splitting identifies the 3-manifold. Yet it has been hard to use Heegaard splittings to obtain information about topology and geometry of the manifold. We develop a new approach to use hyperbolic geometry and in particular deformation theory of compressible ends of hyperbolic manifolds to study closed 3-manifolds. Using this approach, we have been able to prove that a big class of 3-manifolds which admit a Heegaard splitting with what we call "bounded combinatorics" admit a negatively curved metric with sectional curvatures pinched about -1. This answers some unknown questions about these manifolds and in fact gives a coarse description of the geometry of these manifolds equipped with the negatively curved metrics.

The description of these geometries is motivated by work of Minsky in constructing models for hyperbolic manifolds with incompressible boundary. In fact, much of our work is aimed at developing a similar theory in the compressible boundary case.

Contents

1 Introduction

Suppose H^+ and H^- are 3-dimensional handlebodies whose boundaries are identified with an oriented closed surface of genus $g \geq 1$ in a way that the orientation of S agrees with the orientation of ∂H^+ and does not agree with the one of ∂H^-. If we glue the handlebodies along S, we obtain a closed oriented 3-manifold $M = H^+ \cup_S H^-$. Such a decomposition is called a *Heegaard splitting* and it is well known that every closed orientable 3-manifold admits such a splitting. We call the surface S a *Heegaard surface* and two Heegaard splittings for M are equivalent if the associated Heegaard surfaces are isotopic in M. The only 3-manifolds with a Heegaard splitting of genus ≤ 1 are S^3, $S^2 \times S^1$ and Lens spaces. These manifolds are not interesting in our discussions and therefore we always assume that a Heegaard splitting has genus at least 2.

An important problem in studying 3-manifolds is using the combinatorics of the Heegaard splitting and obtain topological and geometrical information about the 3-manifold and its geometries. Hempel [He01] introduced an invariant of the Heegaard splitting, which we call *handlebody distance*, and conjectured that the 3-manifold is hyperbolic (admits a hyperbolic metric) if it has a Heegaard splting with handlebody distance at least 3.

Work of Haken [Ha68], Casson-Gordon [CG87], Hempel [He01], Thompson [Tho99] and Moriah-Scholtens [MS98] shows that every Heegaard splitting of a 3-manifold that is reducible, toroidal or Seifert fibered has handlebody distance at most 2. Therefore Hempel's conjecture agrees with the description of 3-manifolds given by Thurston's Hyperbolization Conjecture.

We introduce a large family of Heegaard splittings which have what we call *R-bounded combinatorics* for some $R > 0$. This definition is motivated by work of Minsky [Min01] for representations of a surface group in $\mathrm{PSL}_2(\mathbb{C})$ and will be more precisely defined in 2.6. Our main theorem here is the following:

Main Theorem. *Given $\epsilon > 0$ and $R > 0$ there exists $n_\epsilon > 0$ depending only on ϵ, R and $\chi(S)$ that if $M = H^+ \cup_S H^-$ has R-bounded combinatorics and handlebody distance $\geq n_\epsilon$ then M admits a Riemannian metric ν such that the sectional curvature of ν is pinched between $-1-\epsilon$ and $-1+\epsilon$. Moreover ν has a lower bound for the injectivity radius independently of the handlebody distance and ϵ.*

This immediately implies that

Corollary 1.1. *If the Heegaard splitting $M = H^+ \cup_S H^-$ has R-bounded combinatorics and sufficiently large handlebody distance, then $\pi_1(M)$ is infinite and word hyperbolic.* $\square$

On the other hand, Tian [Ti90] has claimed a theorem that in presence of the metric constructed in the Main Theorem for ϵ small implies that M is hyperbolic.

Even when we know that a 3-manifold M is hyperbolic, an important question is to be able to describe the geometry of the hyperbolic metric and use it to get topological information about M. The important feature of our construction of the metric for the Main Theorem is that it gives a concrete description of the metric in terms of known hyperbolic manifolds.

In particular, assume $(M_i = H_i^+ \cup_S H_i^-)$ is a sequence of Heegaard splittings with R-bounded combinatorics and handlebody distances tending to infinity as $i \to \infty$. Using the Main Theorem, we can assume that each M_i is equipped with a Riemannian metric ν_i, whose sectional curvatures are pinched in the interval $[-1-\epsilon_i, -1+\epsilon_i]$ and $\epsilon_i \to 0$ as $i \to \infty$. Then we have the following

Theorem 1.2. *Every geometric limit of the sequence $(M_i)_i$ is hyperbolic and either homeomorphic to a genus g handlebody or to the trivial interval bundle $S \times \mathbb{R}$.*

As a matter of fact, we construct a bi-Lipschitz model for the geometry of M_i outside uniform bounded cores of handlebodies H^+ and H^-. The model is described in terms of the *canonical marked hyperbolic surface bundle* over a Teichmüller geodesic, where the Teichmüller geodesic is determined using the combinatorics of the splitting.

Our approach to the proof of the above results is by studying the deformation theory of hyperbolic structures on a handlebody. This approach is highly motivated by works of Minsky and others in proving the Ending Lamination Conjecture and constructing a bi-Lipschitz model for hyperbolic manifolds with incompressible boundary. Once we have a good understanding of the hyperbolic structures on the handlebody, we construct two such structures which are appropriate for our purpose and glue them in a way that we have a manifold homeomorphic to $M = H^+ \cup_S H^-$ and with a Riemannian metric with pinched negative curvature as required by the Main Theorem.

In section 3, we prove a version of Thurston's uniform injectivity theorem for hyperbolic structures on handlebodies. This is a starting point for studying these structurs. We follow this in section 4, by some observations about the pleated surfaces in these structures.

In section 5, we prove the following theorm, which is in fact a joint work with Juan Souto. I am thankful to him for allowing me to present this here.

Theorem 1.3. *Suppose λ is a filling Masur domain lamination on ∂H and λ is not realized in N, where N is a hyperbolic structure on H. Then $\phi(\lambda)$ is the ending lamination of N for some $\phi \in \mathrm{Mod}_0(H)$, where $\mathrm{Mod}_0(H)$ is the subgroup of the mapping class group of ∂H whose elements extend to self-homeomorphisms of H homotopic to identity.*

This is necessary for our construction of geometrically infinite structures with given ending lamination on the handlebody. This theorem answers a question about these structures which we think has been overlooked. We should remind the reader that Ohshika [Oh] has claimed a proof of the above theorem in a special case where N is a strong limit of convex cocompact structures on H.

Then in section 6, we construct the family $\mathcal{B}_0(R)$ of hyperbolic structures on a handlebody which have R-bounded combinatorics for some constant $R > 0$. We prove that this family is compact in the *strong* topology and this is the main tool that helps us make our arguments work.

In sections 7 and 8, similar to Minsky [Min01], we prove a quasi-convexity result for the set of short curves in a hyperbolic structure in $\mathcal{B}_0(R)$ and then we use it to show that all these structures have uniform *bounded geometry.*

We use the bounded geometry in sections 9 and 10 to construct a uniform model for the end of hyperbolic structures in $\mathcal{B}_0(R)$. We use a description of the model which was given by Mosher [Mo03] for the case of hyperbolic structures on $S \times \mathbb{R}$. This gives a description of the structure in terms of the canonical marked hyperbolic surface bundle over a Teichmüller geodesic that is determined by the *end invarinat* of the structure.

We should remark that our results in producing uniform models for this family of hyperbolic structures could not be implied from such descriptions for manifolds with incompressible boundary given by Minsky and others. The question of constructing such models in the general case remain an open question.

Finally in section 11, we use all these to construct appropriate hyperbolic structures on H^+ and H^-. Then we use the model to show that these two are almost isometric on two subsets homeomorphic to $S \times [0, 1]$ and if we glue them along these subsets, we obtain a manifold homeomorphic to $M =$

$H^+ \cup H^-$. All this is provided when the handlebody distance is sufficiently large. This proves the Main Theorem and theorem 1.2 immediately. We briefly describe Tian's result and its consequence in our setting in section 12.

We should point out that the first known examples of Heegaard splittings with sufficiently large handlebody distance were constructed by Luo using an idea of Kobayashi (cf. Hempel [He01]). In our construction in the beginning of the introduction we constructed the manifold by gluing H^+ and H^- along S using the identity map; now suppose f is what we call a *generic pseudo-Anosov*: the stable (resp. unstable) lamination is not limit of meridians of H^+ (resp. H^-). Then the handlebody distance for Heegaard splittings $H^+ \cup_{f^n} H^-$ tends to infinity as $n \to \infty$. In fact, in a joint work with Juan Souto [NS], we proved the same results as our Main Theorem and theorem 1.2 for these examples when n is sufficiently large. One can show that all these Heegaard splittings have some bounded combinatorics depending on f. Therefore those results follow from our theorems here; but the proofs there were more elegant and less involved in the analysis of the ends of hyperbolic structures on handlebodies and construction of uniform models for such structures.

On the other hand, work of Farb-Mosher [FM02] produces many more examples of mapping classes S which satisfy our bounded combinatorics condition once used as a gluing map of a Heegaard splitting $H^+ \cup_f H^-$. In their work, they study what they call *Schottky subgroups* of the mapping class group. Using their work and work of Rafi [Ra05], we can see that if G is such a Schottky subgroup of the mapping class group, there exists $R > 0$, such that every Heegaard splitting $H^+ \cup_f H^-$ has R-bounded combinatorics, where $f \in G$. On the other hand Farb-Mosher [FM02, Thm. 1.4] prove that if $\phi_1, \ldots, \phi_n$ are pseudo-Anosov elements of the mapping class group of S whose axes have pairwise disjoint endpoints in Thurston's compactificiation of Teichmüller space, then for all sufficiently large positive integers $a_1, \ldots, a_n$ the mapping classes $\phi_1^{a_1}, \ldots, \phi_n^{a_n}$ freely generate a Schottky subgroup G. In particular, if these pseudo-Anosovs are generic, then there exists $R > 0$ and we can choose $a_1, \ldots, a_n$ such that $H^+ \cup_f H^-$ satisfies the hypothesis of our theorems for every $f \in G$.

In [NS], we also used our description of the negatively curved metric on $H^+ \cup_S H^-$ to obtain a variety of topological results about the manifolds. Since, all we used was the classification of the geometric limits of these hyperbolic structures and we have a similar classification here, we can prove the same results here.

Theorem 1.4. *If $\Gamma \subset \pi_1(H^+)$ is a finitely generated subgroup of infinite index, then if $M = H^+ \cup_S H^-$ has R-bounded combinatorics and sufficiently large handlebody distance, the map $\Gamma \to \pi_1(M)$ induced by the inclusion $H^+ \hookrightarrow M$ is injective.*

Every minimal generating set for $\pi_1(H^+)$ or $\pi_1(H^-)$ gives a generating set for $\pi_1(M)$ where $M = H^+ \cup_S H^-$ and we call these *standard*.

Theorem 1.5. *The fundamental group of $M = H^+ \cup_S H^-$ has rank g if the Heegaard splitting has R-bounded combinatorics and large handlebody distance. Moreover, every minimal generating set of $\pi_1(M)$ is Nielsen equivalent to a standard generating set.*

For a definition of Nielsen equivalence see [NS].

Theorem 1.6. *For a Heegaard splitting $M = H^+ \cup_S H^-$ with R-bounded combinatorics and large handlebody distance, every proper subgroup $\Gamma \subset \pi_1(M)$ with rank $\leq 2g - 2$ is free.*

Theorem 1.7. *If $M = H^+ \cup_S H^-$ has R-bounded combinatorics and large handlebody distance then the Heegaard genus of M is g and every minimal Heegaard surface is isotopic to S.*

2 Preliminaries

2.1 Coarse geometry

A metric space is *geodesic* if for any x, y there is a rectifiable path p from x to y whose length is equal to $d(x, y)$.

Let X and Y be metric spaces. A map $f : X \to Y$ is (K, c)-*quasi-isometric embedding* if

$$\frac{1}{K} d_X(x, x') - c \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + c$$

for $x, x' \in X$. We say f is *uniformly proper* with respect to a proper, monotonic function $\rho : [0, \infty) \to [0, \infty)$ and constants K and c if

$$\rho(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + c \quad \text{for} \quad x, x' \in X.$$

The function ρ is called a *properness gauge* for f. The map f is c-*coarsely surjective* if for all $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) \leq c$. The map f is a (K, c)-*quasi-isometry* if it is a c-coarsely surjective, (K, c)-quasi-isometric embedding.

Fact 2.1. *Suppose X and Y are geodesic metric spaces. Any coarsely surjective, uniformly proper map $f : X \to Y$ is a quasi-isometry with constants depending only on the constants in the hypothesis.*

Given a geodesic metric space X, a (λ, c)-*quasigeodesic* in X is a (λ, c)-quasi-isometric embedding $\gamma : I \to X$, where I is a closed connected subset of $\mathbb{R}$. When I is a compact interval we have a *quasigeodesic segment*, when I is a half-line we have a *quasigeodesic ray*, and when $I = \mathbb{R}$ we have a quasigeodesic line.

Recall that the Hausdorff distance between two subsets $A, B \subset X$ is the infimum of $r \in \mathbb{R}_+ \cup \{+\infty\}$ such that A is contained in the r-neighborhood of B, and B is contained in the r-neighborhood of A.

Two paths $\gamma : I \to X, \gamma' : I' \to X$ are *asynchronous fellow travellers* with respect to a (K, c)-quasi-isometry $\phi : I \to I'$ if there is a constant A such that $d(\gamma'(\phi(t)), \gamma(t)) \leq A$ for $t \in I$.

Two paths $\gamma : I \to X, \gamma' : I' \to X$ where γ is a quasigeodesic are asynchronous fellow travellers if and only if γ' is a quasigeodesic and the sets $\gamma(I), \gamma'(I')$ have finite Hausdorff distance in X with constants that are uniformly related.

2.2 Laminations and Masur domain

Let S be a closed surface of genus $g > 1$. Let $\mathrm{Diff}(S)$ be the group of diffeomorphisms of S and let $\mathrm{Diff}_0(S)$ be the normal subgroup of homeomorphisms isotopic to the identity. The *mapping class group* of S is $\mathcal{MCG}(S) = \mathrm{Diff}(S)/\mathrm{Diff}_0(S)$.

Suppose S is equipped with a hyperbolic metric τ_0. A *geodesic lamination* on S is a closed subset of S which is a disjoint union of simple geodesics. We denote the space of all these by $\mathcal{GL}(S)$. A *measured lamination* is a geodesic lamination λ together with an invariant (with respect to projection along λ) measure on arcs transversal to λ and supported on λ. $\mathcal{ML}(S)$ is the space of all measured laminations on S and the *projective lamination space* $\mathcal{PML}(S)$ is $(\mathcal{ML}(S)\backslash\{0\})/\mathbb{R}^+$. We identify $\mathcal{PML}(S)$ with the set of measured laminations which have unit length.

We also use the set $\mathcal{UML}(S)$ which is a quotient of $\mathcal{PML}(S)$ obtained by forgetting the measure. We say a geodesic lamination is *filling* or *fills S* if it intersects every essential non-peripheral simple closed curve on S. By a *maximal* lamination we mean an element of $\mathcal{GL}(S)$ whose complementary components are ideal triangles.

It is a standard that the different spaces of laminations defined above do not depend on the hyperbolic metric τ_0. This means that there is a natural homeomorphism from the spaces associated to σ to the ones associated to σ', if σ and σ' are different hyperbolic metrics on S. This homeomorphism is naturally induced from the identification of the circles at infinity of the universal covers $\widetilde{\sigma}$ and $\widetilde{\sigma}'$, via the Gromov boundary of the group $\pi_1(S)$. For more on the spaces of laminations and more see Casson-Bleiler [CB] or Fathi-Laundenbach-Poénorou [FLP].

Also notice that if $\mathcal{C}_0$ represents the set of homotopy classes of essential simple closed curves on S, then there is an embedding $\mathcal{C}_0 \to \mathcal{ML}$, where image of $\alpha \in \mathcal{C}_0(S)$ is a measured lamination whose single leaf is α with total transverse measure 1. This also induces a natural embedding $\mathcal{C}_0 \to \mathcal{PML}$ and an embedding $\mathbb{R}_+ \times \mathcal{C}_0 \to \mathcal{ML}$ whose images are dense. Using the embedding $\mathbb{R}_+ \times \mathcal{C}_0 \to \mathcal{ML}$, we can extend the geometric intersection number for simple closed curves to a continuous intersection number

$$\mathcal{ML} \times \mathcal{ML} \to [0, \infty),$$

denoted $i(\mu_1, \mu_2), \mu_1, \mu_2 \in \mathcal{ML}$. Notice that for elemnts $\mu, \mu' \in \mathcal{PML}$, it makes sense to say $i(\mu, \mu')$ is zero or nonzero. In particular, we say μ and μ'

are *transverse* if $i(\mu, \mu') \neq 0$.

Now assume H is a handlebody of genus g and S its boundary. We denote the group of (isotopy classes of) homeomorphisms of S which extend to homeomorphisms of H, homotopic to identity by $\mathrm{Mod}_0(H)$. In studying, the hyperbolic structures on H a subset of $\mathcal{ML}(S)$ called *Masur Domain* $\mathcal{O}(H)$ appears frequently. Recall that by a *meridian* for H, we mean an essential simple closed curve on ∂H that bounds a disk in H. Let's denote the set of projective measured laminations that are supported on a finite union of meridians by $\mathcal{M} \subset \mathcal{PML}(S)$ and its closure by $\mathcal{M}'$. A measured lamination μ belongs to $\mathcal{O}(H)$ iff it has nonzero intersection with every element of $\mathcal{M}'$. We say a (geodesic) lamination is in Masur domain if every measured lamination supported on that does. The Masur domain has been studied by Masur [Mas86] and Otal [Ota88] and Masur proved the following:

Theorem 2.2. *The Masur domain $\mathcal{O}$ is open and invariant under the action of $\mathrm{Mod}_0(H)$ on $\mathcal{PML}$. Moreover, the action of $\mathrm{Mod}_0(H)$ on $\mathcal{O}$ is properly discontinuous.*

Kerckhoff [Ker90] proved that $\mathcal{O}(H)$ has full measure in $\mathcal{PML}(S)$. Also, Otal [Ota88] proved that

Lemma 2.3. *The complement of a Masur domain multi-curve is incompressible and acylindrical in H.*

2.3 The complex of curves

For a finite type surface $S = S_{g,b}$, the surface of genus g with b boundary components, the *complex of curves* was originally defined by Harvey [Ha81]. Here we usually use the definitions and description used by Masur-Minsky [MM99, MM00]. The definition is slightly different for an annulus $S = S_{0,2}$ but the complex of curves, which we denote by $\mathcal{C}(S)$ is a locally infinite simplicial complex with a path metric on its 1-skeleton when it is nonempty.

If $3g + b \geq 4$, we consider the vertices of $\mathcal{C}(S)$ to be the set of homotopy classes of essential non-peripheral simple closed curves and essential properly embedded arcs relative to ∂S. Here, non-peripheral curves are those which are not boundary parallel and essential arcs which are not homotopic (rel. ∂S) to subarcs of ∂S. A $(k+1)$-tuple of different vertices makes a k-simplex if they have mutually disjoint representatives on the surface.

Notice that $\mathcal{C}_0 = \mathcal{C}_0(S)$ is the set of vertices of $\mathcal{C}(S)$ and when S is a closed surface it will be the set of homotopy classes of essential simple closed curves on S which is the same as our previous defintion for $\mathcal{C}_0$.

When $S = S_{0,2}$ is a compact annulus, we consider the set of vertices of $\mathcal{C}(S)$ to be the homotopy classes of arcs that connect the two boundary components of S relative to their endpoints. This of course will be an uncountable set of vertices; we connect two vertices with an edge when they have representatives with disjoint interiors. For all other surfaces, we define $\mathcal{C}(S)$ to be empty.

To define the metric, we make every edge isometric to the interval $[0, 1]$ and define $d_{\mathcal{C}}(x, y)$ of points x and y in the 1-skeleton of $\mathcal{C}(S)$ to be length of the shortest path in the 1-skeleton that connects them.

Remark 2.1. We should note that what we defined as the curve complex is slightly different from what Masur-Minsky define as the curve complex. (They only allow simple closed curves in the vertices and they call what we define above as the *arc complex*.) Yet it is not hard to see that their complex quasi-isometrically embeds in our complex. Because of this we can translate most of their results about the curve complex to here with possibly different constants.

From now on, by a surface we mean an orientable finite type surface which is an annulus or has negative Euler characteristic. We also assume that every subsurface $Y \subset S$ that we take is *essential*: the map induced on the fundamental groups from the inclusion $Y \hookrightarrow S$ is injective and if Y is an annulus, its core is not peripheral.

Masur-Minsky [MM99] proved that $\mathcal{C}(S)$, when nonempty, has infinite diameter and is hyperbolic in sense of Gromov. In particular, we can define its boundary at infinity in sense of Gromov, which we denote by $\partial\mathcal{C}(S)$.

E. Klarreich [Kla] gave a description of the boundary of $\mathcal{C}(S)$. In this description, $\partial\mathcal{C}(S)$ consists of filling laminations in $\mathcal{UML}(S)$. Then, she proved that a sequence $(\alpha_n) \subset \mathcal{C}_0(S)$ converges to $\mu \in \partial\mathcal{C}(S)$ in sense of Gromov, iff the corresponding sequence in $\mathcal{UML}(S)$ converges to μ.

Following Masur-Minsky [MM00], we also define a projection π_Y from $\mathcal{C}_0(S) \cup \mathcal{UML}(S)$ (where $\mathcal{C}_0(S)$ denotes the 0-skeleton of $\mathcal{C}(S)$) to subsets of $\mathcal{C}_0(Y)$ with diameter at most one, where $Y \subset S$ is an essential subsurface.

We assume S is equipped with a finite area hyperbolic metric. If $\alpha \in \mathcal{C}_0(S) \cup \mathcal{UML}(S)$ does not intersect Y essentially or Y is a three-holed sphere, we define $\pi_Y(\alpha) = \emptyset$. If not we have two cases:

- *Y is non-annular.* Consider $\alpha \cap Y$; this is a set of disjoint curves and arcs. At least one of the components is an essential curve or arc in Y since we assumed α intersects Y essentially. Therefore $\alpha \cap Y$ gives a subset of diameter at most one in $\mathcal{C}(Y)$, which we define to be $\pi_Y(\alpha)$.

- *Y is an annulus.* We can identify the universal cover of S with $\mathbb{H}^2$ and we know that the universal cover has a compactification as a closed disk and the action of $\pi_1(S)$ on its universal cover extends to this compactification. Take the annular cover $\widetilde{Y} = \mathbb{H}^2/\pi_1(Y)$ of S to which Y lifts homeomorphically. The group $\pi_1(Y)$ is a cyclic subgroup of isometries of $\mathbb{H}^2$ with two fixed points at infinity. The quotient of the closed disk minus these two points is a closed annulus $\widehat{Y}$ that compactifies $\widetilde{Y}$ naturally. We identify $\mathcal{C}(Y)$ with $\mathcal{C}(\widehat{Y})$ and define π_Y as a map from $\mathcal{C}_0(S)$ to set of subsets of $\mathcal{C}_0(\widehat{Y})$ with diameter at most one. All lifts of the geodesic representative of α to $\widetilde{Y}$ naturally give properly embedded arcs in the closed annulus $\widehat{Y}$. We define $\pi_Y(\alpha)$ to be the set of those which connect the two boundary components. Again this cannot be empty since α intersects Y essentially and has diameter at most one.

We also denote the distance between projections of α and β in $\mathcal{C}(Y)$ by $d_Y(\alpha, \beta)$ and when Y is an annular neighborhood of the simple closed curve γ, we sometimes use the notations $\mathcal{C}(\gamma)$, π_γ and d_γ instead of $\mathcal{C}(Y)$, π_Y and d_Y.

Masur-Minsky [MM00] also proved the following theorem:

Theorem 2.4. (Bounded geodesic image) *Let Y be a proper subsurface of S which is not a three punctured sphere and let g be a geodesic segment, ray or biinfinite line in $\mathcal{C}(S)$ such that $\pi_Y(v) \neq \emptyset$ for every vertex of g.*

There is a constant M only depending on the Euler characteristic of Y, so that

$$\operatorname{diam}_Y(g) \leq M.$$

2.4 Handlebody distance

Suppose H is a handlebody of genus > 1. The set of meridians of H is a subset of the 0-skeleton of $\mathcal{C}(\partial H)$ which we denote by $\Delta(H)$. Masur-Minsky [MM03] proved that this subset is K-quasi-convex for a constant K that depends only on $\chi(\partial H)$.

When $M = H^+ \cup_S H^-$ is a Heegaard splitting, since we have identified boundaries of H^+ and H^- with S, we can consider $\Delta(H^+)$ and $\Delta(H^-)$ as subsets of $\mathcal{C}_0(S)$. Following Hempel [He01], we define the *handlebody distance* for the splitting to be $d_{\mathcal{C}}(\Delta(H^+), \Delta(H^-))$.

2.5 Pants decompositions and markings

For a surface S, a *multi-curve* is a subset of $\mathcal{C}_0(S)$ whose elements are simple closed curves with pairwise distance 1. In particular, a *pants decomposition* P is a maximal multi-curve on S. Each component of $S \setminus P$ is called a pair of pants. We sometimes consider a multi-curve or a pants decomposition α as an element of $\mathcal{ML}$ or $\mathcal{PML}$; in this case we assume it is a measured lamination or a projectivized measured lamination supported on α where all the components are equipped with equal transverse measure 1 or is the projection of such element in $\mathcal{PML}$.

Suppose $\alpha_1, \ldots, \alpha_k$ are components of a pants decomposition P. One can see that the component of

$$S \setminus (\alpha_2 \cup \cdots \cup \alpha_k)$$

that contains α_1 is either a 1-holed torus or a 4-holed sphere Y. If we replace α_1 with an essential simple closed curve β in Y that $i(\alpha_1, \beta) = 1$ when Y is a 1-holed torus and $i(\alpha_1, \beta) = 2$ when Y is a 4-holed torus, we obtain another pants decomposition $Q = \beta \cup \alpha_2 \cup \cdots \alpha_k$. We say Q is obtained by an *elementary move* on P and we denote this move by $P \to Q$. By an *elementary move sequence*

$$P_1 \to P_2 \to \cdots \to P_k$$

we mean a sequence of pants decomposition, where P_{i+1} is obtained from P_i by an elementary move for every $i = 1, \ldots, k-1$.

The following lemma is easy and we will be using it in section 5.

Lemma 2.5. *Given a path $\alpha_0, \alpha_1, \ldots, \alpha_n$ in $\mathcal{C}(S)$, we can extend it to an elementary-move sequence of pants decompositions $P_0, P_1, \ldots, P_m$ for which:*

every pants decomposition P_i, $0 \le i \le m$, contains an element α_j for some $0 \le j \le n$ and P_0 and P_m are arbitrary pants decompositions that contain α_0 and α_m respectively.

Following Masur-Minsky [MM00], we define a *marking* on a surface S as follows. A *marking* α on S is a pants decomposition P, denoted by $\text{base}(\alpha)$, together with a *transversal* for every component of P. For every component γ of P, a transversal is a simple closed curve δ such that a regular neighborhood of $\gamma \cup \delta$ is either an essential 1-holed torus or an essential 4-holed sphere and δ does not intersect any other component of P.

Notice that what we have defined above are actually *complete clean markings* in [MM00].

By *support* of a marking α, we mean the union of all the components of $\text{base}(\alpha)$ and the transversals. When we consider a marking in $\mathcal{C}_0(S)$ we are in fact considering its support.

We can also extend our definition of π_Y in a way that it includes markings: If Y is an annulus whose core is some $\gamma \in \text{base}(\alpha)$ and δ is the transversal associated to γ, we define $\pi_Y(\alpha) = \pi_Y(\delta)$. In all other cases, $\pi_Y(\alpha) = \pi_Y(\text{base}(\alpha))$. This also defines $d_Y(\alpha, \beta)$ where β is a multi-curve, a geodesic lamination or another marking.

Note that support of a marking *binds* the surface, i.e. interesects every essential simple closed curve. The following follows from work of Kerckhoff [Ker80].

Lemma 2.6. *If α is a marking on a surface S, there exists a unique point $\tau = \tau(\alpha) \in \mathcal{T}(S)$, where length of support of α is minimized in τ. Moreover $\tau(\alpha)$ is ϵ_0-thick, where ϵ_0 is a constant that depends only on $\chi(S)$.*

If H is a handlebody, by a *handlebody pants decomposition*, we mean a pants decomposition on ∂H, whose elements are all in $\Delta(H)$. Also by a *handlebody marking*, we mean a marking α such that $\text{base}(\alpha)$ is a handlebody pants decomposition.

Proposition 2.7. *For a handlebody H, there exists a finite set of handlebody markings $\mathbf{m}_0(\text{H})$ such that every other handlebody marking is obtained by action of $\text{Mod}_0(H)$ on an element of $\mathbf{m}_0(\text{H})$*

2.6 Bounded combinatorics

Suppose α and β are multi-curves, geodesic laminations on S or markings; we say they have *R-bounded combinatorics* for a constant $R > 0$ if for every proper essential subsurface $Y \subset S$ either $d_Y(\alpha, \beta)$ is undefined or $d_Y(\alpha, \beta) \leq R$.

When H is a handlebody and α is either a subset of $\mathcal{C}_0(S)$ or a geodesic lamination, we say α has *R-bounded combinatorics respecto to* H if there exists a handlebody pants decomposition $P \subset \Delta(H)$ such that α and P have R-bounded combinatorics and when α is a subset of $\mathcal{C}_0(S)$, we have

$$d_{\mathcal{C}}(\alpha, \Delta(H)) = d_{\mathcal{C}}(\alpha, P).$$

Finally, when $M = H^+ \cup_S H^-$ is a Heegaard splitting, we say two handlebody pants decompositions $P^+ \subset \Delta(H^+)$ and $P^- \subset \Delta(H^-)$ *realize* the handlebody distance if $d_{\mathcal{C}}(P^+, P^-) = d_{\mathcal{C}}(\Delta(H^+), \Delta(H^-))$. We say this Heegaard splitting has *R-bounded combinatorics* if there exists handlebody pants decompositions $P^+ \subset \Delta(H^+)$ and $P^- \subset \Delta(H^-)$ which realize the handlebody distance and have R-bounded combinatorics.

We can easily see that in the above definition we could replace handlebody pants decompositions with handlebody markings:

Lemma 2.8. *Suppose $M = H^+ \cup_S H^-$ is a Heegaard splitting with R-bounded combinatorics and suppose P^+ and P^- are the handlebody pants decompositions used in the definition of the R-bounded combinatorics. Then we can extend P^+ and P^- to handlebody markings α^+ and α^- which have R-bounded combinatorics.*

Proof. For every $\gamma \in P^+$ consider if $\pi_\gamma(P^-)$ is empty choose an arbitrary transversal for γ; otherwise choose a transversal that belongs to $\pi_\gamma(P^-)$. Repeat the same process for every component of P^+ and then do the same for components of P^- by using projections of P^+. ◻

2.7 Teichmüller space and Thurston's boundary

Like before, assume S is a fixed surface of genus ≥ 2. The *Teichmüller space* of S, denoted $\mathfrak{T}(S)$, is the set of hyperbolic structures on S modulo isotopy, or equivalently the set of conformal structures modulo isotopy. There is a natural actions of $\mathcal{MCG}(S)$ on $\mathfrak{T}(S)$.

The length pairing $\mathfrak{T} \times \mathcal{C}_0 \to \mathbb{R}_+$, assigns to each $\sigma \in \mathfrak{T}$ and $\alpha \in \mathcal{C}_0(S)$ the length of the unique closed geodesic on the hyperbolic surfac σ in the isotopy class of α. This induces a $\mathcal{MCG}$-equivariant embedding $\mathfrak{T} \to [0, \infty)^{\mathcal{C}_0}$ and gives $\mathfrak{T}$ the $\mathcal{MCG}$-equivariant structure of a smooth manifold of dimension $6g-6$ diffeomorphic to $\mathbb{R}^{6g-6}$. The action of $\mathcal{MCG}$ on $\mathfrak{T}$ is properly discontinuous and noncocompact, and so the *moduli space* $\mathcal{M} = \mathfrak{T}/\mathcal{MCG}$ is a smooth, noncompact orbifold of dimension $6g-6$.

The length pairing can be extended to a continuous function:

$$\mathfrak{T} \times \mathcal{ML} \to (0, \infty)$$
$$(\sigma, \mu) \to l_\sigma(\mu)$$

We also have a $\mathcal{MCG}$-equivariant embedding $i : \mathcal{ML} \to [0,\infty)^{\mathcal{C}_0}$ by considering $i(\mu, \alpha)$ for $\mu \in \mathcal{ML}$ and every $\alpha \in \mathcal{C}_0$. This induces an embedding $\mathcal{PML} \to \mathcal{P}[0,\infty)^{\mathcal{C}_0}$, whose image is homeomorphic to a sphere of dimension $6g - 5$. The composed map

$$\mathfrak{T} \to [0,\infty)^{\mathcal{C}_0} \to \mathcal{P}[0,\infty)^{\mathcal{C}_0}$$

is an embedding, the closure of whose image is a closed ball of dimension $6g-6$ with interior $\mathfrak{T}$ and boundary sphere $\mathcal{PML}$ called the *Thurston compactification.*

2.8 Quadratic differentials and Teichmüller geodesics

Suppose S with a conformal structure is given. A *quadratic differential* associates to each conformal coordinate z an expression $q(z)dz^2$ with q holomorphic, such that whenever z, w are two overlapping conformal coordinates we have $q(z) = q(w)(\frac{dw}{dz})^2$. For such a quadratic differential, we have the *area form* expressed in the conformal coordinate $z = x + iy$ as $|q(z)|\,|dx|\,|dy|$, and the integral of this form is a positive number $\|q\|$ called the *area.* We say q is *normalized* if $\|q\| = 1$.

Away from zeros of q, there is a canonical conformal coordinate $\zeta = x + iy$, defined locally up to translation and sign, such that $q = d\zeta^2$ in this coordinate. The lines $\{y = x\}$ and $\{x = c\}$ are thus consistently defined and form what are known as the *horizontal and vertical foliations*, respectively or q_h and q_v. The metric $|q| = |d\zeta|^2 = dx^2 + dy^2$ is also canonically defined, and is Euclidean with isolated singularities at the zeros of q where there is concentrated negative curvature.

Now for every $t \in \mathbb{R}$ consider a new conformal structure obtained by taking the singular Euclidean metric $e^{2t}dx^2 + e^{-2t}dy^2$ and let $g(t)$ be the associated point of $\mathfrak{T}(S)$. This gives path in $\mathfrak{T}$ which we call a *Teichmüller geodesic.* Teichmüller's theorem states that any two points $\sigma \neq \sigma' \in \mathfrak{T}$ lie on a Teichmüller line g, and that line is unique up to an isometry of the parameter lin $\mathbb{R}$. Moreover, if $\sigma = g(s)$ and $\sigma' = g(t)$, then $d_{\mathfrak{T}}(\sigma, \sigma') = |s - t|$ defines a proper, geodesic metric on $\mathfrak{T}$, called the *Teichmüller metric.*

For a Teichmüller geodesic constructed as above, the horizontal and vertical foliations of q correspond to two tansverse elements of $\mathcal{PML}(S)$, called the *negative and positive ending laminations* or *ideal endpoints* of g. It turns out that the image of g is completely determined by this pair. Also given $\sigma \in \mathfrak{T}$ and $\mu \in \mathcal{PML}$, there is a unique geodesic ray with finite endpoint σ and given by the above description in a way that μ is associated to the vertical foliation and we call μ the *ideal endpoint* of the ray.

The group $\mathcal{MCG}$ acts isometrically on $\mathfrak{T}$, and so the Teichmüller metric descends to a proper geodesic metric on the moduli space $\mathcal{M}$. A subset $A \subset \mathfrak{T}$ is said to be *cobounded* if its projection to $\mathcal{M}$ has bounded image. When $\mathcal{K}$ a bounded subset of $\mathcal{M}$ is given, we say A is $\mathcal{K}$-cobounded if the projection of A to $\mathcal{M}$ is contained in $\mathcal{K}$.

Mumford's theorem provides a criterion for coboundedness. Given $\epsilon > 0$, let $\mathfrak{T}_\epsilon$ be the set of hyperbolic structures σ whose shortest closed geodesic has length $\geq \epsilon$ (we sometimes say σ is *ϵ-thick*) and define $\mathcal{M}_\epsilon$ to be the projcted image of $\mathfrak{T}_\epsilon$. Mumford's theorem says that the sets $\mathcal{M}_\epsilon$ are all compact and their union is evidently all of $\mathcal{M}$. It follows that a subset $A \subset \mathfrak{T}$ is cobounded if and only if it is contained in some $\mathfrak{T}_\epsilon$.

2.9 Canonical bundles over Teichmüller space

For the closed surface S of genus ≥ 2, there is a smooth fiber bundle $\mathcal{S} \to \mathfrak{T}(S)$ whose fiber $\mathcal{S}_\sigma$ over $\sigma \in \mathfrak{T}$ is a hyperbolic surface representing the point $\sigma \in \mathfrak{T}$. More precisely, as a smooth fiber bundle we identify $\mathcal{S}$ with $S \times \mathfrak{T}$, and we impose smoothly varying hyperbolic structures on the fiberes $\mathcal{S}_\sigma = S \times \sigma$, $\sigma \in \mathfrak{T}$, such that under the canonical homeomorphism $\mathcal{S}_\sigma \to S$ the hyperbolic structure on $\mathcal{S}_\sigma$ represents the point $\sigma \in \mathfrak{T}$. The action of $\mathcal{MCG}$ on $\mathfrak{T}$ lifts to an fiberwise isometric action of $\mathcal{MCG}$ on $\mathcal{S}$. Each fiber $\mathcal{S}_\sigma$ is a *marked* hyperbolic surface, i.e. it comes equipped with an isotopy class of homeomorphisms to S. The bundle $\mathcal{S} \to \mathfrak{T}$ is called the *canonical marked hyperbolic surface bundle* over $\mathfrak{T}$.

The *canonical hyperbolic plane bundle* $\mathcal{H} \to \mathfrak{T}$ is defined as the composition $\mathcal{H} \to \mathcal{S} \to \mathfrak{T}$ where $\mathcal{H} \to \mathcal{S}$ is the universal covering map. Each fiber $\mathcal{H}_\sigma$, $\sigma \in \mathfrak{T}$, is isometric to the hyperbolic plane, with hyperbolic structures varying smoothly in σ. The group $\pi_1(S)$ acts as deck transformations of the covering map $\mathcal{H} \to \mathcal{S}$ and this action preserves each fiber $\mathcal{H}_\sigma$ with quotient $\mathcal{S}_\sigma$. The action of $\pi_1(S)$ on $\mathcal{H}$ extends to a fiberwise isometric action of $\mathcal{MCG}(S, p)$ on $\mathcal{H}$, such that the covering map $\mathcal{H} \to \mathcal{S}$ is equivariant with respect to the group homomorphism $\mathcal{MCG}(S,p) \to \mathcal{MCG}(S)$. Bers [Be73] proved that $\mathcal{H}$ can be identified with the Teichmüller space of the once-punctured surface (S,p), and the action of $\mathcal{MCG}(S,p)$ on $\mathcal{H}$ is identified with the natural action of the mapping class group on Teichmüller space.

Suppose $T\mathcal{S}$ denotes the tangent bundle of $\mathcal{S}$ and $T_v\mathcal{S}$ denotes the *vertical subbundle* of $T\mathcal{S}$, i.e. the kernel of the derivative of the fiber bundle projection $\mathcal{S} \to \mathfrak{T}$. It follows from standard methods that there exists an $\mathcal{MCG}$-equivariant connection on $\mathcal{S}$. Choose a locally finite, equivariant open cover of $\mathfrak{T}$, and an equivariant partition of unity dominated by this cover. For each $\mathcal{MCG}$-orbit of this cover, choose a representative $U \subset \mathfrak{T}$ and choose a linear retraction $T\mathcal{S}_U \to T_v\mathcal{S}_U$. Use the action of $\mathcal{MCG}$ to define this retraction on all elements of orbit of U and take a linear combination using the partition of unity to obtain an equivariant linear retraction $T\mathcal{S} \to T_v\mathcal{S}$. The kernel of this retraction is one such connection. Also by lifting to $\mathcal{H}$ we obtain a connection on the bundle $\mathcal{H} \to \mathfrak{T}$, equivariant with respect to the action of the group $\mathcal{MCG}(S,p)$. We fix a choice of such a connection once and forever.

The connection obtains a smooth sub-bundle $T_h\mathcal{S}$ of $T\mathcal{S}$ which is complementary to $T_v\mathcal{S}$: $T\mathcal{S} = T_h\mathcal{S} \oplus T_v\mathcal{S}$. Lifting to $\mathcal{H}$ we also have a sub-bundle $T_h\mathcal{H}$ of the bundle $T\mathcal{H}$.

By a *closed interval*, we mean a closed connected subset of $\mathbb{R}$. Given a closed interval $I \subset \mathbb{R}$, a path $\gamma : I \to \mathfrak{T}$ is *affine* if it satisfies $d_{\mathfrak{T}}(\gamma(s), \gamma(t)) = K|s-t|$ for some constant $K \geq 0$, and γ is *piecewise affine* if γ is affine restricted to pieces of a decomposition of I into subintervals. In particular γ is $\mathbb{Z}$*-piecewise affine* if it is affine restricted to $[n, n+1] \cap I$ for every integer n.

Given an affine path $\gamma : I \to \mathfrak{T}$, by pulling back the canonical marked hyperbolic surface bundle $\mathcal{S} \to \mathfrak{T}$ and its connection $T_h\mathcal{S}$, we obtain a marked hyperbolic surface bundle $\mathcal{S}_\gamma \to I$ and a connection $T_h\mathcal{S}_\gamma$. This connection cannonically determines a Riemannian metric on $\mathcal{S}_\gamma$ as follows. Without loss of generality, assume $K = 1$ in the defintion of the affine path γ. Since $T_h\mathcal{S}_\gamma$ is 1-dimensional, there is a unique vector field V on $\mathcal{S}_\gamma$ parallel to $T_h\mathcal{S}_\gamma$ such that the derivative of the map $\mathcal{S}_\gamma \to I \subset \mathbb{R}$ takes each vector in V to the positive unit vector in $\mathbb{R}$. The fiberwise Riemannian metric on $\mathcal{S}_\gamma$ now extends uniquely to a Riemannian metric on $\mathcal{S}_\gamma$ such that V is everywhere orthogonal to the fibration and has unit length.

Even when γ is piecewise affine, the above construction gives a Riemannian metric over each affine subpath, and at any point $t \in I$ where two such subpaths meet, the metrics agree along the fibers, thereby producing a piecewise Riemannian metric on $\mathcal{S}_\gamma$.

We can lift the above construction to $\mathcal{H}_\gamma$ to produce an $\mathcal{MCG}$-equivariant (piecewise affine) Riemannian metric such that the covering map $\mathcal{H}_\gamma \to \mathcal{S}_\gamma$ is local isometry. One can see that these path metrics are proper geodesic metrics.

A *connection line* in either of the bundles $\mathcal{S}_\gamma \to I$, $\mathcal{H}_\gamma \to I$ is a piecewise smooth section of the projection map which is everywhere tangent to the connection. By construction, given $s, t \in I$, a path p from a point in the fiber over s to a point in the fiber over t has length $\geq |s-t|$, with equality only if p is a connection path. It follows that the min distance and the Hausdorff distance between these fibers are both equal to $|s-t|$. By moving points along connection paths, for each $s, t \in I$ we have well-defined maps $\mathcal{S}_s \to \mathcal{S}_t$, $\mathcal{H}_s \to \mathcal{H}_t$, both denoted h_{st}. By a result of Farb-Mosher [FM02, Lem. 4.1], for a bounded set $\mathcal{K} \subset \mathcal{M}$ and $\rho \geq 1$ there exists K such that if $\gamma : I \to \mathfrak{T}$ is a $\mathcal{K}$-cobounded, ρ-Lipschitz, piecewise affine path, then for each $s, t \in I$ the connection map h_{st} is $K^{|s-t|}$-bi-Lipschitz.

2.10 Singular SOLV spaces

When $\gamma : I \to \mathfrak{T}$ is a geodesic there is another pair of natural geometries, the singular SOLV space $\mathcal{S}_\gamma^{\mathrm{SOLV}}$ and its universal cover $\mathcal{H}_\gamma^{\mathrm{SOLV}}$. Recall that a Teichmüller geodesic $\gamma(t)$ (parametrized by length) is given by a quadratic differential q and a family of singular Euclidean metrics

$$ds^2_{\gamma(t)} = e^{2t}|dx|^2 + e^{-2t}|dy|^2$$

where $|dy|$ and $|dx|$ are associated to the horizontal and vertical measured foliations of q and the conformal class of $ds_{\gamma(t)}$ represents $\gamma(t) = \mathcal{S}_t$.

We can use the above to define the *singular* SOLV *metric* on $\mathcal{S}_\gamma$ by

$$ds^2 = e^{2t}|dx|^2 + e^{-2t}|dy|^2 + dt^2$$

and we denote this metric space by $\mathcal{S}_\gamma^{\mathrm{SOLV}}$. The lift of this metric to the universal cover ch_γ produces a metric space denoted by $\mathcal{H}_\gamma^{\mathrm{SOLV}}$.

Farb-Mosher [FM02, Porp. 4.2] proved the following:

Proposition 2.9. *For any $\rho \geq 1$, any bounded subset $\mathcal{K} \subset \mathcal{M}$, and any $A \geq 0$ there exists $K \geq 1, c \geq 0$ such that the following holds. If $\gamma, \gamma' : I \to \mathfrak{T}$*

are two ρ-Lipschitz, $\mathcal{K}$-cobounded, piecewise affine paths defined on a closed interval I, and if $d(\gamma(s), \gamma'(s)) \leq A$ for all $s \in I$, then there exists a map $\mathcal{S}_\gamma \to \mathcal{S}_{\gamma'}$ taking each fiber $\mathcal{S}_{\gamma(t)}$ to the fiber $\mathcal{S}_{\gamma'(t)}$ by a homeomorphism in the correct isotopy class, such that any lifted map $\mathcal{H}_\gamma \to \mathcal{H}_{\gamma'}$ is a (K, c)-quasi-isometry.

If γ' is a geodesic, the same is true with $\mathcal{S}_{\gamma'}$, $\mathcal{H}_{\gamma'}$ replaced by the singular SOLV *spaces $\mathcal{S}_{\gamma'}^{\mathrm{SOLV}}, \mathcal{H}_{\gamma'}^{\mathrm{SOLV}}$.*

2.11 Hyperbolic Manifolds

By a *hyperbolic manifold*, we always mean a Riemannian 3-manifold with finitely generated fundamental group and constant sectional curvature -1. A hyperbolic manifold N is also recognized by the conjugacy class of a discrete and faithful representation

$$\pi_1(N) \to \mathrm{PSL}_2(\mathbb{C}).$$

We recall the definition of the *injectivity radius* of N at a point x, denoted by $\mathrm{inj}(x)$, is half the length of shortest (homotopically nontrivial) loop through x. By Margulis lemma, there exists a universal constant $\epsilon_M > 0$, such that for any $\epsilon \leq \epsilon_M$, every component of the *ϵ-thin part* of N

$$N^{<\epsilon} := \{x \in N | \ \mathrm{inj}(x) < \epsilon\}$$

is either

1. a torus cusp: a horoball in $\mathbb{H}^3$ modulo a parabolic action of $\mathbb{Z} \oplus \mathbb{Z}$,
2. a rank one cusp: a horoball in $\mathbb{H}^3$ modulo a parabolic action of $\mathbb{Z}$, or
3. a solid torus neighborhood of a geodesic

(see Thurston [Thu79] or Benedetti and Petronio [BP].) We also denote the complement of $N^{<\epsilon}$ by $N^{\geq\epsilon}$ that is the ϵ-thick part of N and the complement of all cuspidal parts of the thin part by $\hat{N}^\epsilon$. We call the components of type (1) and (2), *ϵ-cusps* or simply *cusps* of N and we call the components of type (3), *ϵ-Margulis-tubes* or simply *Margulis-tubes*.

Suppose N is a hyperbolic manifold and

$$\rho : \pi_1(N) \to \mathrm{PSL}_2(\mathbb{C})$$

is the associated representation which gives a discrete subgroup $\Gamma = \rho(\pi_1(N))$ of the group of isometries of $\mathbb{H}^3$. We can consider the limit set $\Lambda(\Gamma)$ and its convex hull $\mathcal{CH}(\Gamma)$; the projection of this set gives a subset $\mathcal{CH}(N) \subset N$, which we call the *convex core* of N. We say N is *convex cocompact* if $\mathcal{CH}(N))$ is comapct.

The following lemma is an easy observation using hyperbolic geometry.

Lemma 2.10. *Let N be a hyperbolic manifold and α a homotopically non-trivial closed curve in N and α^* its geodesic representative. Then*

$$\cosh d_N(\alpha, \alpha^*) \leq l_N(\alpha)/l_N(\alpha^*),$$

where $l_N(\alpha)$ is length of α as a curve in N.

2.12 Geometrically finite and infinite

A hyperbolic manifold N is *geometrically finite* if its convex core has finite volume; otherwise it is *geometrically infinite.* Works of Bers, Maskit [Mas71], Kra [Kr72] and Sullivan give a description of the space of geometrically finite structures on a 3-manifold in terms of the Teichmüller space of its boundary.

Even when N is geometrically infinite, there exists a compact submanifold C of $\hat{N}^\epsilon$, called the *relative compact core* such that the inclusion of C into N is a homotopy equivalence, C intersects each component of $\partial\hat{N}_\epsilon$ in an annulus, if the corresponding component is a rank one cusp, or in a torus, if the corresponding component is a torus cusps (see Feighn-McCullough [FMc87]). The ends of $\hat{N}^\epsilon$ are in one-to-one correspondence with components of $\partial C \backslash P$, where $P := \partial\hat{N}^\epsilon \cap C$ is the *parabolic locus.* In general $\hat{N}^\epsilon$ can have several ends. Each end is either a *geometrically finite end* when it intersects the convex core in a bounded set or a *geometrically infinite end* otherwise. Canary [Can89], [Can93b] proved that if the manifold is *topologically tame,* (it is homeomorphic to the interior of a compact manifold), then each geometrically infinite end is *simply degenerate*: it has a neighborhood U homeomorphic to $\overline{R} \times [0, \infty)$ (where $\overline{R}$ is a compact surface) and there exists a sequence of *simplicial hyperbolic surfaces* $\{h_n : \overline{R} \to U\}$ leaving every compact set such that for each n, $h_n(\overline{R})$ is homotopic to $\overline{R} \times \{0\}$ within U. (We will define *simplicial hyperbolic surfaces* in 2.14.) A (geodesic) *current* on a hyperbolic manifold M (in any dimension) is a (positive) transverse invariant measure on the geodesic flow of M whose support is contained within the projective tangent bundle of the *convex core.* (The convex core of a hyperbolic manifold is the smallest convex submanifold such that the inclusion is a homotopy equivalence.) Equivalently, if $M = \mathbb{H}^n/\Gamma$, we may think of a current as a Γ-invariant measure on $L_\Gamma \times L_\Gamma \backslash \Delta$, where L_Γ is the limit set of Γ and Δ is the diagonal. We denote the space of currents on M by $\mathscr{C}(M)$. When the support of a current c is a closed geodesic, we define its length, $l_M(c)$, to be the length of its support times the transverse measure of c. This extends to a continuous map $l_M : \mathscr{C}(M) \to \mathbb{R}_+ \cup \{0\}$, which is continuous when M is *convex cocompact*: it has a compact convex core. If S is a hyperbolic surface and α and β two closed geodesics, we define their geometric intersection number $i(\alpha, \beta)$ to be the number of points in $\alpha \cap \beta$. This extends to a symmetric, bilinear map

$$i : \mathscr{C}(S) \times \mathscr{C}(S) \to \mathbb{R}_+ \cup \{0\}$$

which is again continuous if S is convex cocompact. Note that $\mathcal{ML}(S)$ is naturally a subset of $\mathscr{C}(S)$ consisting of the currents c such that $i(c, c) = 0$. (See Bonahon [Bo86] for a discussion of currents and for the above definitions and facts.)

Canary [Can93b, Thm. 10.1] proved that if E is a simply degenerate end of $\hat{N}^\epsilon$ with a neighborhood homeomorphic to $\overline{R} \times [0, \infty)$, there exists a unique geodesic lamination $\mu(E)$ called *the ending lamination*. The main properties of the ending lamination $\mu(E)$ are the following (let R be the interior of $\overline{R}$):

1. $\mu(E)$ is supported on R and fills R,

2. if $\{\gamma_i^*\}$ is a collection of closed geodesics exiting E which are homotopic (within U) to curves γ_i on R, then every limit of the sequence $(\frac{\gamma_i}{l_0(\gamma_i)})$ in the space of currents is a measured lamination supported on $\mu(E)$. Here, $l_0(\gamma_i)$ is its length in a fixed finite area hyperbolic metric on R.

2.13 Hyperbolic structures on handlebodies

A *hyperbolic structure* on the handlebody H (or simply a *structure*) is a complete hyperbolic manifold N with a homeomorphism $\phi : H \to N$. Two structures (N_1, ϕ_1) and (N_2, ϕ_2) are equivalent if there exists an isometry $f : N_1 \to N_2$ such that $\phi_2^{-1} \circ f \circ \phi_1$ is homotopic to identity. Equivalently, a hyperbolic structure on H is given by the conjugacy class of a representation $\rho : \pi_1(H) \to \mathrm{PSL}_2(\mathbb{C})$. (The equivalence, in fact, follows from the recent proof of the Tameness Conjecture (Agol [Ag] and Calegari-Gabai [CG].) If we choose a *base frame* (a base point together with a basis for the tangent space at the base point), this uniquely determines a representation of $\pi_1(H)$. By a *hyperbolic structure with a base frame*, we mean a hyperbolic structure with a choice of a base frame.

A *marked hyperbolic structure* on the handlebody H (or simply a *marked structure*) is a complete hyperbolic manifold N and an embedding $j : \partial H \to N$, called *marking*, such that j can be extended to an embedding $\bar{j} : H \to N$ and $N \setminus \bar{j}(H)$ is homeomorphic to $\partial H \times \mathbb{R}$. Two marked structures (N_1, j_1) and (N_2, j_2) are equivalent if there exists an isometry $f : N_1 \to N_2$ such that $f \circ j_1$ and j_2 are isotopic. We can think of a marked hyperbolic structure on H as a complete hyperbolic metric on the interior of H defined up to deformations induced by self-homeomorphisms of H isotopic to identity. In this case, the marking j is simply any embedding of ∂H into the interior of H isotopic to the inclusion $\partial H \hookrightarrow H$. When we speak of a marked structure N, always a choice of a marking $j : \partial H \to N$ is implicit. Also when we have a compact core $C \subset N$, we can isotope the marking j and assume that $j(\partial H)$ does not intersect C and the component of $N \setminus j(\partial H)$ that gives a neighborhood of the end of N does not intersect C either. Notice that the class of embeddings that are homotopic to j in $N \setminus C$ is included in the class of maps that are isotopic to j.

Remark 2.2. In the literature, usually a marked hyperbolic structure is a hyperbolic manifold with a marking for the fundamental group (a choice of an isomorphism from a fixed group to the fundamental group of the manifold). Here our markings not only mark the fundamental group but mark the isotopy class of identification of the manifold with a fixed copy of the manifold.

We say a sequence of hyperbolic structures with base frame converge *algebraically* to a hyperbolic manifold N, if the associated representations do. A sequence of hyperbolic structures converge algebraically if by choosing a base frame for each of the elements of the sequence, they converge algebraically (equivalenty if the associated representations converge algebraically up to conjugation). Also a sequence of hyperbolic structures (N_i) converge *geometrically* to a hyperbolic manifold N if with an appropriate choice of base points $p_i \in N_i$ and $p \in N$, the pointed manifolds (N_i, p_i) converge to (N, p) in the Gromov-Hausdorff topology. In other terms, there exists a sequence of maps

$$\kappa_i : (\mathcal{N}_i(p), p) \to (N_i, p_i) \quad i \geq 0,$$

called the *approximating maps*, where $\mathcal{N}_i(p)$ is the ball of radius i about p, such that on every compact subset of N the maps κ_i converge to an isometry in the C^∞ topology as $i \to \infty$.

We use the geometric topology and the approximating maps in many places of our arguments; in order to shorten the arguments we are usually careless and consider the approximating maps by

$$\kappa_i : N \to N_i.$$

It should be understood that the approximating maps do not have to be defined on all of N and by the above notation we simply mean that κ_i is defined on larger and larges neighborhoods of the base point as $i \to \infty$.

We say a sequence of hyperbolic structures convege *strongly* to N if for an appropriate choice of base frames, they converge both algebraically and geometrically to N. For a sequence (N_i) of marked structures on H, we say (N_i) converge *strongly*, if there exists a marked hyperbolic structure N and basepoints $p_i \in N_i$ and $p \in N$ such that $(N_i, p_i) \to (N, p)$ strongly as a sequence of hyperbolic manifolds and if

$$\kappa_i : (N, p) \to (N_i, p_i)$$

are the approximating maps, then $\kappa_i \circ j$ is isotopic to j_i, where j_i is the marking of N_i and j is the marking of N.

When a (marked) hyperbolic structure N on H is convex cocompact, the associated representation and subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is called a *Schottky group.* Suppose $N = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic structure on H and $j : \partial H \to N$ is a marking. If Ω is the domain of discontinuity for the action of Γ on the boundary at infinity, then we know that $(\mathbb{H}^3 \cup \Omega)/\Gamma$ is homeomorphic to H and gives a compactification for N. Also we know that Ω/Γ has a conformal structure which is induced by the Poncaré metric on Ω. The marking j can be used to obtain a marking for Ω/Γ and this together with the conformal structure uniquely determines a point τ of $\mathfrak{T}(\partial H)$, which we call the *conformal structure at infinity.* It follows from works of Bers, Maskit [Mas71], Kra [Kr72] and Sullivan that this map gives a parametrization of the space of marked convex cocompact hyperbolic structures on H by $\mathfrak{T}(\partial H)$, the Teichmüller space of ∂H. (This also can be used to show that the space of unmarked convex cocompact hyperbolic structures on H is parametrized by $\mathfrak{T}(\partial H)/\mathrm{Mod}_0(H)$.) We sometimes say τ is *associated* to the marked structure N.

In this case, also recall that $\partial\mathcal{CH}(N)$, the boundary of the convex core of N, is equipped with a hyperbolic metric induced by N and there is a natural nearest point retraction from the conformal structure at infinity to $\partial\mathcal{CH}(N)$. The following follows from a theorem of Bridgeman and Canary [BC03]:

Theorem 2.11. *Given ϵ_0 there exists $J > 0$ such that if N is a convex cocompact hyperbolic structure on handlebody H associated to the conformal structure at infinity τ and the injectivity radius of the hyperbolic metric correspondent to τ is bounded below by ϵ_0, then the nearest point retraction*

$$r : (\partial H, \tau) \to \partial\mathcal{CH}(N)$$

is J-Lipschitz and has a J-Lipschitz homotopy inverse.

On the other hand, when a marked hyperbolic structure N on H is geometrically infinite, we can choose a relative compact core $C \subset N$ which is homeomorophic to H and $N \setminus C$ is homeomorphic to $\partial H \times \mathbb{R}$ and is a neighborhood of the end of N. The marking $j : \partial H \to N$ determines a homotopy equivalence from ∂H to $N \setminus C$ and gives an identification of ∂C with ∂H up to isotopy. By Canary's theorem [Can93b] all geometrically infinite ends of N are simply degenerate. In this case P, the parabolic locus, can be represented by a set of disjoint non-parallel essential simple closed curves on ∂H, which we still call the parabolic locus. The ends of $\hat{N}^\epsilon$ correspond to the components of the complement of the parabolic locus in ∂H.

In particular, when N is geometrically infinite without parabolics, $\hat{N}^\epsilon$ has exactly one end with a neighborhood $N \setminus C$. Since we have an identificiation of ∂C with ∂H (using the marking j), this uniquely determines an ending lamination on ∂H, which we call the ending lamination for the marked structure N. (Notice that for unmarked structures, the ending lamination is defined only up to actions of Mod_0.) Even more, Canary [Can93b, Cor. 10.2] proved that in this case, the ending lamination is in the Masur domain of H and fills ∂H.

2.14 Hyperbolic surfaces in 3-manifolds

Following Thurston [Thu79] (cf. Canary-Epstein-Green [CEG87]), we define a *pleated surface* in a hyperbolic 3-manifold N to be a map $f : S \to N$ together with a hyperbolic metric σ_f on S, called the *induced metric*, and a σ_f-geodesic lamination λ on S, so that the following holds: f is length-preserving on paths, maps leaves of λ to geodesics and is totally geodesic on the complement of λ. We say f realizes λ' if λ' is a sublamination of λ. With an abuse of notation, when we consider a pleated surface $f : S \to N$, we usually assume that S is equipped with the induced metric already.

When N is a hyperbolic structure on H, by $\mathbf{pleat}_N$ we denote the set of pleated surfaces $f : \partial H \to N$ which induce the same map as $\partial H \hookrightarrow H$ on the level of fundamental groups. If μ is a geodesic lamination on ∂H, by $\mathbf{pleat}_N(\mu)$ we denote the subset of $\mathbf{pleat}_N$ whose elements realize μ.

Similar to Bonahon [Bo86] and Canary [Can96], we define simplicial hyperbolic surfaces and recall some facts about them.

First recall a generalized definition of a triangulation for a surface (cf. Harer [Ha86] and Hatcher [Ha91]). Let S be a closed surface and Let $\mathcal{V}$ denote a finite collection of points in S. (We often restrict to the case where $\mathcal{V}$ is a single point.) A *curve system* $\{\alpha_1, \ldots, \alpha_m\}$ is a collection of arcs with disjoint interiors and endpoints in $\mathcal{V}$, no two of which are ambient isotopic (rel $\mathcal{V}$), and none of which is homotopic to a point (rel $\mathcal{V}$). A *triangulation* $\mathcal{T}$ of $(S, \mathcal{V})$ is simply a maximal curve system for $(S, \mathcal{V})$. We say two triangulations are equivalent if they are ambient isotopic (rel $\mathcal{V}$).

Suppose N is a hyperbolic 3-manifold. A continuous map $f : S \to N$ from a closed surface S into N is said to be a *simplicial pre-hyperbolic surface* if there exists a triangulation $\mathcal{T}$ of S such that image of each face of $\mathcal{T}$ is an immersed, totally geodesic, non-degenerate triangle. The map f induces a piecewise Riemannian metric on S, and f is said to be a *simplicial hyperbolic surface* if the angle about each vertex of $\mathcal{T}$ is at least 2π. We say a simplicial hyperbolic surface *realizes* a multi-curve α on S if there exists a subset of the 1-skeleton of $\mathcal{T}$ homotopic to α, and f maps each component of α to a closed geodesic in N..

Here, we only use a special class of simplicial hyperbolic surfaces where all the vertices of $\mathcal{T}$ are contained on a subset of the 1-skeleton that is homotopic to a multi-curve and that multi-curve is realized by the simplicial hyperbolic surface.

We say a complete Riemannian 3-manifold has *pinched negative curvature* if there exist nonzero constants $-a^2$ and $-b^2$ such that the sectional curvatures of N lie between the two constants. When N has pinched negative curvature, instead of simplicial hyperbolic surfaces, we use *simplicial ruled surfaces*. Recall that a *ruled triangle* is constructed by taking 3 totally geodesic arcs e_1, e_2 and e_3 which form a triangle in N and taking the collection of geodesics (in the appropriate homotopy class) with one endpoint v_{12}, the mutual endpoint of e_1 and e_2, and the other endpoint on e_3. A map

$f : S \to N$ is called a *simplicial ruled surface* if there exists a triangulation $\mathcal{T}$ of S, such that each face of the triangulation is taken to a non-degenerate ruled triangle and the total angle about each vertex is at least 2π. We say a simplicial ruled surface f *realizes* a simple closed α if there is a closed loop in the 1-skeleton of the triangulation associated to f which is homotopic to α and is mapped to a closed geodesic in N.

Suppose an incompressible map $f : S \to N$ and a multi-curve α on S are given such that $f(\alpha)$ is freely homotopic to a a set of closed geodesics. Then it follows from work of Bonahon [Bo86] that there exists a simplicial ruled surface which realizes α.

For N hyperbolic or with pinched negative curvature by $N^{<\epsilon}$ we mean the set of points of N where the injectivity radius is less than ϵ and $N^{\geq\epsilon}$ denotes its complement. Also, for subsets $X, Y \subset N$, by $d_N^{\geq\epsilon}(X, Y)$ we denote the infimum of length of $P \cap N^{\geq\epsilon}$ among all paths that connect X to Y and by $\mathrm{diam}_N^{\geq\epsilon}(X)$, we denote the supremum of $d_N^{\geq\epsilon}(x, y)$ for points $x, y \in X$. Various versions of the next theorem have been proved by Thurston, Bonahon [Bo86] and Canary [Can93b, Can96].

Lemma 2.12. (Bounded diameter lemma) *Let $f : S \to N$ be a pleated surface or a simplicial hyperbolic surface or a simplicial ruled surface, where in the last case we assume N has pinched negative curvature for constants $-a^2$ and $-b^2$ and otherwise N is hyperbolic. Also assume $f(\gamma)$ has length at least ϵ if γ is a compressible curve on S. Then*

$$\mathrm{diam}_N^{\geq\epsilon}(f(S)) \leq D,$$

where D depends only on ϵ and $\chi(S)$ and the pinching constants in case N has pinched negative curvature.

3 Uniform injectivity for handlebodies

Our purpose in this section is to obtain a parallel version of Thurston's uniform injectivity theorem and the efficiency of pleated surfaces for hyperbolic structures on a handlebody H. All we are doing is assuming that our pleated surfaces are incompressible in a nice neighborhood of the end and then once they are far from a compact core, we can argue similar to Thurston [Thu86, Thu98].

Suppose N is a hyperbolic structure on H. Here in this section we assume that a compact core $C \subset N$ is already chosen. Following Canary-Minsky [CM96], we say a continuous map $f : \partial H \to N$ is *end-homotopic*, if there exists a neighborhood of the end homeomorphic to $\partial H \times (0, \infty)$ and $f(\partial H)$ is homotopic to $\partial H \times \{0\}$ within $N \setminus C$. We concentrate on end-homotopic pleated surfaces in N. In fact, our proofs work in a more general setting. It is enough to have a closed subset C of any hyperbolic manifold N and consider pleated surfaces which are acylindrical in $N \setminus C$. Then once we are far from C the conclusions of our theorems hold.

Fact 3.1. *Suppose N is a hyperbolic structure on a handlebody H and $C \subset N$ is a compact core. Then $H \setminus C$ is acylindrical. In particular, if*

$$f : \partial H \to N \setminus C$$

is end-homotopic, we have

(a) *every disk in N whose boundary is the f-image of an essential curve of ∂H intersects C,*

(b) *every homotopy in N between $f(\alpha)$ and $f(\beta)$, where α and β are non-homotopic closed curves on ∂H hits C and*

(c) *every homotopy between $f(\alpha)$ and a non-trivial power in N intersects C, if α is primitive in ∂H.*

By $\mathcal{N}_a(C)$ we denote the set of points of N which have distance at most a from C.

Lemma 3.2. *Suppose N is a hyperbolic structure on H with a compact core C and $f : \partial H \to N$ is an end-homotopic pleated or simplicial hyperbolic surface such that $f(\partial H) \subset N \setminus \mathcal{N}_a(C)$. Then every compressible curve on $f(\partial H)$ has length at least a.*

Proof. The proof is similar to Canary-Minsky [CM96, Lem. 4.1]. Suppose $f(\alpha)$ bounds a disk in N and α is an essential closed curve on ∂H. Suppose ∂H is equipped with the metric induced by f. By subdividing α into its intersections with faces of the triangulation and straightening, we may replace it by a homotopic curve α' such that $f(\alpha')$ is a polygonal curve with length at most that of $f(\alpha)$. Obviously $f(\alpha')$ is compressible and bounds an immersed disk D, which we may assume is triangulated by totally geodesic triangles whose vertices lie on $f(\alpha')$. Thus D inherits a hyperbolic metric. Becuase of fact 3.1, D must intersect C in some point x. This shows that D is a hyperbolic disk such that every point on the boundary has distance at least a from x. This implies that $f(\alpha')$ has length at least $2\pi \sinh a > a$. ☐

The next lemma is a variation of an observation of Thurston about pleated surfaces.

Lemma 3.3. *Suppose N is a hyperbolic structure on the handlebody H and C is a compact core of N. There exists D_0 such that the following holds: for any ϵ there exists $\delta = \delta(\epsilon, \chi(\partial H)) < \epsilon$ with*

$$f((\partial H)^{<\epsilon}) \subset N^{<\epsilon}, \quad f((\partial H)^{\geq\epsilon}) \subset N^{\geq\delta}$$

for every end-homotopic $f \in \mathbf{pleat}_N$ that $f(\partial H) \subset N \setminus \mathcal{N}_{D_0}(C)$.

Proof. Using lemma 3.2, the first statement

$$f((\partial H)^{<\epsilon}) \subset N^{<\epsilon}$$

is immediate once $f(\partial H)$ has distance ϵ or more from C because the f-image of a curve of length $< \epsilon$ will be an essential curve of length $< \epsilon$.

Suppose, x is a point in the ϵ_0-thick part of ∂H then it has two loops through it of length not exceeding some constant $a/4$, depending only on $\chi(\partial H)$, such that the two loops generate a free subgroup of rank 2 in $\pi_1(\partial H)$. The commutator of these two loops also will have length at most a. But by what we said above, if $f(\partial H)$ does not intersect $\mathcal{N}_a(C)$and α is a closed curve of length at most a on ∂H, then $f(\alpha)$ is essential in N by lemma 3.2. Therefore the loops that we considered on ∂H and their commutator map to nontrivial loops in N. This provides two loops of length at most a based at $f(x)$ whose representatives do not commute. Because of Margulis lemma, this immediately implies that the injectivity radius of N at $f(x)$ is greater than some $\delta_0 > 0$ which depends on a.

Now if x is in the ϵ-thick part of ∂H for any ϵ, its distance from the ϵ_0-thick part of ∂H is bounded depending only on ϵ. Hence the distance of $f(x)$ from the δ_0-thick part of N is bounded with the same bound as well. From this, one can easily see that $f(x)$ has to be in the δ-thick part of N for some δ depending only on ϵ and $\chi(\partial H)$. $\square$

If $f : S \to N$ is a pleated surface with pleating locus λ, it naturally lifts to a map $\mathbf{P}\mathrm{f}$ of λ into the tangent line bundle $\mathbf{P}\mathrm{N}$ of the target hyperbolic manifold.

Theorem 3.4. (Uniform injectivity) *Let H be a handlebody and ϵ_0 a given constant. For every hyperbolic structure N on H, a compact core $C \subset N$ and an end-homotopic pleated surface $f : \partial H \to N$, that realizes a geodesic lamination λ, the map*

$$\mathbf{P}\mathrm{f} : \lambda \to \mathbf{P}\mathrm{N}$$

is uniformly injective on the ϵ_0-thick part of ∂H, provided that $d_N(f(\partial H), C)$ is large. That is, for every $\epsilon > 0$, there is D and $\delta > 0$ such that for any N, $C \subset N$, λ and an end-homotopic $f \in \mathbf{pleat}_N$ with $d_N(f(\partial H), C) \geq D$, if x and $y \in \lambda$ are given whose injectivity radii are greater than ϵ_0,

$$d_{\sigma_f}(x, y) \geq \epsilon \Longrightarrow d_{\mathbf{P}\mathrm{N}}(\mathbf{P}\mathrm{f}(\mathrm{x}), \mathbf{P}\mathrm{f}(\mathrm{y})) \geq \delta.$$

Proof. Suppose we are given a sequence of hyperbolic structures N_i on H, and for every i, we have a compact core $C_i \subset N_i$ and an end-homotopic pleated surfaces $f_i : \partial H \to N_i$ realizing a geodesic laminations λ_i. Also assume for every i, there are points x_i and $y_i \in \lambda_i$ with $\mathrm{inj}(x_i), \mathrm{inj}(y_i) \geq \epsilon_0$ and $d_{\mathbf{P}\mathrm{N_i}}(\mathbf{P}\mathrm{f_i}(\mathrm{x_i}), \mathbf{P}\mathrm{f_i}(\mathrm{y_i})) \to 0$ and $d_{N_i}(f_i(\partial H), C_i) \to \infty$. Theorem 3.4 (Uniform injectivity) will follow when we show that $d_{\sigma_i}(x_i, y_i) \to 0$, where σ_i is the metric induced by f_i.

By lemma 3.3, we know that $\mathrm{inj}(f_i(x_i))$ and $\mathrm{inj}(f_i(y_i))$ are bigger than $\delta_0 = \delta(\epsilon_0, \chi(\partial H))$ for $i \gg 0$. We take x_i and $f_i(x_i)$ to be base points for $(\partial H, \sigma_i)$ and N_i. Therefore these pleated surfaces and the domain and target manifolds converge in the geometric topology (after passing to a subsequence). Suppose Σ, N and $f : \Sigma \to N$ are limits of $(\partial H, \sigma_i)$, N_i and f_i respectively. Notice that Σ and N are not necessarily hyperbolic structures on ∂H and H anymore, but we know that $\chi(\Sigma) \geq \chi(\partial H)$. By taking a further subsequence, we can also assume that the laminations λ_i converge in the Hausdorff topology and λ is the limit lamination on Σ.

Recall Thurston's [Thu86] notion of *weakly doubly incompressible* surfaces: if Σ is a hyperbolic surface of finite area and if $f : \Sigma \to N$ is a map to a hyperbolic 3-manifold which takes cusps to cusps, then f is weakly doubly incompressible if

(a) $f_* : \pi_1(\Sigma) \to \pi_1(N)$ is injective,

(b) homotopy classes of maps $(I, \partial I) \to (\Sigma, \text{cusps}(\Sigma))$ relative to cusps map injectively to homotopy classes of maps $(I, \partial I) \to (N, \text{cusps}(N))$,

(c) for any cylinder $c : S^1 \times I \to N$ with a factorization of its boundary $\partial c = f \circ c_0 : \partial(S^1 \times I) \to \Sigma$ through Σ, if $\pi_1(c)$ is injective then either the image of $\pi_1(c_0)$ consists of parabolic elements of $\pi_1(\Sigma)$, or c_0 extends to a map of $S^1 \times I$ into Σ and

(d) Each maximal cyclic subgroup of $\pi_1(S)$ is mapped to a maximal cyclic subgroup of $\pi_1(N)$.

Lemma 3.5. *The limit pleated surface $f : \Sigma \to N$ is weakly doubly incompressible.*

Proof. The proof is very similar to Thurston's proof [Thu86, Lem. 5.10], where he proves that limits of *doubly incompressible* pleated surfaces are weakly doubly incompressible. The main difference is that here we do not have doubly incompressibility of the maps f_i, but instead using fact 3.1 and the fact that image of f_i is a closed surface, we know that f_i is doubly incompressible in $N_i \setminus C_i$ and its distance from C_i tends to infinity as $i \to \infty$.

Suppose f is not π_1-injective. Then there exists a closed geodesic α on Σ such that $f(\alpha)$ bounds a disk D in N. Since arbitrary large compact subsets of N are approximately isometric to subsets of N_i for large i, we obtain similar disks D_i in N_i. But since D_i has bounded diameter for every i, we conclude that D_i does not intersect C_i for $i \gg 0$. Then fact 3.1 shows that, there is a disk $D_i' \subset \partial H$, whose f_i-image has the the same boundary as D_i. Because ∂D_i has bounded length, we can assume D_i' has bounded diameter in σ_i and therefore they converge to a disk $D' \subset \Sigma$ with boundary α and we have a contradiction.

To check condition (c), suppose we have an incompressible cylinder $A : S^1 \times I \to N$ in the limit manifold, with a factorization of its boundary through f. Again by using the approximating maps, we can push this cylinder to obtain similar cylinders A_i in N_i with bounded diameter. By fact 3.1 (for

$i \gg 0$) there is a cylinder A_i' in ∂H, whose f_i-image has the same boundary as A_i. The core curve of these cylinders cannot be inessential in N_i, otherwise we can take a sequence of bounded diameter compressing disks for A_i and in the limit we get a compressing disk for A, which contradicts incompressibility of A. If the lengths of the core curves of these cylinders tend to 0, then the boundary components of A are parabolics and (c) is satisfied. Otherwise since length of ∂A_i is bounded, there is a bounded diameter homotopy in $(\partial H, \sigma_i)$ between boundaries of A_i and in the limit we will have a homotopy in Σ between boundaries of A and (c) is satisfied.

For condition (d), suppose α is a non-trivial element of $\pi_1(\Sigma)$, and $f_*(\alpha) = \beta^k$ for some $\beta \in \pi_1(N)$. If we take representatives $a \subset \Sigma$ and $b \subset N$ for α and β respectively, together with a cylinder C giving the homotopy from b^k to $f(a)$, we can push this configuration to approximates N_i. Again by fact 3.1, it follows that we can push the homotopy to $f_i(\partial H)$. Therefore, if a_i is the approximation to a on $(\partial H, \sigma_i)$, there is a loop c_i on ∂H such that c_i^k is homotopic to a_i. In fact, we can assume that c_i is contained in a small neighborhood of a_i independently of i and the homotopy between c_i^k and a_i does not leave this neighborhood either. Then in the limit a will be a kth power in Σ and this proves (d).

To prove (b), let α and β be two arcs on Σ with ends in cusps(Σ) which represent different homotopy classes of maps

$$(I, \partial I) \to (\Sigma, \text{cusps}(\Sigma))$$

relative to cusps. Suppose that they are mapped to the same element of the homotopy classes of maps $(I, \partial I) \to (N, \text{cusps}(N))$ relative to cusps. This means that there are arcs μ and ν in cusps(N) such that

$$f(\alpha) * \nu * f(\beta^{-1}) * \mu$$

is null-homotopic in N. Note that if we push α, β, $f(\alpha)$ and $f(\beta)$ to the approximates, we get arcs

$$\alpha_i, \beta_i : (I, \partial I) \to (\partial H, \text{thin}(\sigma_i))$$

and

$$f_i(\alpha_i), f_i(\beta_i) : (I, \partial I) \to (N_i, \text{thin}(N_i)).$$

Even more, we can push μ and ν to μ_i and $\nu_i \subset \text{thin}(N_i)$ in the approximates and

$$f_i(\alpha_i) * \nu_i * f_i(\beta_i^{-1}) * \mu_i$$

bounds a disk with bounded diameter in N_i.

Suppose P is the component of cusps(N) which contains $\alpha(0)$ and $\beta(0)$ and P_i is the corresponding component of thin(N_i) in an approximate and note that P is either a rank 1 cusp or a rank 2 cusp and for every i, P_i is either a rank 1 cusp or a Margulis tube. Now we have two different cases:

Case 1: P is a rank 2 cusp. Then ∂P has bounded diameter and in the approximates ∂P_i has bounded diameter as well and therefore P_i is a Margulis tube in N_i whose distance to Γ_i tends to infinity as $i \to \infty$. Because the thin components of $(\partial H, \sigma_i)$ corresponding to $\alpha_i(0)$ and $\beta_i(0)$ map to P_i, some power of their cores are homotopic within P_i. Because of fact 3.1 and lemma 3.3, it is possible only when these cores are homotopic in ∂H and they represent the same component of thin(σ_i). In addition, it easily follows that we can connect $\alpha_i(0)$ and $\beta_i(0)$ with an arc μ_i' such that $f(\mu_i')$ and μ_i are homotopic relative their endpoints with a homotopy that stays in P_i and therefore does not intersect C_i.

Case 2: P is a rank 1 cusp. The components of cusps(Σ) corresponding to $\alpha(0)$ and $\beta(0)$ map to the same component of cusps(N): P. Hence by condition (d), the images of their cores are homotopic to the core of P and there is a cylinder $A \subset P$ that gives the homotopy. One can easily assume that μ is in A. Now if we push A to the approximates N_i, we get cylinders A_i whose boundaries are on $f_i(\partial H)$ and because they have bounded diameter they stay in $N_i \setminus C_i$ for $i \gg 0$. Also note that $\mu_i \subset A_i$. Again fact 3.1 implies that there is a homotopy within $N_i \setminus C_i$, which fixes ∂A_i and pushes A_i to $f_i(\text{thin}(\sigma_i))$. In particular, μ_i is homotopic (rel. endpoints) to an arc $f_i(\mu_i')$ with a homotopy that does not intersect C_i, where $\mu_i' \subset \text{thin}(\sigma_i)$.

In both these cases, we could find an arc $\mu_i' \subset \text{thin}(\sigma_i)$ such that $f_i(\mu_i')$ is homotopic to μ_i (rel. endpoints) within $N_i \setminus C_i$ for $i \gg 0$. We can do the same for ν_i and find and arc $\nu_i' \subset \text{thin}(\sigma_i)$ such that $f_i(\nu_i')$ is homotopic to ν_i (rel. endpoints) within $N_i \setminus C_i$. Then $f_i(\alpha_i * \nu_i' * \beta_i^{-1} * \mu_i')$ bounds a disk in $N_i \setminus C_i$ and by fact 3.1, α_i and β_i are homotopic as maps from $(I, \partial I)$ to $(\partial H, \text{thin}(\sigma_i))$.

We claim that we can assume μ_i' and ν_i' have bounded length. We certainly can assume that μ_i' and ν_i' are geodesics (since thin(σ_i) is convex).

Also consider geodesics α'_i and β'_i homotopic to α_i and β_i (rel. endpoints) respectively. Lengths of α'_i and β'_i are bounded uniformly. If μ'_i and ν'_i had very long length then because of hyperbolicity of σ_i they have to get arbitrary close and this means that the two components of $\text{thin}(\sigma_i)$ get arbitrary close using an arc in the homotopy class (rel. cusps) of α and β. Which is impossible since we always assume that components of the thin part have distance at least 1 from each other.

Using the above claim, we can assume that the homotopy between α_i and β_i has bounded diameter in $\text{thick}(\sigma_i)$ and therefore in the limit it gives a homotopy between α and β relative to $\text{cusps}(\Sigma)$ and we have a contradiction. $\square$

Thurston proved [Thu86, Thm. 5.6] that for a weakly doubly incompressible map $f : \Sigma \to N$ which takes each leaf of λ, a geodesic lamination on Σ, to a geodesic in N, the canonical lifting $\mathbf{P}\mathrm{f} : \lambda \to \mathbf{P}\mathrm{N}$ is an embedding. Once we know this and the above lemma, we can conclude that the limit points $x = \lim_i x_i$ and $y = \lim_i y_i$ must be equal, since their images in $\mathbf{P}\mathrm{N}$ are equal. Therefore, $d_{\sigma_i}(x_i, y_i) \to 0$ and we have proved theorem 3.4. $\square$

If λ is a lamination in S, a *bridge arc* for λ is an arc in S with end points on λ, which is not deformable rel endpoints into λ. A *primitive bridge arc* is a bridge arc whose interior is disjoint from λ. If σ is a hyperbolic metric on S and τ is a bridge arc for λ, let $[\tau]$ denote the homotopy class of τ with endpoints fixed, and let $l_\sigma([\tau])$ denote the length of the minimal representative of $[\tau]$.

For a lamination λ and two homotopic maps f, f' that realize λ in a hyperbolic manifold N, we say that f and f' are *homotopic relative to* λ if there is a homotopy between them fixing λ point-wise. One can see that after precomposing f' by a homeomorphism isotopic to identity, we can obtain a map that is homotopic to f relative to λ. (Cf Minsky [Min00, Lem. 3.3].)

Similar to Minsky [Min92, Min00], we can strengthen uniform injectivity as follows:

Corollary 3.6. (Short bridge arcs) *Fix the handlebody H and ϵ_0. Given $\delta_1 > 0$ there exists $\delta_2 > 0$ and D such that the following holds: Let N be a hyperbolic structure on H and $C \subset N$ a compact core. Also let $g \in \mathbf{pleat}_N$ be an end-homotopic map with $d_N(g(\partial H), C) \geq D$ that realizes a lamination λ.*

Suppose that τ a bridge arc for λ is either primitive or contained in the ϵ_0-thick part of σ_g. Then

$$l_{\mathbf{P}\mathrm{N}}(\mathbf{P}\mathrm{g}(\tau)) \leq \delta_2 \Longrightarrow \mathrm{l}_{\sigma_\mathrm{g}}([\tau]) \leq \delta_1.$$

Moreover if f is another map that realizes λ, chosen so it is homotopic to f relative to λ, then

$$l_{\sigma_f}([\tau]) \leq \delta_2 \Longrightarrow l_{\sigma_g}([\tau]) \leq \delta_1.$$

We can use our uniform injectivity theorem and similar to Thurston we can prove a version of efficiency of pleated surfaces. Recall the notion of *alternation number* $a(\lambda, \gamma)$ where λ is a lamination with finitely many leaves and γ is a simple closed curve. This is defined carefully in Thurston [Thu98] and Canary [Can93a]. For our purpose, we need only to know that if γ does not intersect the recurrent part of λ then $a(\lambda, \gamma)$ is bounded by the number of intersection points of λ and γ.

Theorem 3.7. (Efficiency of pleated surfaces) *Given $\epsilon > 0$ smaller than the Margulis constant, there exists constants $c > 0$ and $D > 0$ depending only on $\chi(\partial H)$ and ϵ_0 such that the following holds. Let N be a hyperbolic structure on H with a compact core $C \subset N$ and $f \in \mathbf{pleat}_N$ is end-homotopic realizing a maximal finite leaved lamination λ such that image of every component of λ that is a closed curve has length at least ϵ and $d_N(f(\partial H), C) \geq D$. Also suppose γ is a simple closed curve on ∂H. Then*

$$l_N(\gamma) \leq l_{\sigma_f}(\gamma) \leq l_N(\gamma) + ca(\gamma, \lambda). \tag{3.1}$$

Sketch of proof. The proof follows Thurston's idea in his original proof. Notice that here we are not assuming γ to be incompressible in N.

Similar to Thurston, we can find a plygonal representative γ' of γ on ∂H, equipped with σ_f, by concatentating segments of leaves of λ and short segments between pairs of leaves of λ. we alse now that the number of sides of this polygon is bounded proportional to $a(\gamma, \lambda)$. We can construct a closed curve γ'' in N by taking $f(\gamma')$ and replacing image of every jump by its geodesic representative (rel. endpoints). The lengths of $f(\gamma')$ and γ'' differ by a small numbed dpending on ϵ and $a(\gamma, \lambda)$.

Now we have two cases depending on whether $f(\gamma)$ is essential or inessential in N. If $f(\gamma)$ is essential then it has a geodesic representative γ^* and there exists a pleated annulus A which represents the free homotopy between

γ'' and γ^*. The area of A is bounded by a multiple of $a(\gamma, \lambda)$ (in fact we can assume it is bounded by $2\pi a(\gamma, \lambda)$). If there is a significant difference between lengths of γ'' and γ^* then one can see that there has to be two sides of γ'' coming from $f(\lambda)$ which are close to each other for a long time. Using the Uniform injectivity theorem, we can see that this cannot be the case and we are done.

When $f(\gamma)$ is inessential, we can use a similar argument. Choose a plygonal closed curve γ' that rperesents γ on ∂H and construct γ'' in N as before. In this case, we know that γ'' bounds a pleated disk and again by using Gauss-Bonnet theorem, we know that the area of this disk is bounded proportional to $a(\gamma, \lambda)$. As before, if γ'' has very long length, two of its sides come close and spend a long time close to each other which contradicts the uniform injectivity as in the previous case. □

4 Pleated surfaces in handlebodies

For handlebodies, Canary [Can89, Can93b] proved the following:

Proposition 4.1. *Let H be a handlebody. There is a collection Γ of disjoint simple closed curves on ∂H with the following properties:*

1. *Γ intersects at least three times every essential simple closed compressible curve on ∂H,*

2. *Γ intersects the boundary of every essential and properly embedded annulus $(A, \partial A) \subset (H, \partial H)$ and*

3. *$0 = [\Gamma] \in H_1(H; \mathbb{Z})$.*

For a hyperbolic structure N on H, by a *diskbusting geodesic* in N, we mean the geodesic representative of a collection of curves which satisfy the conclusion of the above. In fact, Canary proved that we can always choose Γ such that none of its components represent parabolic elements of N and therefore there always exist a diskbusting geodesic.

Let N be a marked hyperbolic structure on H and Γ a diskbusting geodesic in N which we assume is fixed. Suppose we have chosen a compact core $C \subset N$ which contains a diskbusting geodesic and as always, we assume that image of the marking j does not intersect C and the component of $N \setminus j(\partial H)$ that is a neighborhood of the end of N does not intersect C either. We call such a compact core a *useful* compact core for N. If α is a multi-curve on ∂H, by a geodesic representative of α in $N \setminus C$, we mean a closed geodesic which is freely homotopic to $j(\alpha)$ with a homotopy that stays in $N \setminus C$.

Lemma 4.2. *Given $\epsilon > 0$ and $d > 0$ there exists a constant $D_1 > 0$ depending only on ϵ, d and $\chi(\partial H)$ such that if N is a marked hyperbolic structure on H with a useful compact core C and if α is a simple closed curve on ∂H which has a geodesic representative α^* in $N \setminus C$ with*

$$d_N^{\geq \epsilon}(\alpha^*, C) \geq D_1,$$

then there exists $f \in \mathbf{pleat}_N(\mu)$ that is homotopic to j within $N \setminus C$ and

$$d_N^{\geq \epsilon}(f(\partial H), C) \geq d,$$

for every finite leaved lamination μ that contains α.

Proof. Suppose μ is a finite leaved maximal lamination that contains α and let γ be the recurrent part of μ. Consider a diskbusting geodesic $\Gamma \subset C$. The idea is to lift to the 2-fold branched cover of N, branched along Γ, which we call $\hat{N}$. Similar to Canary [Can93b], one can see that $\hat{N}$ has a compact core $\hat{C}$ with incompressible boundary which is lift of C. We also know that we can put a Riemannian metric with pinched negative curvature on $\hat{N}$, which is hyperbolic and is lift of the metric of N outside $\hat{C}$. By construction, there is a component of $\hat{N} \setminus \hat{C}$ which is isometric to $N \setminus C$ and we can lift j and α^* to $\hat{j}$ and $\hat{\alpha}^*$. Notice that $\hat{j}$ is incompressible. (Note that there is a minor problem, when Γ has self intersection, but as Canary [Can93b] showed, we can get around this problem by perturbing the metric of N in a small neighborhood of Γ without affecting the metric in $N \setminus C$ and pursue as before.)

As we mentioned in 2.14, it follows from work of Bonahon [Bo86] and Canary [Can93b] that there exists a simplicial ruled surface $\hat{g} : \partial H \to \hat{N}$ homotopic to $\hat{j}$ that realizes γ. Note that $\hat{\alpha}^*$ has to be in the image of this surface. Since the covering map is an isometry when restricted to $\hat{N} \setminus \hat{C}$, we have

$$d_{\hat{N}}^{\geq \epsilon}(\hat{\alpha}^*, \hat{C}) = d_N^{\geq \epsilon}(\alpha^*, C) \geq D_1.$$

But $\operatorname{diam}_{\hat{N}}^{\leq \geq \epsilon}(\hat{g}(\partial H))$ is bounded from above depending only on ϵ and $\chi(\partial H)$ because of lemma 2.12 (Bounded diameter lemma). Therefore if we assume D_1 is large enough depending only on ϵ and $\chi(\partial H)$, the image of $\hat{g}$ has to be contained in $\hat{N} \setminus \hat{C}$. Even more, we can see that there is a homotopy between $\hat{g}$ and $\hat{j}$ that stays within $\hat{N} \setminus \hat{C}$. (Note that in our statement of the bounded diameter lemma, D depended on the curvature bounds; but here since $\hat{N}$ is hyperbolic outside $\hat{C}$, it is enough to assume $D_1 \geq D + 1$ where D is the upper-bound for the hyperbolic case.)

Then we can project $\hat{g}$ down to $N \setminus C$ to obtain a simplicial ruled surface $g : \partial H \to N$ which is homotopic to j within $N \setminus C$ and realizes γ. Replace ruled triangles of $g(\partial H)$ with totally geodesic triangles and similar to Thurston's construction of pleated surfaces [Thu79], spiral vertices of the associated triangulation about components of γ in a way that it approximates μ. In the limit, we get a pleated surface that realizes μ and one can see that during this process, we stay in a small neighborhood of $g(\partial H)$ and the obtained pleated surface has all the required properties of our statement. □

For $f \in \mathbf{pleat}_N$, we define $\mathbf{short}(f, B)$ to be the set of essential simple closed curves on ∂H whose length in the induced metric does not exceed B.

Theorem 4.3. *Given $\epsilon > 0$ and d there exists $D_2 > d$ and $A > 0$ depending only on d, R and $\chi(\partial H)$ such that the following holds. Let N be a marked hyperbolic structure on H and $C \subset N$ a useful compact core. If α has a geodesic representative α^* in $N \setminus C$ with $d_N^{\geq\epsilon}(\alpha^*, C) \geq D_2$, then for every $\beta \in \mathcal{C}_0(S)$ with $d_{\mathcal{C}}(\alpha, \beta) \leq 1$:*

(a) *there exists $f \in \mathbf{pleat}_N(\beta)$ homotopic to j within $N \setminus C$*

(b) *there exists $f \in \mathbf{pleat}_N(\alpha) \cap \mathbf{pleat}_N(\beta)$ homotopic to j within $N \setminus C$*

(c) *every $f \in \mathbf{pleat}_N(\beta)$ has $d_N^{\geq\epsilon}(f(\partial H), C) \geq d$*

(d) *for every end-homotopic f and $g \in \mathbf{pleat}_N(\beta)$, the set*

$$\mathbf{short}(f, B) \cup \mathbf{short}(g, B)$$

has diameter bounded by A in $\mathcal{C}(\partial H)$.

Proof. Using lemma 4.2, we know that for every $d > 0$ if we assume D_2 is bigger than the constant obtained there and α, β and N as in the hypothesis, there exists $f \in \mathbf{pleat}_N(\alpha \cup \beta)$ homotopic to j within $N \setminus C$ and with

$$d_N^{\geq\epsilon}(f(\partial H), C) \geq d.$$

This already proves (a) and (b). In particular, this implies that β has a geodesic representative β^* in $N \setminus C$ with

$$d_N^{\geq\epsilon}(\beta^*, C) \geq d.$$

By assuming that d is larger than the constant in lemma 2.12 (Bounded diameter lemma) for ϵ, this implies (c) too. In fact, given $d > 0$, we can choose D_2 large such that for every $f \in \mathbf{pleat}_N(\beta)$

$$d_N^{\geq\epsilon}(f(\partial H), C) \geq d.$$

For part (d), the idea is to use an argument similar to Minsky's [Min01, Lem. 3.2]. There, the pleated surfaces are doubly incompressible and in particular π_1-injective; but here we use theorem 3.4 and lemma 3.3 instead. Before explaining the proof, note that

$$\operatorname{diam}_{\mathcal{C}}(\mathbf{short}(f, B)) < B' \qquad (4.1)$$

for every pleated surface f, where B' depends only on $\chi(\partial H)$ and B.

Suppose f and g are as in the statement and σ_f and σ_g are the hyperbolic metrics induced on ∂H. After possibly precomposing g with a self-homeomorphism of ∂H homotopic to identity, we can assume that f and g are homotopic via a homotopy that fixes image of β point-wise. As we showed in proving part (c), by assuming $d > D_0$ we can make sure that both $f(\partial H)$ and $g(\partial H)$ have distance more than D_0 from Γ_N, where D_0 is the constant in lemma 3.3.

Let ϵ' be a constant smaller than the Margulis constant and the injectivity radii within distance D_1 of Γ_N and let $\delta = \delta(\epsilon', \chi(\partial H))$ be the constant chosen in lemma 3.3. Suppose β meets the δ-thin part of either σ_f or σ_g, say σ_f. Then its f-image meets the δ-thin part of N and so does its g-image (since they agree). But then because of our choice of the constant δ from lemma 3.3, the pleated surface $g(\partial H)$ intersects this component of the thin part of N in a component of its ϵ'-thin part. The images of the cores of the thin part components of σ_f and σ_g are homotopic (up to taking a power) within $N \setminus C$. By fact 3.1 these have to be homotopic in ∂H and therefore there is a simple closed curve γ that is short in both σ_f and σ_g; hence

$$\gamma \in \mathbf{short}(f, B) \cap \mathbf{short}(g, B).$$

Together with (4.1), this implies a bound on

$$\operatorname{diam}_{\mathcal{C}}(\mathbf{short}(f, B) \cup \mathbf{short}(g, B)).$$

Hence we can assume that β stays in the δ-thick part of σ_f and σ_g. By the second part of corollary 3.6, there exists a $\delta_2 > 0$ and D such that if $d_N(g(\partial H), \Gamma_N) \geq D$ and τ is a bridge arc for β in the δ-thick part of σ_f and whose σ_f-length is at most δ_2, then τ is homotopic rel endpoints to an arc of σ_g-length ϵ'.

Given this δ_2, we may construct a simple closed curve γ_{δ_2} in the ϵ'-thick part of σ_f, whose σ_f length is at most a constant L depending only on δ_2 and $\chi(\partial H)$, and which is composed of at most two arcs of β and at most two bridge arcs of length δ_2 or less (Cf. Minsky [Min01, Lem. 8.5]).

The bridge arcs can be homotoped to have σ_g-length at most ϵ', and hence γ_{δ_2} can be realized in σ_g with length at most $L + 2\epsilon'$. In each surface, this bounds its $\mathcal{C}$-distance from the curves of length B, and together with (4.1) we again obtain a bound on

$$\operatorname{diam}_{\mathcal{C}}(\mathbf{short}(f, B) \cup \mathbf{short}(g, B)).$$

Hence by simply assuming $d > D$ and choosing D_2 accordingly, we will be done. $\square$

For a choice of a useful compact core $C \subset N$, following Canary-Minsky, we say a continuous map $f : \partial H \to N$ is *end-homotopic* if f is homotopic to $j \circ \phi$ within $N \setminus C$ for a homeomorphism $\phi : \partial H \to \partial H$. We relax our definition of realization of a multi-curve P by considering

$$\mathbf{good}_N(P; A)$$

to denote the set of end-homotopic pleated maps $f \in \mathbf{pleat}_N$ such that

$$l_{\sigma_f}(\alpha) \leq l_N(\alpha) + A$$

for all components α of P, where $l_N(\alpha)$ is length of the geodesic representative of α if there exists one and is zero otherwise.

Also if σ is a hyperbolic metric on ∂H and $\alpha \in \mathcal{C}_0(\partial H)$, we define $\mathbf{collar}(\alpha, \sigma)$ to be the set of points which have distance $\leq \omega(l_\sigma(\alpha))$ from the geodesic representative of α on σ, where

$$\omega := \max(\omega_0/2, \omega_0 - 1)$$

and

$$\omega_0(t) := \sinh^{-1}(\frac{1}{\sinh(t/2)}.$$

It is well known that this set is always an embedded annulus, and f $\alpha_1, \ldots, \alpha_k$ are disjoint and homotopically distinct then $\mathbf{collar}(\alpha_i)$ are pairwise disjoint with a definite distance between every pair. (C.f. Minsky [Min01, Sec. 8] or Buser [Buser, Chap. 4].)

Let $f, g \in \mathbf{pleat}_N$ and let P be a curve system. We say f and g admit a (K, ϵ)-*good homotopy* with respect to P if there is a homotopy $F : \partial H \times [0, 1] \to N$ such that

(a) F_0 and F_1 are respectively the same as f and g up to precomposing with a homeomorphism of ∂H isotopic to identity.

(b) $\mathbf{collar}(P, \sigma_0) = \mathbf{collar}(P, \sigma_1)$ where σ_i denotes the metric induced by F_i for $i = 0, 1$.

(c) The metrics σ_0 and σ_1 are locally K-bi-Lipschitz outside $\mathbf{collar}(P, \sigma_0)$.

(d) Suppose P_0 denotes the subset of P consisting of curves α with $l_N(\alpha) < \epsilon$. The tracks $F(p \times [0,1])$ are bounded in length by K when $p \notin \mathbf{collar}(P_0, \sigma)$.

1. For each $\alpha \in P_0$, the image $F(\mathbf{collar}(\alpha, \sigma_0) \times [0,1])$ is contained in a K-neighborhood of the Margulis tube $T_\alpha(\epsilon)$.

Similar to Minsky [Min01, Lem. 4.1], we have:

Lemma 4.4. (Homotopy bound) *Given A and $\epsilon > 0$, there exists $K = K(A, \epsilon)$ and $D > 0$ so that for any marked hyperbolic structure N on the handlebody H, a useful compact core $C \subset N$ and a maximal curve system P on ∂H, if*

$$f, g \in \mathbf{good}_N(P; A)$$

and $f(\partial H) \cap \mathcal{N}_D(C) = g(\partial H) \cap \mathcal{N}_D(C) = \emptyset$ then f and g admit a (K, ϵ)-good homotopy with respect to P that stays outside of C.

Sketch of proof. Suppose α is a component of P and $f(\alpha)$ is compressible in N. Then $l_N(\alpha) = 0$ and therefore $l_{\sigma_f}(\alpha) \leq A$. Then using lemma 3.2, we can see that $f(\alpha)$ is not compressible by assuming that $f(\partial H) \cap \mathcal{N}_{A+1}(C) = \emptyset$.

Once we know that components of $f(P)$ are all incompressible and also by assuming that K is sufficiently large such that we have uniform injectivity for f and g by using theorem 3.4, then we can argue similar to Minsky [Min01, Lem. 4.1].

First, we can replace g with $g \circ h$, where $h : \partial H \to \partial H$ is a homeomorphism isotopic to identity, such that $\mathbf{collar}(P, \sigma_f) = \mathbf{collar}(P, \sigma_g)$ (which from now on we call it just $\mathbf{collar}(P)$), and σ_f and σ_g are locally K-bi-Lipschitz off $\mathbf{collar}(P)$, and have bounded additive length distortion on $\partial\mathbf{collar}(P)$, with K depending only on A.

Then define $F : \partial H \times [0,1] \to N$ to be the homotopy between f and g whose tracks $F|_{\{x\}\times[0,1]}$ are geodesics parametrized at constant speed and we will bound tracks of F on successively larger parts of the surface.

Let Y be a component of $\partial H \setminus P$ and let $Y_0 = Y \setminus \int(\mathbf{collar}(\partial Y))$. Note that length of any boundary component of Y_0 is at most 2 more than its corresponding geodesic in ∂H and we have

$$l_\sigma(\gamma) \leq l_N(\gamma) + A + 2$$

for $\sigma = \sigma_f$ or σ_g.

□

Lemma 4.5. (Halfway surface) *Given ϵ, there exists constants A_1 and D such that given pants decompositions P and Q on ∂H which differ by an elementary move and a hyperbolic structure N on H with a useful compact core C then*

$$\mathbf{good}_N(P, A_1) \cap \mathbf{good}_N(Q, A_2) \neq \emptyset$$

if there exists an end-homotopic $f \in \mathbf{pleat}_N(P)$ with

$$d_N^{\geq \epsilon}(f(\partial H), C) \geq D).$$

Proof. In [Min01, Lem. 4.2] Minsky considers a finite leaved lamination μ whose recurrent part is $P \cap Q$. To be able to use lemma 4.2, assume we have changed the marking of N to one which is homotopic to f within $N \setminus C$. If $d_N^{\geq \epsilon}(f(\partial H), C)$ is bigger than the constant in lemma 4.2, since $f(P \cap Q)$ is a geodesic representative of $P \cap Q$ it follows that there exists a pleated surface $h \in \mathbf{pleat}_N$ that realizes μ. Because f and h have the same image restricted to $P \cap Q$, and $\mathrm{diam}_N^{\geq \epsilon}(h(\partial H))$ is bounded depending only on ϵ and $\chi(\partial H)$, by assuming that D is large, we can guarantee that $d_N(h(\partial H), C)$ is bigger than the constant in theorem 3.7 (Efficiency of pleated surfaces).

Suppose $\alpha_0 \in P$ and $\alpha_1 \in Q$ are curves that are exchanged by the elementary move. Similar to Minsky, we can see that

$$a(\mu, \alpha_i) \leq 4$$

for $i = 0, 1$. Hence by using theorem 3.7, we have

$$l_{\sigma_h}(\alpha_i) \leq l_N(\alpha_i) + A$$

for $i = 0, 1$ and a uniform A. Thus $h \in \mathbf{good}_N(P, A) \cap \mathbf{good}_N(Q, A)$ and the lemma is proved. ☐

Using the above lemmas, we can prove the following corollary.

Corollary 4.6. *For given ϵ there exists $D > 0$ and $K > 0$ such that for a hyperbolic structure N on H and a useful compact core C the following holds. Let $P_0 \to P_1 \to \cdots \to P_n$ be an elementary-move sequence of pants decompositions on ∂H and let $f_0 \in \mathbf{pleat}_N(P_0)$ be end-homotopic and with*

$$d_N^{\geq \epsilon}(f_0(\partial H), C) \geq D$$

then either

1. *there exists* $F : \partial H \times [0, n] \to N \setminus C$ *such that*

   - $F_0 = f_0$,
   - $F_i = F|_{\partial H \times \{i\}} \in \mathbf{pleat}_N(P_i)$,
   - $F_{i-1/2} = F_{\partial H \times \{i-1/2\}} \in \mathbf{pleat}_N(P_{i-1}) \cap \mathbf{pleat}_N(P_i)$ *and*
   - F *is a* (K, ϵ)*-good homotopy restrictd to* $\partial H \times [i-1, i-\frac{1}{2}]$ *and* $\partial H \times [i - \frac{1}{2}, i]$

   for every $i = 1, \ldots, n$ *or*

2. *there exists* $F : \partial H \times [0, k] \to N \setminus C$ *for some* $0 \leq k \leq n$ *such that*

   - $F_0 = f_0$,
   - $F_i = F|_{\partial H \times \{i\}} \in \mathbf{pleat}_N(P_i)$,
   - $F_{i-1/2} = F_{\partial H \times \{i-1/2\}} \in \mathbf{pleat}_N(P_{i-1}) \cap \mathbf{pleat}_N(P_i)$,
   - F *is a* (K, ϵ)*-good homotopy restrictd to* $\partial H \times [i-1, i-\frac{1}{2}]$ *and* $\partial H \times [i - \frac{1}{2}, i]$ *and*
   - $d_N^{\geq \epsilon}(F_k(\partial H), C) < D$

   for every $i = 1, \ldots, k$.

5 Non-realizability and ending laminations

Assume H is a handlebody and let N be a hyperbolic structure on H. Suppose $C \subset N$ is a relative compact core homeomorphic to H, choose a marking $j : \partial H \to N \setminus C$ and as usual we assume that the component of $N \setminus J(\partial H)$ that is a neighborhood of the end does not intersect C.

As we explained in 2.12, Canary [Can93b] proved that if N is geometrically infinite then there exists an ending lamination which is not realizable in N. Recall that a geodesic lamination λ on ∂H is realizable, if there exists a pleated surface $f : \partial H \to N$ homotopic to j such that f is totally geodesic restricted to leaves of λ. He also proved that when N has no prabolics, the ending lamination is filling and in the Masur domain.

One can ask the converse question:

Question. Suppose λ is a filling lamination in the Masur domain of ∂H and λ is nonrealizable in N, is λ or some $\phi(\lambda)$ the ending lamination of N, where $\phi \in \mathrm{Mod}_0(H)$?

Recall that the ending lamination for hyperbolic structures on H is defined only up to actions of $Modo(H)$ and for marked structures where we have a well-defined and uniqe ending lamination, the translates of the ending lamination are unrealizable too. This is why we have to state our conclusion up to actions of $\mathrm{Mod}_0(H)$.

Aside from being an interesting problem, to prove that given a Masur domain filling lamination λ, there exists a hyperbolic structure with ending lamination λ one needs to know a solution to the above question or a variation of that. In fact, this is what we will prove and use in section 6. Work of Kleineidam-Souto [KS03] answers a similar question for hyperbolic structures on a compression body that is not a handlebody.

In case of handlebodies, we think this problem was not noticed before. Here we give an affirmative answer to the above question in case of handlebodies and as we explained in the introduction, the proof is a joint work with Juan Souto. We should also point out that Ohshika [Oh] has also recently claimed an answer to the above question when N is the strong limit of a sequence of convex cocompact structures on H.

Theorem 1.3 *Suppose λ is a filling Masur domain lamination on ∂H and λ is not realized in N, where N is a hyperbolic structure on H. Then $\phi(\lambda)$ is the ending lamination of N for some $\phi \in \mathrm{Mod}_0(H)$.*

The main issue that makes the treatment of the above theorem different from the other cases, where N is a hyperbolic structure on a manifold with incompressible boundary or on a compression body, is that general pleated surfaces in handlebodies are not end-homotopic and it is hard to distinguish the ones that are end-homotopic from the ones which are not. So simply having a sequence of pleated surfaces that exit the end of N does not show much about the geometry of the end of N and the position of the pleated surfaces.

Here our idea is to use the fact that λ is a Masur domain filling lamination and seek a contradiction using topological arguments.

Let (α_n) be a sequence of essential simple closed curves on ∂H that converges to λ in $\mathcal{PML}(\partial H)$. The following is an easy consequence of work of Otal [Ota88] (cf. Kleineidam-Souto [KS03]).

Fact 5.1. *For n sufficiently large, α_n is realized in N.*

Suppose $f_n : \partial H \to N$ realizes α_n for n. Since limit of the sequence (α_n) is a non-realizable Masur domain lamination, it follows from Kleineidam and Souto [KS03] that the sequence of maps (f_n) *exits the end of* N, i.e. every compact subset of N intersects at most a finite number of pleated surfaces $f_n(\partial H)$.

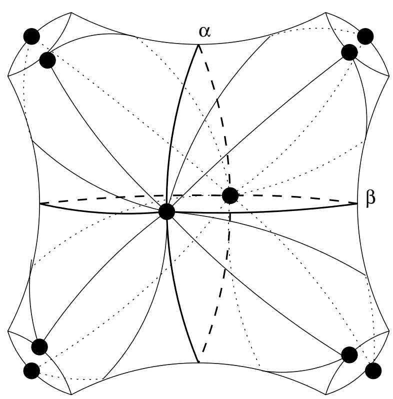
Claim 5.1. We can assume, perhaps after considering sufficiently large indexes, that f_n and f_m are homotopic in $N \setminus C$ for every m and n.

Proof. Using Klarreich's work [Kla], we know that λ represents a point on the Gromov boundary of the curve complex of ∂H and α_n converges to this point in sense of Gromov. Also we can assume that the sequence (α_n) is an infinite path in the curve complex, i.e. $d_{\mathcal{C}}(\alpha_n, \alpha_{n+1}) = 1$ for every n. Then we can extend it to an elementary-move sequence of pants decompositions (P_n) as in lemma 2.5. Notice that still every limit of the sequence (P_n) in $\mathcal{PML}$ is supported on λ.

Now for $n \gg 0$ choose a maximal lamination μ_n that contains P_n as a sublamination and such that all noncompact leaves of μ_n that approach a component γ of P_n (from either side) spiral about γ in the same direction. It is not hard to see that such a lamination exists. Then as above by using Otal's work [Ota88], we can realize μ_n for $n \gg 0$ and conclude that the sequence of realizations exits the end of N. Suppose f_n realizes μ_n and f_{n+1} realizes μ_{n+1}. It will be enough to prove that for n sufficiently large, there exists a homotopy between f_n and f_{n+1} that stays away from C.

Suppose $\alpha, \alpha_1, \ldots, \alpha_k$ and $\beta, \alpha_1, \ldots, \alpha_k$ are components of P_n and P_{n+1} respectively and let $Y \subset \partial H$ be the closure of a component of $\partial H \setminus \{\alpha_1, \ldots, \alpha_k\}$ that contains α and β. We know that Y is either a 4-holed sphere or a 1-holed torus.

First we construct a triangulation $\mathcal{T}$ on ∂H as follows. Restricted to Y, we assume that $\mathcal{T}$ is one of the triangulations in figure (1). Then extend this to a triangulation for the entire ∂H in a way that all the vertices are on components of $P_n \cap P_{n+1}$.



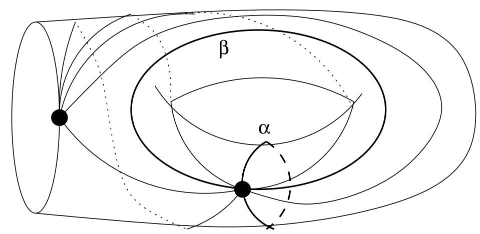


Figure 1: The triangulation on Y.

What is important for us about this triangulation is that P_n and P_{n+1} are homotopic to subgraphs of the 1-skeleton of $\mathcal{T}$ and every triangle on Y has at least one vertex on ∂Y.

For any choice for images of vertices of $\mathcal{T}$ on geodesic representative of P_n (resp. P_{n+1}) that preserves their ordering, we can construct a simplicial hyperbolic surface with associated triangulation $\mathcal{T}$ that realizes P_n (resp. P_{n+1}). The construction is standard, simply make the map identical to f_n (resp. f_{n+1}) restricted on P_n (reps. P_{n+1}). Change the triangulation by an ambient isotopy such that the vertices get mapped to the chosen points on the image of P_n (resp. P_{n+1}). Extend this to the 1-skeleton of $\mathcal{T}$ by sending each edge to a geodesic in the homotopy class of its f_n (resp. f_{n+1}) image (rel. endpoints). Finally extend the map to the entire surface by maping the 2-simplices totally geodesically.

Using an idea of Thurston, we can construct a continuous family g_n^t (resp. g_{n+1}^t of simplicial hyperbolic surfaces as above that converge to f_n (resp. f_{n+1}) in Hausdorff topology. This is possible by starting from one such map

g_n^0 (resp. g_{n+1}^0) and for each component γ of P_n (resp. P_{n+1}) continuously twist the images of vertices on γ about the geodesic representative of γ in the direction that noncompact leaves of μ_n (resp. μ_{n+1}) spiral about γ when approching γ and then construct the simplicial hyperbolic surface g_n^t (resp. g_{n+1}^t) as above. (Here we are assuming that the image of P_n (resp. P_{n+1}) is fixed and we are twisting the vertices of the triangulation about the components.)

Notice that a maximal lamination containing P_n (resp. P_{n+1}) is identified by the direction that its noncompact leaves spiral about components of P_n (resp. P_{n+1}) on each side. Hence the limit of the simplicial hyperbolic surfaces g_n^t (resp. g_{n+1}^t) has to be a pleated surface that realizes μ_n and since μ_n is maximal it has to be identical with f_n (resp. f_{n+1}) up to precomposition with a self-homeomorphism of ∂H isotopic to the identity. From here we can see that when t is large there is a homtopy with bounded length tracks between f_n and g_n^t (resp. f_{n+1} and g_{n+1}^t). Fix t large such that this homotopy stays away from C and denote $h = g_n^t$, $h' = g_{n+1}^t$

It will be enough to show the existance of a homotopy between h and h' whose image is contained in a uniformly bounded neighborhood of $h(\partial H) \cup h'(\partial H)$. First of all, we precompose h or h' with a self-homeomorphism of ∂H isotopic to identity to make h and h' identical restricted to $P_n \cap P_{n+1}$. We know that there is a homotopy between h and h' and we can consider this as a map from $\partial H \times [0,1] \to N$, where restricted to $\partial H \times \{0\}$ and $\partial H \times \{1\}$ the map induces h and h'. The simplicial structure of h and h' makes $\partial H \times \{0,1\}$ triangulated with two triangulations which are isotopic to $\mathcal{T}$ on ∂H. Extend this to a triangulation of $\partial H \times [0,1]$ first connecting every vertex on $\partial H \times \{0\}$ to the corresponding vertex on $\partial H \times \{1\}$. Then add faces homeomrophic to rectangles where two opposite sides of the rectangle are corresponding edges of the triangulations on $\partial H \times \{0\}$ and $\partial H \times \{1\}$. Finally we are left with regions that are homeomrophic to a triangle times an interval, we call them *prisons*, and simply divide each of these to 3 tetrahedra arbitrarily. Now we can assume that the homotopy is totally geodesic restricted to the 1-skeleton and 2-skeleton of the constructed triangulation and extend it to the 3-skeleton (the prisons) by coning of from a vertex of each tetrahedra and map every line segment geodesically.

It will be enough to show that image of every prison stays in a bounded diameter neighborhood of $h(\partial H) \cup h'(\partial H)$. In fact it is enough to do this for faces of the prison. Every prison Q has two triangular faces D and D' which we call them *horizontal* and we call the other faces and edges that connect these horizontal faces *vertical*. The image of horizontal faces are contained in $h(\partial H) \cup h'(\partial H)$. In our construction of the triangulation $\mathcal{T}$, each triangle has at least one vertex on a component of $P_n \cap P_{n+1}$ and the image of the vertical edge e associated to this vertex will be a single point v on the geodesic representative of $P_n \cap P_{n+1}$. The picture of image of a prison is suggested in (2).

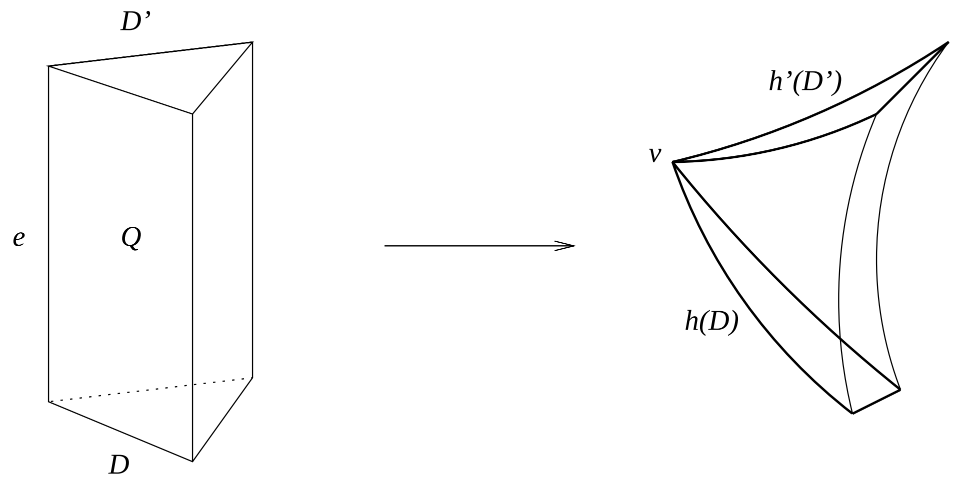


Figure 2: The image of a prison.

Every point on the image of a face of the prison is contained in a rectangle with two sides on D and D' and from this and hyperbolicity of N we can see that it has to have bounded distance from $D \cup D'$ and therefore from $h(\partial H) \cup h'(\partial H)$ and we are done.

$\square$

Note that every $\alpha_n^* := f_n(\alpha_n)$ is homotopic to a closed curve $j(\beta_n)$ in $N \setminus C$, because $j : \partial H \to N \setminus C$ is a homotopy equivalence. Then because the sequence α_n^* exits the end and the homotopy stays outside of the compact core C, the sequence of closed curves (β_n) has to converge in the *space of projectivized currents* to μ: an ending lamination for some end of N. (See 2.13.) On the other hand, because maps (f_n) are homotopic outside of C and they are homotopic to the inclusion $\partial H \hookrightarrow H$ in H, there exists a single map $g : \partial H \to \partial H$ that extends to a map $\overline{g} : H \to H$ homotopic to identity and $g(\alpha_n)$ is homotopic to β_n in ∂H for every n.

Remark 5.2. Note that if g is homotopic to a homeomorphism then we are done. This is because g extends to the handlebody and therefore $g \in \mathrm{Mod}_0(H)$ and $\mu = g(\lambda)$ is the ending lamination. Therefore it will be enough to prove that g is homotopic to a homeomorphism on ∂H.

Without loss of generality, we can assume that the sequence (α_n) converges to a lamination $\bar{\lambda}$ in the Hausdorff topology. Note that $\bar{\lambda}$ contains λ as a sublamination.

Fix a marked convex cocompact structure N_0 on H with a marking $j_0 : \partial H \to N_0$. By a theorem of Otal [Ota88], $\bar{\lambda}$ is realized in N_0 by a map $f_0 : \partial H \to N_0$ homotopic to j_0 and for $\epsilon > 0$ arbitrary, we can construct a train track τ in ∂H that fully carries $\bar{\lambda}$ and all its routes are $(1+\epsilon, \epsilon)$-quasigeodesics in σ_0, the metric induced by f_0. In fact, we are free to split the train track as much as we want and make the branches of the train track long in σ_0. Obviously, τ carries α_n for n sufficiently large and we assume from now on that ∂H is equipped with the hyperbolic metric induced by f_0. Also notice that f_0 and $f_0 \circ g$ are homotopic in N_0. Suppose $F : \partial H \times [0,1] \to N_0$ is this homotopy.

Claim 5.3. The image of every route of τ by g is a (K, c)-quasigeodesic in σ_0 for constants K, c.

Proof. Consider a simple closed curve β which is carried by τ and spends a long time on a route of τ. Then $f_0(\beta)$ is nearly geodesic and therefore its length is nearly the smallest among the closed curves in its free homotopy class in N_0. In particular since $f_0(g(\beta))$ is freely homotopic to $f_0(\beta)$ in N_0, its length has to be bigger than $l_0(\beta) - \epsilon'$ where ϵ' is small depending on ϵ and l_0 denotes the length in σ_0. On the other hand g has bounded Lipschitz constants and increases the length in bounded proportion.

This shows that $g(\beta)$ is a quasigeodesic with constants that depend on ϵ. Using this we can see that images of longer and longer subroutes of a given route are quasigeodesic with the same constants and an easy observation shows that image of the entire route has to be a quasigeodesic. ☐

Now consider $\widetilde{g} : \mathbb{H}^2 \to \mathbb{H}^2$ the lift of g and assume $\widetilde{\lambda}$ is the lift of λ. The above claim shows that $\widetilde{g}(l)$ is a quasigeodesic for every leaf l of $\widetilde{\lambda}$. Hence $\widetilde{g}(l)$ uniquely determines a geodesic in $\mathbb{H}^2$. We consider $\bar{g}$ to be the map from leaves of $\widetilde{l}$ to geodesics in $\mathbb{H}^2$ induced by g.

Lemma 5.2. *If l_1 and l_2 are leaves of $\widetilde{\lambda}$ then $\bar{g}(l_1)$ and $\bar{g}(l_2)$ do not intersect. Also if l_1 and l_2 are asymptotic leaves of $\widetilde{\lambda}$ then their $\bar{g}$-images are asymptotic geodesics. (In fact, the same is true about rays which are subsets of leaves of $\widetilde{\lambda}$.)*

Proof. Suppose they intersect. By going back to the train track τ and splitting it, we can assume that there are two branches b_1 and b_2 whose images by g intersect. We can assume these branches are as long as we want and the intersection point is far from the endpoints of b_1 and b_2. Since g takes every route to a K-quasigeodesic, it is not hard to see that images of every two routes of τ that contain b_1 and b_2, intersect essentially. Now assume n is sufficiently large, so that α_n is carried by τ and assume $b_1(\alpha_n)$ and $b_2(\alpha_n)$ denote the number of times that α_n passes through b_1 and b_2 respectively. Note that by what we just said, we can see that

$$i(\beta_n, \beta_n) \geq b_1(\alpha_n) \cdot b_2(\alpha_n)$$

for $n \gg 0$.

Because the sequence (α_n) converges in $\mathcal{PML}$ to λ and λ passes through b_1 and b_2,

$$\frac{b_1(\alpha_n)}{l_0(\alpha_n)} \text{ and } \frac{b_2(\alpha_n)}{l_0(\alpha_n)}$$

converge to positive numbers (the measure deposited on b_1 and b_2 by λ) and are bigger than some $c > 0$ for $n \gg 0$. On the other hand, like in the proof of the previous claim, because g has bounded Lipschitz constants, there exists $c' > 0$ such that

$$l_0(\beta_n) \leq c' l_0(\alpha_n).$$

Hence

$$\frac{i(\beta_n, \beta_n)}{l_0(\beta_n) \cdot l_0(\beta_n)} \geq \frac{1}{(c')^2} \frac{b_1(\alpha_n) \cdot b_2(\alpha_n)}{l_0(\alpha_n) \cdot l_0(\alpha_n)} \geq \frac{c^2}{(c')^2} > 0.$$

But this implies that every limit of the sequence (β_n) has to have self-intersection and cannot be a lamination, which is a contradiction.

The second statement is obvious, because two sub-rays of $\widetilde{\lambda}$ are asymptotic iff they have bounded Hausdorff distance from each other and again since $\widetilde{g}$ is Lipschitz, their $\widetilde{g}$-images will have bounded Hausdorff distance as well and have to be asymptotic.

□

Lemma 5.3. *Even more, $\bar{g}$ is continuous as a map from leaves of $\widetilde{\lambda}$ to the set of geodesics of $\mathbb{H}^2$ with Hausdorff topology.*

Sketch of proof. The idea of the proof is similar to the proof of the first statement of lemma 5.2. Notice that $\widetilde{\lambda}$ is closed in the Hausdorff topology. Also note that every leaf of λ gives a biinfinite route of τ and if l is a leaf of $\widetilde{\lambda}$, the train track routes of other leaves that are close to l (in Hausdorff topology) will share a long segment with the route associated to l. This shows that in the image also images of these leaves are close to image of l on a long subsegment. Then using the fact that images of all these leaves are (K, c)-quasigeodesic implies that their endpoints cannot be far from each other and they are close in the image with Hausdorff topology. $\square$

Lemma 5.4. *If l_1 and l_2 are leaves of $\widetilde{\lambda}$ and $\bar{g}(l_1) = \bar{g}(l_2)$, then there exists some $h \in \mathrm{Ker}(j_* : \pi_1(\partial H) \to \pi_1(H))$ such that $h(l_1) = l_2$.*

Sketch of the proof. If distinct leaves, l_1 and l_2, of $\widetilde{\lambda}$ project to the same leaf l' of λ in ∂H, then there exists an element $h \in \pi_1(\partial H)$ such that $h(l_1) = l_2$. In this case, we claim that $h \in \mathrm{Ker}(g_*)$ and since $j \circ g$ is homotopic to j, we have to have $h \in \mathrm{Ker}(j_*)$. Suppose $g_* h$ is nontrivial. Since $\bar{g}$ is induced from lifting g we have

$$\bar{g}(l_2) = \bar{g}(h(l_1)) = g_* h(\bar{g}(l_1))$$

and our assumption implies that $g_* h$ preserves $\bar{g}(l_1)$. Since $g_* h$ is non-trivial, it has to be a hyperbolic isometry of $\mathbb{H}^2$ with axis $\bar{g}(l_1)$. This implies that $g(l')$ fellow travels the closed curve that represents $g_* h$. Since $f_0 \circ g$ and f_0 are homotopic in N_0, this implies that $f_0(l')$ fellow travels a closed curve that represents $g_* h$ in N_0, but $f_0(l')$ is a geodesic and this shows that it is a closed geodesic. We knew that λ has no isolated leaves and because of this l' is noncampact. Otal [Ota88] proved that if a pleated surface f_0 realizes a Masur domain lamination λ, then

$$\mathbf{P}\mathrm{f}_0 : \lambda \to \mathbf{P}\mathrm{N}$$

is a homeomorphism to its image. (Recall that $\mathbf{P}\mathrm{N}$ is the projective tangent bundle of N.) This contradicts the possibility that f_0 takes a noncompact leaf of λ to a closed geodesic.

On the other hand, suppose l_1 and l_2 project to distinct leaves of λ: l'_1 and l'_2. Since $\bar{g}(l_1) = \bar{g}(l_2)$, the images $g(l'_1)$ and $g(l'_2)$ are asymptotic in ∂H (they have bounded Hausdorff distance) and we have a homotopy with bounded tracks between them. If we concatenate this homotopy with $F|_{l'_1\times[0,1]}$ and $F|_{l'_2\times[0,1]}$, we get a homotopy with bounded tracks in N_0 between $f_0(l'_1)$ and $f_0(l'_2)$. Since $f_0(l'_1)$ and $f_0(l'_2)$ are geodesics, this is impossible unless $f_0(l'_1) = f_0(l'_2)$. This again contradicts Otal's theorem which we mentioned above and we are done.

□

We know that every complementary component of $\widetilde{\lambda}$ is an ideal polygon.

Lemma 5.5. *If P is a complementary component of $\widetilde{\lambda}$ then $\bar{g}$ is injective on sides of P.*

Proof. Suppose $l_1, l_2, \ldots, l_k$ are sides of P. We want to show that $\bar{g}(l_i) \neq \bar{g}(l_j)$ for $i \neq j$. If this is not the case then lemma 5.4 shows that there exists $h \in \ker j_*$ such that $h(l_i) = l_j$. But then if we consider $h(P)$, it is a complementary component of $\widetilde{\lambda}$ too and it cannot be P, since h is not elliptic. The complementary components P and $h(P)$ share a side: l_j. This shows that l_j is an isolated leaf of $\widetilde{\lambda}$ which is impossible, since λ has no isolated leaf. □

The above lemma and lemma 5.2 show that $\bar{g}$ takes complementary domains to complementary domains. Let $\mathcal{T}_\lambda$ be the dual tree to $\widetilde{\lambda}$ and also consider $\mathcal{T}$ to be the dual tree for the $\bar{g}$-image of $\widetilde{\lambda}$. We have, in fact, shown that g induces a $\pi_1(\partial H)$-equivariant morphism $G : \mathcal{T}_\lambda \to \mathcal{T}$.

Claim 5.4. The map $G : \mathcal{T}_\lambda \to \mathcal{T}$ is locally injective.

First assume that x is correspondent to a component of $S \setminus \lambda$. This component is an ideal polygon P with sides $l_1, l_2, \ldots$ and l_k. It will be enough to show that $\bar{g}(l_i) \neq \bar{g}(l_j)$ for $i \neq j$. Because once we know the injectivity of $\bar{g}$ on the sides of P, it follows from the continuity of $\bar{g}$ that the image of P is an ideal polygon P'. Also we know that leaves which are very close to l_i will be mapped to leaves close to $\bar{g}(l_i)$ and therefore they all stay on a side of $\bar{g}(l_i)$ which is opposite to P'. Now suppose $a_n \neq b_n$ are leaves of λ which approach l_i as $n \to \infty$ and $\bar{g}(a_n) = \bar{g}(b_n)$ for every n. Because of the above lemma for every n there exists some $h_n \in \ker i_*$ such that $h_n(a_n) = b_n$. One can see that the axis of h_n also approaches l_i as $n \to \infty$ and this will contradict the fact that λ is in the Masur domain.

Now we want to show that $\bar{g}$ is injective on sides of P. In fact, if we have $\bar{g}(l_i) = \bar{g}(l_j)$, again by above lemma it implies that there exists $h \in \ker i_*$

such that $h(l_i) = l_j$. But then if we look at $h(P)$ it will be a polygon on the other side of l_j (it is because h is not elliptic) and therefore l_j is an isolated leaf, which is impossible.

Claim 5.5. The map $\mathcal{T}_g : \mathcal{T}_\lambda \to \mathcal{T}$ is locally injective.

Proof. We will need the following lemma for the proof:

Lemma 5.6. *Suppose l is a leaf of $\widetilde{\lambda}$ then $\bar{g}(l_1) \neq \bar{g}(l_2)$ for distinct leaves l_1, l_2 of $\widetilde{\lambda}$ which have very small Hausdorff distance from l.*

Proof. Suppose $a_n \neq b_n$ are pairs of leaves of $\widetilde{\lambda}$ that converge to l in Hausdorff topology as $n \to \infty$ and $\bar{g}(a_n) = \bar{g}(b_n)$. Lemma 5.4 shows that there exists $h_n \in \ker j_*$ such that $h_n(a_n) = b_n$. Because of discreteness of action of $\pi_1(\partial H)$, we can see that length of h_n goes to infinity as $n \to \infty$ and from this one can see that the axis of h_n converges to l in the Hausdorff topology. In other terms, we have a sequence of closed curves (β_n) on ∂H, which are null-homotopic in H and their Hausdorff limit has zero intersection with λ. We want to show that this contradicts the fact that λ is in the Masur domain.

We need to find an "efficient disk system" for λ. Here, a *disk system* is a set of disjoint disks in H which cut H into a ball. A *wave* for a disk system $\mathbf{D}$ is an essential arc $k \subset \partial H$ (essential on ∂H relative to $\mathbf{D}$), that does not intersect $\mathbf{D}$ in its interior and both its end point are on the boundary of a single component D of $\mathbf{D}$ and finally k is homotopic in H to a subarc of ∂D relative to its end points. Note that one can easily see that if we orient k, it intersects D in opposite directions at its end points. For a lamination λ, we say a disk system $\mathbf{D}$ is *efficient*, if all its waves and components intersect λ. The following is a version of Starr's theorem [Sta] for Masur domain measured laminations. (For a similar proof for Starr's theorem see [Wu96].)

Lemma 5.7. *If λ is a Masur domain lamination, then there exists a disk system $\mathbf{D}$ such that λ intersects every wave for $\mathbf{D}$. Equivalently, $\mathbf{D}$ is efficient for λ.*

Proof. Take a non-zero measured lamination supported on λ, which we still denote by λ. We also consider $\mathcal{M}$, the set of meridians, and its closure $\mathcal{M}'$ (in $\mathcal{PML}$) as subsets of the set of measured laminations that have total measure 1. We know that λ has nonzero intersection with every element of $\mathcal{M}'$. Since $\mathcal{M}'$ is a compact set, this means that there exists $c > 0$ such that $i(\lambda, D) > c$ for every meridian D. Now take an arbitrary disk system $\mathbf{D}$. If λ intersect all the meridians of $\mathbf{D}$, then we are done. Otherwise we show that we can replace $\mathbf{D}$ with another disk system $\mathbf{D'}$ such that $i(\lambda, \mathbf{D'}) < i(\lambda, \mathbf{D}) - c$. (By $i(\lambda, \mathbf{D})$, we mean the sum $\Sigma_{D \in \mathbf{D}} i(\lambda, D)$.) Therefore continuing this a finite number of times we should get to a disk system that satisfies the conclusion.

Suppose there exists an arc k which is a wave for $\mathbf{D}$ and it does not intersect λ. The end points of k are on the boundary of the same disk $D \in \mathbf{D}$ and divide it into two arcs α and β. $D_1 = k \cup \alpha$ and $D_2 = k \cup \beta$ are both meridians and since k does not intersect λ, we have

$$i(\lambda, D) = i(\lambda, D_1) + i(\lambda, D_2).$$

On the boundary of the three ball obtained by cutting the handlebody along $\mathbf{D}$, there are two copies of D. Either D_1 or D_2, say D_1, separates these two. Now if we take $\mathbf{D'} := \mathbf{D} \backslash \{D\} \cup \{D_1\}$, it will be a new disk system. But because

$$i(\lambda, D_1) = i(\lambda, D) - i(\lambda, D_2) < i(\lambda, D) - c,$$

this new disk system has the desired property and we can pursue as we explained. ☐

On the other hand, if β is an essential closed curve on ∂H which is null-homotopic in H, then for every disk system $\mathbf{D}$, there exists a subarc of β which is a wave for $\mathbf{D}$. (Cf. Masur [Mas86].)

Now let $\mathbf{D}$ be a disk system that satisfies the above lemma for λ. What we just said, shows that every β_n contains a wave for $\mathbf{D}$. From this one can see that every Hausdorff limit of β_n has to contain a subarc which is a wave for $\mathbf{D}$ and therefore it cannot be disjoint from λ, which is a contradiction. ☐

Now every $x \in \mathcal{T}_\lambda$ either corresponds to a complementary component of $\widetilde{\lambda}$ or to a leaf of $\widetilde{\lambda}$, which is a limit of leaves of $\widetilde{\lambda}$ from both sides. We have to prove local injectivity in small neighborhood of x in both cases.

Suppose x corresponds to a component of $\mathbb{H}^2 \setminus \widetilde{\lambda}$ which is an ideal polygon P with sides $l_1, l_2, \ldots, l_k$. Using lemma 5.5, we know that $\bar{g}(l_i) \neq \bar{g}(l_j)$ when $i \neq j$. Using continuity of $\bar{g}$ from lemma 5.3, we know that leaves which are very close to l_i, will be mapped to leaves which are very close to $\bar{g}(l_i)$. Therefore, images of leaves that are close to l_i cannot be identified with images of those that are close to l_j for $i \neq j$. Finally, using lemma 5.6, we know that $\bar{g}$ is injective in a very small Hausdorff neighborhood of each leaf l_i and this proves the claim in this case.

On the other hand if x corresponds to a leaf l of $\widetilde{\lambda}$ that is a limit from both sides, lemma 5.6 immediately implies that $\bar{g}$ is injective in a small Hausdorff neighborhood of l and we have finished proof of the claim. □

The following lemma shows that $G : \mathcal{T}_\lambda \to \mathcal{T}$ is in fact injective:

Lemma 5.8. *A morphism $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ between $\mathbb{R}$-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ is injective if and only if it is locally injective.*

Proof. Obviously injectivity implies local injectivity. On the other hand, assume ϕ is locally injective but it is not injective. Suppose $\phi(x) = \phi(x')$ for distinct points $x, x' \in \mathcal{T}_1$. Consider the unique geodesic $l : [0,1] \to \mathcal{T}_1$ in $\mathcal{T}_1$ that connects x and x': $l(0) = x$ and $l(1) = x'$. The path $\phi \circ l : [0,1] \to \mathcal{T}_2$ is a closed path in $\mathcal{T}_2$ and it is enough to show that $\phi \circ l$ is not locally injective (because l is injective). If $\phi \circ l$ is locally injective, then there exists a non-degenerate interval $[a,b] \subset [0,1]$ such that $\phi \circ l(a) = \phi \circ l(b)$ but $\phi \circ l$ does not identify any other two points of $[a,b]$. But then we obtain two paths with disjoint interiors between $\phi \circ l(a)$ and $\phi \circ l(\frac{a+b}{2})$ which contradicts the fact that $\mathcal{T}_2$ is an $\mathbb{R}$-tree. □

We use the injectivity of G to show that $g_* : \pi_1(\partial H) \to \pi_1(\partial H)$ is an injective map. Otherwise, assume $h \in \ker g_*$ is nontrivial. If l is a leaf of $\widetilde{\lambda}$, then $h(l) \neq l$; otherwise l would project to a closed curve, but this contradicts our assumption about λ. Since h is in $\ker g_*$, we can see that lift of g to the universal covers takes l and $h(l)$ to the same geodesic and we have $\bar{g}(l) = \bar{g}(h(l))$. Let $x, x' \in \mathcal{T}_\lambda$ be the points that correspond to the leaves l and $h(l)$. We know that $x = x'$ if and only if l and $h(l)$ are sides of a complementary component of $\widetilde{\lambda}$ but then we have a contradiction with lemma 5.5. Hence $x \neq x'$ and we have $G(x) = G(x')$ which contradicts injectivity of G. Therefore g_* is injective.

It is a standard fact that every proper subgroup of a surface group is free, there fore $g_*(\pi_1(\partial H))$ cannot be a proper subgroup of $\pi_1(\partial H)$ and therefore g_* is surjective also and in fact g is a π_1-isomorphism. But again, it is a standard fact about surfaces that every homotopy equivalence from ∂H to itself is homotopic to a homeomorphism. Hence g is homotopic to a homeomorphism and we are done by remark 5.2.

6 Hyperbolic structures with bounded combinatorics

Lemma 6.1. *Let H be a handlebody and $\Delta \subset \mathcal{C}(\partial H)$ its handlebody subcomplex. There exists d_0 only depending on $\chi(\partial H)$ that if $\alpha \subset \mathcal{C}(\partial H)$ is a multi-curve with distance bigger than d_0 from Δ then α is in the Masur domain. In fact, we can consider $d_0 = d + 2$ where d is the quasi-convexity constant of Δ in theorem 7.11.*

Proof. If α is not in the Masur domain, then it has zero intersection with μ, an element of $\bar{\Delta}$. (As usual the closure is taken in $\mathcal{PML}$.) The lamination μ cannot be filling, since it has zero intersection with a simple closed curve. Therefore, $Y = \text{supp}(\mu)$ is a proper subsurface of ∂H. Of course ∂Y has zero intersection with α too: $d_{\mathcal{C}}(\alpha, \partial Y) \leq 1$. We claim that ∂Y has distance at most $d + 1$ from Δ and this together with what we said before shows that $d_{\mathcal{C}}(\alpha, \Delta) \leq d + 2$ which contradicts our hypothesis. To prove the claim, consider a sequence $(\beta_i) \subset \Delta$ that converges to $\mu \subset Y$ in $\mathcal{PML}$. One can see that the sequence $(\pi_Y(\beta_i))$ also has to converge to μ in $\mathcal{PML}(Y)$ and therefore in $\mathcal{C}(Y) \cup \partial\mathcal{C}(Y)$. Hence $d_Y(\beta_0, \beta_i) \to \infty$ as $i \to \infty$. Then theorem 2.4 (Bounded image theorem), implies that for i sufficiently large, any geodesic connecting β_i and β_0 intersects the one-neighborhood of ∂Y. On the other hand because of quasi-convexity of Δ, theorem 7.11, this geodesic is in the d-neighborhood of Δ and this proves the claim. ☐

Let $R > 0$ be given and let $A(R)$ be the set of full markings with R-bounded combinatorics respect to H and distance at least d_0 from $\Delta(H)$, where d_0 is the constant obtained in lemma 6.1. Recall that when we consider a marking in the curve complex, we take its support instead. We also consider the subset $A_0(R) \subset A(R)$ to be the subset of $A(R)$ whose elements have R-bounded combinatorics with respect to an element of $\mathbf{m}_0(\mathrm{H})$. Notice that every element of $A(R)$ can be translated to $A_0(R)$ by an action of $\text{Mod}_0(H)$.

Proposition 6.2. ($A_0(R)$ is compact in $\mathcal{C}(\partial H) \cup \partial\mathcal{C}(\partial H)$.) *Let $(\alpha_n) \subset A_0(R)$ be a sequence of elements of $A_0(R)$. Then there exists a subsequence (α_{n_k}) that either all its elements are the same or their supports converge to a lamination $\mu \in \partial\mathcal{C}(S)$ which has $(R+1)$-bounded combinatorics respect to an element of* $\mathbf{m}_0(\mathrm{H})$.

Proof. Let $(\alpha_n) \subset A_0(R)$ be given and suppose P_n is the pants decomposition associated to α_n for every n. After passing to a subsequence, we can assume that all elements of (α_n) have R-bounded combinatorics with respect to an element $\beta_0 \in \mathbf{m}_0(\mathrm{H})$.

Consider the sequence (P_n) as a sequence in $\mathcal{PML}$ by putting equal measures on components of P_n for every n. If there is a subsequence of (P_n) whose elements are all equal to a single pants decomposition P_0, then we claim that there is a subsequence of (α_n) whose elements are all the same. This is because for every α_n, once we know the pants decomposition $P_n = P_0$ we only have the freedom to choose the transversals. But for each $\gamma \in P_0$, we have only a finite number of choices for the transversal to γ, since $d_\gamma(\alpha_n, \beta_0) \leq R$ and the claim follows.

Now suppose we have extracted a subsequence such that (P_n) converges to a lamination μ in $\mathcal{PML}$. Using Klarreich's description of $\partial\mathcal{C}(\partial H)$ it will be enough to show that μ is filling. Let $Y \subset \partial H$ be the smallest essential subsurface that contains support of μ.

Case 1. Y has a non-annular component $Y' \subsetneq \partial H$. Let μ' be the component of μ that is contained in Y' and notice that μ' fills Y'. We know that the sequence $\pi_{Y'}(P_n)$ also converges to the point associated to μ' in $\partial\mathcal{C}(Y')$. In particular, $d_{Y'}(P_0, P_n) \to \infty$ which contradicts the fact that $d_{Y'}(\alpha_n, \alpha_0) \leq 2R$.

Case 2. All components of Y are annuli. Either infinitely many elements P_n are contained in Y or there exists a subsequence of (P_n) whose elements all intersect a component $Y' \subset Y$. If the former happens, Y is the union of annular neighborhoods of components of a pants decomposition P_0 and by what we explained earlier there is subsequence of (α_n) whose elements are all equal. If the latter happens, similar to the first case, we can see that $d_{Y'}(P_0, P_n) \to \infty$ as $n \to \infty$ and after passing to a subsequence. This is again a contradiction to the fact that α_n has R-bounded combinatorics respect to β_0 for every n.

The fact that every such limit has R-bounded combinatorics respect to β_0 is easy by observing that if α_n converges to $\mu \in \mathcal{C}(\partial H)$ then $d_Y(\mu, \alpha_n) \leq 1$ for $n \gg 0$. $\square$

With an abuse of notation, we denote the subset of $\partial\mathcal{C}(S)$ whose elements are limits of sequences of $A_0(R)$ by $\partial A_0(R)$. For every $\alpha \in A(R)$, we can take an element $\alpha' \in \mathcal{PML}(S)$, supported on the pants decomposition associated to α and equal measures on the components of this pants decomposition. The set of all such points provides a subset $A'(R) \subset \mathcal{PML}(S)$ associated to $A(R)$ and we can also consider the set $A'_0(R) \subset A'(R)$ the subset that corresponds to $A_0(R)$.

Proposition 6.3. *The set $A'_0(R)$ has compact closure inside the Masur domain and the accumulation points are projectivized measure laminations supported on elements of $\partial A(R)$.*

Proof. Suppose $(P_i) \subset A'_0(R)$ converges to μ in $\mathcal{PML}$. Using proposition 6.2, one can see that either the elements P_i are all equal for $i \gg 0$ or μ is filling. If the first case happens, there is nothing more to do since we already know that each P_i is in the Masur domain. Therefore, we can assume that μ is filling and we need to prove that μ is in the Masur domain.

If μ is not in the Masur domain then it has zero intersection with an element $\lambda \in \bar{\Delta}$. Since μ is filling it has to have the same support as λ and λ is also filling and they represent the same point in $\partial\mathcal{C}(\partial H)$ which we call λ. For each P_i choose a component $a_i \in \mathcal{C}_0(\partial H)$ and notice that the sequence (a_i) also converges to μ in $\partial\mathcal{C}(\partial H)$. Also let $(b_i) \subset \Delta$ be a sequence that converges to λ in sense of Gromov. Klarreich's theorem [Kla] shows that the geodesic segments $[\alpha_i, \beta_i]$ connecting α_i and β_i in $\mathcal{C}(\partial H)$ get further and further from x_0, where x_0 is a fixed element point in $\mathcal{C}(\partial H)$. Consider the geodesic triangle with vertices $\{a_i, b_i, x_0\}$; this triangle is δ-thin, where δ is the hyperbolicity constant in theorem 12.3. Since the segment $[a_i, b_i]$ is far from x_0, the other two sides $[x_0, a_i]$ and $[x_0, b_i]$ have to be δ-close on subsegment of length D_i of their initial part. The length D_i is comparable to the distance between x_0 and the segment $[a_i, b_i]$ and in particular $D_i \to \infty$ as $i \to \infty$. Because of quasi-convexity of Δ, theorem 7.11, the segment $[x_0, b_i]$ is in the d-neighborhood of Δ. Therefor

$$d_{\mathcal{C}}(a_i, \Delta) \leq d_{\mathcal{C}}(a_i, [b_i, x_0]) + d \tag{6.1}$$
$$\leq d_{\mathcal{C}}(a_i, x_0) - D_i + \delta + d \tag{6.2}$$

is much shorter than $d_{\mathcal{C}}(x_0, a_i)$ for $i \gg 0$. Therefore for $i \gg 0$, $d_{\mathcal{C}}(a_i, \Delta)$ is shorter than the distance between a_i and every element of $\mathbf{m}_0$(R) as well. But this contradicts our choice of $a_i \in A_0(R)$ whose distance from Δ is realized between a_i and an element of $\mathbf{m}_0$(H). ☐

Every element $\alpha \in A_0(R)$ gives a hyperbolic metric $\tau(\alpha) \in \mathfrak{T}(S)$ and we can consider $\tau(A_0(R))$ to be the set of all such points. In parallel to propositions 6.2 and 6.3 we have:

Proposition 6.4. *The set $\tau(A_0(R))$ has compact closure in Thurston's compactification of Teichmüller space of ∂H and the limit points on the boundary are supported on elements of $\partial A_0(R)$ and are in the Masur domain.*

We can take $B_0(R)$ to be the set of marked convex cocompact hyperbolic structures on H, which are associated to elements of $\tau(A_0(R))$. Notice that, we could start with $A(R)$ and construct the associated convex cocompact structures; but those would be the same as the structures in $B_0(R)$ up to changing the marking.

Theorem 6.5. *The set $B_0(R)$ with an appropriate choice of base points is precompact in the set of marked hyperbolic structures on H with strong topology. The accumulation points are degenerate hyperbolic structures on H, whose ending laminations are in $\partial A_0(R)$.*

Proof. Let $(N_i) \subset B_0(R)$ be a sequence of marked structures on H, where each N_i is obtained from $\tau_i = \tau(\alpha_i) \in \tau(A_0(R))$. Using proposition 6.4, we know that either the sequence (τ_i) stabilizes up to taking a subsequence, where there is nothing to prove, or after taking a subsequence we can assume that (τ_i) converges to λ in Thurston's compactification of Teichmüller space, which we assume is the case. Let $\rho_i : \pi_1(H) \to \mathrm{PSL}_2(\mathbb{C})$ be the representation associated to N_i: N_i is isometric to $\mathbb{H}^3/\rho_i(\pi_1(H))$. Then using Kleineidam and Souto's work [KS02], we know that up to passing to a subsequence and conjugation the sequence (ρ_i) converges algebraically to a representation $\rho : \pi_1(H) \to \mathrm{PSL}_2(\mathbb{C})$.

The recent proof of the Tameness Conjecture (Agol [Ag], Calegari-Gabai [CG] and Brock-Souto [BS]), we know that $N = \mathbb{H}^3/\rho(\pi_1(H))$ is tame and is homeomorphic to the interior of a compact 3-manifold. An elementary argument shows that N has to be homeomorphic to the interior of H (cf. Hempel [He86]).

Claim 6.1. The lamination λ is not realized in N.

Proof. Let a_i be a component of the support of α_i. We know that $l_{\tau_i}(a_i) \leq B_0$, where B_0 is the Bers' constant. Since the sequence (a_i) converges to a Masur domain lamination in $\mathcal{PML}$, for all $A > 0$ there is, by the continuity of the intersection form, some i_A such that $l_{\tau_i}(m) > A$ for all $i \geq i_A$ and all meridians m. Then by a theorem due to Canary [Can91], there is $K > 0$ such that for all $i \geq i_A$

$$l_{N_i}(a_i) \leq K l_{\tau_i}(a_i) \leq K B_0 \tag{6.3}$$

where $l_{N_i}(a_i)$ is the length of the geodesic freely homotopic to a_i in N_i.

Let $\bar{\lambda}$ be the Hausdorff-limit of the sequence (a_i). The geodesic lamination $\bar{\lambda}$ contains λ as a sublamination and $\bar{\lambda} \setminus \lambda$ consists of a finite number of biinfinite geodesics. Seeking a contradiction, assume λ is realized in N. Then $\bar{\lambda}$ is also realized in N by work of Otal [Ota88] (cf. [KS03, Lem. 4.2]). This implies that there is a compact set $K \subset N$ such that for all i the curve a_i is freely homotopic in N to a geodesic a_i^* contained in K. In particular we have $l_N(a_i) \asymp l_{N_0}(a_i)$, i.e. there is a constant $c > 1$ with

$$c^{-1} l_N(a_i) \leq l_{N_0}(a_i) \leq c l_N(a_i)$$

for all i.

Since (ρ_i) converges to ρ algebraically, on the level of manifolds there are smooth homotopy equivalences $h_i : N \to N_i$, compatible with ρ_i and ρ, such that on any compact subset of N, h_i tends C^∞ to a local isometry for all $i \gg 0$. (Cf. McMullen [McM].)

For large i, the curves $h_i(a_i^*)$ have small geodesic curvature and we have

$$l_{N_i}(a_i) \asymp l_N(a_i) \asymp l_{N_0}(a_i).$$

But $l_{N_0}(a_i) \to \infty$ as $i \to \infty$ since $\bar{\lambda}$ is realized in N_0 and similar to the arguments of Canary [Can93a], we can see that $l_{N_0}(a_i)$ is comparable to the length of a_i on this pleated surface and has to go to infinity. This contradicts (6.3) and we have proved the claim. □

Once we know that λ is not realized in N, we can use theorem 1.3 and see that $\phi(\lambda)$ is the ending lamination for N, where $\phi \in \mathrm{Mod}_0(H)$.

Next, we want to prove that the convergence $N_i \to N$ is geometric too. We know that we can conjugate the representations ρ_n so that they converge to ρ; we assume from now on that this is the case. Choosing once and forever a point $p_{\mathbb{H}^3}$ in $\mathbb{H}^3$, let p_i and p be their projections of $p_{\mathbb{H}^3}$ to $N_i = \mathbb{H}^3/\rho_i(\pi_1(H))$ and $N = \mathbb{H}^3/\rho(\pi_1(H))$.

The Margulis lemma implies that the injectivity radius of the manifold N_i at the points p_i is uniformly bounded from below. In particular, we obtain from Gromov's compactness theorem that every subsequence of (ρ_i) contains a subsequence (ρ_{i_j}) such that the pointed manifolds $(N_{i_j}, p_{i_j})_j$ converge geometrically to a pointed manifold (N_G, p_G). The relation between geometric convergence of hyperbolic manifolds and convergence in the Chabauty topology of discrete groups of $\mathrm{PSL}_2(\mathbb{C})$ [BP] implies that $N_G = \mathbb{H}^3/\Gamma_G$ where Γ_G

is the group formed by all elements $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ such that there is a sequence (γ_j) in $\pi_1(H)$ with $\gamma = \lim_j \rho_{i_j}(\gamma_j)$. In particular $\rho(\pi_1(H)) \subset \Gamma_G$ and hence the manifold N is a Riemannian cover of N_G. Thurston and Canary's Covering Theorem [Can96] implies that the cover

$$N \to N_G$$

is finitely sheeted. We claim that this covering is trivial, or equivalently that $\Gamma_G = \rho(\pi_1(H))$.

The proof is based on an idea of Thurston. Suppose $\beta \in \Gamma_G$ then $\beta^k \in \rho(\pi_1(H))$ for some $k > 1$ and is equal to $\rho(\gamma)$ for some $\gamma \in \pi_1(H)$. Since $\beta \in \Gamma_G$ there exists a sequence of elements $\alpha_j \in \pi_1(H)$ such that $\beta = \lim_j \rho_{i_j}(\alpha_j)$. So

$$\lim_j \rho_{i_j}(\alpha_j^k) = \beta^k = \rho(\gamma) = \lim_j \rho_{i_j}(\gamma).$$

Because of discreteness and faithfulness of the representations, $\alpha_j^k = \gamma$ for all $j \gg 0$. But in a free group we have at most one k-th root for every element and therefore $\alpha_j = \alpha$ for some fixed $\alpha \in \pi_1(H)$ and $j \gg 0$. Hence $\beta = \lim_j \rho_{i_j}(\alpha) \in \rho(\pi_1(H))$ and we have proved that every subsequence of (N_i, p_i) has a subsequence that converges to (N, p) geometrically. Therefore the entire sequence (N_i, p_i) converges to (N, p) geometrically and the convergence $N_i \to N$ is strong.

Now it only remains to choose a marking $j : \partial H \to N$ for N such that λ is the ending lamination for the marked structure (N, j) and $N_i \to N$ as a sequence of marked structures.

Let $j : \partial H \to N$ be a marking such that λ is the ending lamination for the marked structure (N, j). (This is always possible by precomposing an arbitrary marking with an element of $\mathrm{Mod}_0(H)$.) Also let C be a useful compact core of N and choose a sequence (Q_n) of pants decompositions that converges to a lamination supported on λ in $\mathcal{PML}$. Since λ is the ending lamination, for n sufficiently large, Q_n has a geodesic representative in $N \setminus C$ and these representatives exit the end of N as $n \to \infty$. Hence by using lemma 4.2, we know that for $n \gg 0$, there exists a pleated surface $f_n : \partial H \to N$ that realizes Q_n and is homotopic to j within $N \setminus C$. This sequence of pleated surfaces has to exit the end of N because of lemma 2.12 (Bounded diameter lemma). (Notice that in both these, we are using the fact that the function $d_N^{\geq \epsilon}(\cdot, p)$ is proper.) We assume n is sufficiently large such that $d_N^{\geq \epsilon}(f_n(\partial H), C) \geq D + 1$ where D is the constant obtained in corollary 4.6.

Fix n large and let $\kappa_i : (N, p) \to (N_i, p_i)$ be the approximating maps for the geometric convergence $N_i \to N$. It is not hard to see that $C_i = \kappa_i(C)$ is a useful marking for N_i for $i \gg 0$. Let $j_i : \partial H \to N_i \setminus C_i$ be a representative of the marking of N_i. We know that κ_i approaches an isometry on every compact subset of N. Using this we can show that for i sufficiently large depending on n, there exists $h_{i,n} \in \mathbf{pleat}_{N_i}$ that realizes b_n in N_i, is ϵ-close to $\kappa_i \circ f_n$ and

$$d^{\geq\epsilon}_{\bar{N}_i}(h_{i,n}(\partial H), C_i) \geq D. \tag{6.4}$$

It will be enough to show that $\kappa_i \circ j$ or equivalently $\kappa_i \circ f_n = h_{i,n}$ is homotopic to j_i in $N_i \setminus \kappa_i(C)$ for i sufficiently large.

Suppose $g_i : \partial H \to N_i$ parametrizes $\partial\mathcal{CH}(N_i)$, the boundary of the compact core of N_i, and g_i is homotopic to j_i in $N_i \setminus C_i$. Let P_i be the pants decomposition associated to α_i. By our construction, we know that P_i has length bounded by B_0 in τ_i and we know that the injectivity radius of τ_i is bounded from below by ϵ_0. Hence, we can use Bridgeman-Canary's theorm 2.11 and conclude that P_i has length at most JB_0 on σ_{g_i}, where J depends only on ϵ_0.

We obviously know that $d_{N_i}(\partial\mathcal{CH}(N_i), C_i) \to \infty$ as $i \to \infty$. Therefore for $i \gg 0$, if there exists $g' \in \mathbf{pleat}_{N_i}$ that is end-homotopic, realizes P_i in N_i and

$$g'(\partial H) \cap \mathcal{N}_D(C_i) = \emptyset$$

where D is the constant obtained in lemma 4.4 (Homotopy bound) for $A = JB_0$ and ϵ small, then it follows that there is a (K, ϵ) good homotopy with respect to P_i between g_i and g' that stays away from C_i.

Consider the geodesic that connects Q_n and P_i in $\mathcal{C}(\partial H)$. As in lemma 2.5, we can extend this to an elementary-move sequence

$$Q_n = P^i_0 \to P^i_1 \to \cdots \to P^i_{m_i} = P_i.$$

Using corollary 4.6, since $h_{i,n}$ is end-homotopic realizes Q_n and satisfies (6.4), we know that either

(1) there exists $g' \in \mathbf{pleat}_{N_i}$ that realizes P_i and is homotopic to $h_{i,n}$ within $N_i \setminus C_i$ or

(2) there exists a pleated surface $g'' \in \mathbf{pleat}_{N_i}$ homotopic to $h_{i,n}$ within $N_i \setminus C_i$ realizing P^i_k for some $0 \leq k \leq m_i$ and with

$$d^{\geq\epsilon}_{\bar{N}_i}(g''(\partial H), C_i) \leq D.$$

If case (1) happens, by what we said earlier, g' is homotopic to g_i within $N_i \setminus C_i$ and therefore $h_{i,n}$ is homotopic to g_i within $N \setminus C_i$ and finally $h_{i,n}$ is homotopic to j_i within $N \setminus C_i$. This is what we wanted and if this happens for all sufficiently large i then we are done. So, we seek a contradiction if this is not the case for every n. Therefore suppose there exists $i(n) \geq n$ such that the case (2) happens. Then we get a pleated surface $g''_n \in \mathbf{pleat}_{N_{i(n)}}$ that realizes an element γ_n on the $\mathcal{C}(\partial H)$-geodesic path between Q_n and $P_{i(n)}$, is homotopic to $h_{i(n),n}$ and equivalently to $\kappa_{i(n)} \circ j$ within $N_i \setminus C_i$ and

$$d^{\geq \epsilon}_{N_{i(n)}}(g''_n(\partial H), C_{i(n)}) \leq D.$$

If K is a compact subset of N and i sufficiently large,

$$d^{\geq \epsilon}_{N}(x, C) - 1 \leq d^{\geq \epsilon}_{N_i}(\kappa_i(x), C_i) \leq d^{\geq \epsilon}_{N}(x, C) + 1.$$

Using this together with lemma 2.12 (Bounded diameter lemma), we can see that $g''_n(\partial H)$ is contained in a bounded neighborhood of $p_{i(n)}$ in $N_{i(n)}$ independently of n. Therefore, $\kappa^{-1}_{i(n)} \circ g''_n$ is ϵ close to a pleated surface f'_n that realizes γ_n in N and intersects a compact subset of N.

When $n \to \infty$ by definition $i(n) \to \infty$. Therefore the sequences (Q_n) and $P_{i(n)}$ both converge to elements supported on λ in $\mathcal{PML}$. Since λ is in $\partial\mathcal{C}(\partial H)$ and γ_n is on the geodesic connecting Q_n and $P_{i(n)}$, it follows from Klareich's theorem that the sequence (γ_n) converges to λ as well. But since f'_n realizes γ_n and intersects a compact subset of N, it follows from Kleineidam-Souto [KS03, Prop. ?] that λ is realized in N and we have a contradiction with claim 6.1.

□

Remark 6.2. Notice that in the above statement, we have chosen appropriate base point p_N for every $N \in B_0(N)$ and from now on, whenever we speak of N we consider it as a pointed manifold (N, p_n).

Again, with an abuse of notation, we denote the set of all degenerate hyperbeolic structures, which are limits of sequences in $B_0(R)$ to be $\partial B_0(R)$. Apparently $\partial B_0(R)$ is also compact in the set of marked structures on H with strong topology. To simplify our notations we denote the set $B_0(R) \cup \partial B_0(R)$ by $\mathcal{B}_0(R)$ and the set $A_0(R) \cup \partial A_0(R)$ bu $\mathcal{A}_0(R)$.

Since elements of $\mathcal{B}_0(R)$ are marked structures on H, the conformal structure at infinity or the ending lamination are defined uniquely on $\mathfrak{T}(\partial H)$ or

$\partial\mathcal{C}(\partial H)$. Hence we have a map

$$\mathcal{E} : \mathcal{B}_0(R) \to \mathcal{A}_0(R), \tag{6.5}$$

and we call $\mathcal{E}(N)$ the *end invariant* of N.

7 Quasiconvexity

Let H be a handlebody as before and $\mathcal{B}_0(R)$ the set of marked hyperbolic structures on H introduced in the last section. In our discussions, we usually consider an element $N \in \mathcal{B}_0(R)$ to be the interior of H equipped with a complete hyperbolic metric and we assume the marking $j : \partial H \to N$ is simply an embedding isotopic to the inclusion $\partial H \hookrightarrow H$. Therefore, we use the same marking j for all structures in $\mathcal{B}_0(R)$; yet, because j is defined up to isotopy, we feel free to isotope j whenever is needed. In particular, when we make a choice of a compact core, we always assume that $j(\partial H)$ and the isotopy between j and $\partial H \hookrightarrow H$ stays away the compact core.

We also assume that we have fixed a choice of Γ in H that satisfies proposition 4.1. Then for every $N \in \mathcal{B}_0(R)$, we take Γ_N to be the geodesic representative of $j(\Gamma)$ in N. (We are using the fact that elements of $\mathcal{B}_0(R)$ have no parabolics.) From now on, when we speak of the diskbusting geodesic for $N \in \mathcal{B}_0(R)$, we are referring to Γ_N. The next proposition follows easily from compactness of $\mathcal{B}_0(R)$ in the strong topology.

Proposition 7.1. (Uniform compact core) *There exists a constant $d_0 > 1$ such that for every $N \in \mathcal{B}_0(R)$.*

(1) *the diskbusting geodesic Γ_N has total length at most d_0 and is contained in the d_0-neighborhood of the base point p_N and*

(2) *there exists a compact core $C \subset N$ homeomorphic to H that contains a 1-neighborhood of Γ_N and is contained in the $(d_0 - 1)$-neighborhood of Γ_N.*

For now on, we fix a choice of d_0 that satisfies the above proposition and is bigger than the constant D_0 in lemma 3.3 and for $N \in \mathcal{B}_0(R)$ we always assume that a useful compact core is one that satisfies the second part of the above proposition and in particular has uniformly bounded diameter. We also define the set $\overline{\mathbf{pleat}}_N$ to be the set of pleated surfaces $f : \partial H \to N$ homotopic to j within $N \setminus C$ for some useful compact core $C \subset N$ and with $d_N(f(\partial H), \Gamma_N) \geq d_0$. Notice that these pleated surfaces satisfy the conclusion of lemma 3.3 by our assumption about d_0:

$$f((\partial H)^{\geq \epsilon}) \subset N^{\geq \delta} \tag{7.1}$$

for every $f \in \overline{\mathbf{pleat}}_N$ and where $\delta = \delta(\epsilon, \chi(\partial H))$.

If α is a multi-curve on ∂H and $N \in \mathcal{B}_0(R)$, by a *(geodesic) representative for α in $N \setminus \Gamma_N$*, we mean a closed (geodesic) curve freely homotopic within $N \setminus C$ to $j(\alpha)$ for some compact core C that contains Γ_N.

The purpose of this section is prove a result in parallel to Minsky's [Min01, Thm. 3.1] that shows for every $N \in \mathcal{B}_0(R)$ and B bigger than the Bers' constant, the set

$$\mathcal{C}(B, N) := \bigcup_{f \in \overline{\mathbf{pleat}}_N} \mathbf{short}(f, B)$$

is L-quasiconvex for some constant L that depends only on R and $\chi(\partial H)$.

When N is geometrically infinite, it follows from the definition and description of the ending lamination of N in Canary's work [Can93b] that there exists a subsequence of $\mathcal{C}(B, N)$ which converges to $\mathcal{E}(N) \in \partial\mathcal{C}(\partial H)$.

By $\overline{\mathbf{pleat}}_N(\mu)$, we denote the set of pleated surfaces in $\overline{\mathbf{pleat}}_N(\mu)$ whose pleating locus contains μ. If $f \in \overline{\mathbf{pleat}}_N(\mu)$ is given, we say f *realizes* μ in $N \setminus \Gamma_N$. Similar to our definition in section 3, we define $\overline{\mathbf{pleat}}_N^{<D}$ to be the subset of $\overline{\mathbf{pleat}}_N$ whose elements have distance less than D from Γ_N.

Lemma 7.2. *There exists a monotonic function $\rho : [0, \infty) \to [0, \infty)$ such that $d_N(x, p_N) \leq \rho(d_N^{\geq\epsilon}(x, p_N))$ for every $N \in \mathcal{B}_0(R)$ and $\epsilon \leq \epsilon_0$.*

Proof. It will be enough to show that for every $a > 0$ there exists $b > 0$ such that if $d_N^{\geq\epsilon}(x, p_N) \leq a$ then $d_N(x, p_N) \leq b$ for every $N \in \mathcal{B}_0(N)$, $x \in N$ and $\epsilon \leq \epsilon_0$. Also note that it is enough to prove it for $\epsilon = \epsilon_0$.

Suppose, this is not the case and we have a sequence of counter examples (N_i, x_i) such that $N_i \in \mathcal{B}_0(R)$ and $d_{N_i}^{\geq\epsilon}(x_i, p_{N_i}) \leq a$ but $d_{N_i}(x_i, p_{N_i}) \to \infty$. We can assume that the sequence (N_i) converges strongly to $N \in \mathcal{B}_0(R)$ and suppose

$$\kappa_i : (N, p_N) \to (N_i, p_{N_i})$$

are the approximating maps. In N, the function $d_N^{\geq\epsilon}(\cdot, p_N)$ is proper. This is because each component of $N^{<\epsilon}$ is compact and they are uniformly separated. Therefore, there exists a compact set $K \subset N$ such that for every $x \notin K$, $d_N^{\geq\epsilon}(x, p_N) \geq a+1$. When i is sufficiently large κ_i is very close to an isometry and therefore the injectivity radius of $x \in K$ and $\kappa_i(x)$ are extremely close. Take $x \in \partial K$ and a path P_i between $\kappa_i(x)$ and p_{N_i}. Then except for a subset of very small length which is contained in a small neighborhood of boundary of ϵ-thin components every point $y \in N^{\geq\epsilon} \cap \kappa_i^{-1}(P)$ maps to $N_i^{\geq\epsilon}$ and therefore length of $P \cap N_i^{\geq\epsilon}$ is $\geq a + 1/2$. This shows that

$$d_{N_i}^{\geq\epsilon}(x, p_{N_i}) \geq a + 1/2$$

for every $x \notin \kappa_i(K)$ and we have a contradiction. □

Note that $\rho(x) \geq x$ for every $x \in \overline{[0, \infty)}$. This result helps us to get an actual diameter bound for elements of $\overline{\mathbf{pleat}}_N$ close to Γ_N.

Lemma 7.3. (Uniform bounded diameter lemma) *For every $D > 0$, there exists $L(D) > 0$ such that $\operatorname{diam}_N(f(\partial H)) \leq L(D)$ for every $f \in \overline{\mathbf{pleat}}_N^{<D}$ and $N \in \mathcal{B}_0(R)$.*

Proof. The result is immediate by knowing lemma 2.12 (Bounded diameter lemma), lemma 7.2 and noticing that length of compressible curves in σ_f is at least d_0 because of lemma 3.2.

□

We can also translate our results in lemmas 4.2 and 4.3 by replacing the distance in the thick part of manifolds to the actual distance. Note that we also use the fact that we have a bounded diameter useful compact core for every element of $\mathcal{B}_0(R)$.

Lemma 7.4. *Given $d > 0$ there exists a constant $D_1 > 0$ depending only on d, R and $\chi(\partial H)$ such that if $N \in \mathcal{B}_0(R)$ and if α is a simple closed curve on ∂H with a geodesic representative α^* in $N \setminus \Gamma_N$ that*

$$d_N(\alpha^*, \Gamma_N) \geq D_1,$$

then $\overline{\mathbf{pleat}}_N(\mu)$ is nonempty and

$$d_N(f(\partial H), \Gamma_N) \geq d$$

for every $f \in \overline{\mathbf{pleat}}_N(\mu)$, where μ is any finite leaved lamination that contains α.

Theorem 7.5. *Given d there exists $D_2 > d$ and $A > 0$ depending only on d, R and $\chi(\partial H)$ such that the following holds. If $N \in \mathcal{B}_0(R)$ and α has a geodesic representative α^* in $N \setminus \Gamma_N$ with $d_N(\alpha^*, \Gamma_N) \geq D_2$, then for every $\beta \in \mathcal{C}_0(S)$ with $d_{\mathcal{C}}(\alpha, \beta) \leq 1$:*

(a) $\overline{\mathbf{pleat}}_N(\beta) \neq \emptyset$,

(b) $\overline{\mathbf{pleat}}_N(\alpha) \cap \overline{\mathbf{pleat}}_N(\beta) \neq \emptyset$

(c) *every $f \in \mathbf{pleat}_N(\beta)$ has $d^(_N f(\partial H), C) \geq d$*

(d) f *and* $g \in \overline{\mathbf{pleat}}_N(\beta)$*, the set*

$$\mathbf{short}(f, B) \cup \mathbf{short}(g, B)$$

has diameter bounded by A *in* $\mathcal{C}(\partial H)$*.*

Next, we need to have some control over the pleated maps that are nearby the diskbusting geodesic.

Lemma 7.6. *For every* D *there exists* K *such that for every two pleated surfaces*

$$f \in \overline{\mathbf{pleat}}_N^{<D} \text{ and } g \in \overline{\mathbf{pleat}}_M^{<D}$$

the induced metrics σ_f *and* σ_g *on* ∂H *are* K*-bi-Lipschitz up to isotopy for every* $M, N \in \mathcal{B}_0(R)$*.*

Proof. Suppose we have a sequence of counter examples $(N_i, M_i, f_i, g_i)_{i\geq 1}$ that satisfy the hypothesis. We can assume that the sequences (N_i) and (M_i) converge in the geometric topology to N and $M \in \mathcal{B}_0(R)$ respectively. Because of lemma 7.3, the pleated surfaces $f_i(\partial H)$ and $g_i(\partial H)$ are contained in a bounded neighborhood of the base points and therefore they also converge to pleated surfaces f and g in N and M.

Suppose $\kappa_i : N \to N_i$ are the approximating maps for the convergent sequence (N_i). We claim that the sequence (f_i) is convergent as a sequence of *marked pleated surfaces.* (For a description of different types of convergence for pleated surfaces, see Canary-Epstein-Green [CEG87].) In our situation it means that in addition to the fact that $f_i(\partial H)$ converges to $f(\partial H)$ in the geometrically, $\kappa_i \circ f$ is very close to $f_i \circ \phi_i$ as a map for $i \gg 0$, where ϕ_i is a self-homeomorphism of ∂H whose isotopy class does not depend on i.

Let C be a useful compact core for N. Then it easily follows that f is π_1-injective into $N \setminus C$. Otherwise image of a compressing disk would give a compressing disk for $f_i(\partial H)$ in $N_i \setminus \Gamma_{N_i}$. Hence there exists a homotopy between $f \circ \psi$ and j within $N \setminus C$ for some ψ a self-homeomorphism of ∂H. By applying κ_i on these, we get a homotopy between $\kappa_i \circ f \circ \psi$ and $\kappa_i \circ j$ within $N_i \setminus \Gamma_{N_i}$. We know that $\kappa_i \circ j$ is isotopic to j and therefore they are homotopic outside a useful compact core and in the complement of Γ_{N_i} (the convergence $N_i \to N$ was in the strong topology for marked structures). On the other hand, $\kappa_i \circ f(\partial H)$ is extremely close to $f_i(\partial H)$ and they have distance at least 1 from Γ_N for $i \gg 0$. Therefore, there is a homotopy between $f_i \circ \phi_i$ and $\kappa_i \circ f$ for a self-homeomorphism ϕ_i of ∂H (within $N_i \setminus \Gamma_{N_i}$). These show that

$$j \sim \kappa_i \circ j, \quad \kappa_i \circ j \sim \kappa_i \circ f \circ \psi, \quad \kappa_i \circ f \circ \psi \sim f_i \circ \phi_i \circ \psi,$$

where $\sim$ denotes homotopy and all the above homotopies take place in $N_i \setminus \Gamma_{N_i}$. But j and f_i were homotopic and π_1-injective therefore $\phi_i \circ \psi$ is isotopic to identity and we have proved the claim. We assume that we have replaced f with $f \circ \psi$ and then the sequence (f_i) converges as a sequence of marked pleated surfaces to f. Once this is the case, one can easily see that the metrics induced by f and f_i are very close up to isotopy for $i \gg 0$. This is because $\kappa_i \circ f$ and $f_i \circ \phi_i$ are very close as maps, where ϕ_i is isotopic to identity and also that κ_i is very close to an isometry for $i \gg 0$.

The same argument shows that after possibly precomposing g with a homeomorphism of ∂H, $g_i \to g$ as a sequence of marked pleated surfaces. Then, we know that the metrics induced by g_i and g are 1-bi-Lipschitz up to isotopy for $i \gg 0$. But the metrics induced by f and g are bi-Lipschitz and this shows that the metrics induced by f_i and g_i (up to isotopy) are bi-Lipschitz with a bounded bi-Lipschitz constant. □

Corollary 7.7. *Suppose $\mathcal{B}_0(R)$ is as before. For every D and $B > 0$ there exists a finite set $A_{D,B} \subset \mathcal{C}_0(\partial H)$ such that if $\alpha \in$ **short**(f, B) for some $N \in \mathcal{B}_0(R)$ and some $f \in \overline{\mathbf{pleat}}_N^{<D}$, then $\alpha \in A_{D,B}$.*

Proof. This is immediate after lemma 7.6. Choose a fixed pleated surface $g \in \overline{\mathbf{pleat}}_M^{<D}$ for some $M \in \mathcal{B}_0(R)$. Now if f and α are as in the hypothesis, since the metrics induced by f and g are K-bi-Lipschitz, we know that

$$l_{\sigma_g}(\alpha) \leq KB.$$

But for fixed g, there is only a finite set of closed curves whose length do not exceed KB and we are done. □

In the next lemma, we show that there are pleated surfaces in $\overline{\mathbf{pleat}}_N$ within a uniformly bounded distance from Γ_N. As a matter of fact, if $d_N(\partial\mathcal{CH}(N), \Gamma_N)$ is small, it is false. But in such a case $\mathcal{CH}(N)$ will have bounded diameter and most of the things that we need become trivial. In particular, using the above corollary, we can see that $\mathcal{C}(B, N)$ is finite depending on the diameter of $\mathcal{CH}(N)$ and theorem 7.11 (Quasiconvexity) is obvious. Hence, in all our discussions, we assume $d_N(\partial\mathcal{CH}(N), \Gamma_N)$ is uniformly large, when appropriate.

Lemma 7.8. *If D is large enough independently of N, then $\overline{\mathbf{pleat}}_N^{<D}$ is nonempty for every $N \in \mathcal{B}_0(R)$. Even more, suppose $d > 0$ is given then if D is sufficiently large, there exists one whose distance from Γ_N is at least d.*

Proof. Again the idea of the proof is taking a geometric limit. Suppose $(N_i) \subset \mathcal{B}_0(R)$ is a sequence such that for every i, every $f \in \overline{\mathbf{pleat}}_N$ has distance at least i from Γ_{N_i} or $d_N(f(\partial H), \Gamma_N) \leq d$. After extracting a subsequence, which we still call (N_i), we can assume (N_i) converges strongly to $N \in \mathcal{B}_0(R)$. In N, take a pleated surface $f \in \overline{\mathbf{pleat}}_N$ that has distance at least $d+1$ from Γ_N. If we use the approximating maps to push $f(\partial H)$ to N_i, the image has to be close to a pleated surface for $i \gg 0$ with distance more than d from Γ_{N_i}. The obtained pleated surface has to be in $\overline{\mathbf{pleat}}_N$ and we have a contradiction. ☐

Fix a constant D_1 that satisfies lemma 7.4 for $d = d_0$ and let D_2 be the constant obtained in theorem 7.5 for $d = D_1$ and let $\eta > 0$ be a lower bound for the injectivity radius in the D_1-neighborhood of Γ_N for every N. Finally fix D_3 to be large enough to satisfy the conclusion of the above lemma and be bigger than

$$\max\{D_2, \cosh^{-1}(\frac{B}{\eta}) + B + D_1\}.$$

Now we define a projection from $\mathcal{C}(\partial H)$ to $\mathcal{C}(B, N)$ as follows:

$$\Pi_N(\alpha, B) := \bigcup_{f \in \overline{\mathbf{pleat}}_N(\alpha)} \mathbf{short}(f, B),$$

if α has a geodesic representative α^* in $N \setminus \Gamma_N$ with $d_N(\alpha^*, \Gamma_N) \geq D_1$ and

$$\Pi_N(\alpha, B) := A_{D_3,B} \cap \mathcal{C}(B, N),$$

otherwise.

The first part of the above definition always gives nonempty projections, since $\overline{\mathbf{pleat}}_N(\alpha)$ is nonempty by lemma 4.2 and B is bigger than the Bers' constant. Also because of lemma 7.8 and our assumption about D_3, the second part gives nonempty projections as well.

Similar to Minsky [Min01], we can prove that Π is a coarse Lipschitz projection:

Proposition 7.9. (Coarse Projection) *There exists $c > 0$ depending only on $\chi(\partial H)$, R and B such that*

(i) (Coarse idempotence) *If $\alpha \in \mathcal{C}(B, N)$ then $\alpha \in \Pi_N(\alpha, B)$.*

(ii) (Coarse Lipschitz) *For* α *and* $\beta \in \mathcal{C}_0(\partial H)$ *with* $d_{\mathcal{C}}(\alpha, \beta) \leq 1$*,*

$$\operatorname{diam}_{\mathcal{C}}(\Pi_N(\alpha, B) \cup \Pi_N(\beta, B)) \leq c.$$

Proof. Proof of part (i) is easy. Notice that there is always a useful compact core within distance d_0 of Γ_N. If α has a geodesic representative α^* with $d_N(\Gamma_N, \alpha^*) \geq D_1$, by lemma 7.4 there exists $f \in \overline{\mathbf{pleat}}_N$ that realizes α and then

$$\alpha \in \mathbf{short}(f, B) \subset \Pi_N(\alpha, B).$$

If not assume $\Gamma_N \in \mathbf{short}(f, B)$, then either $f(\alpha)$ is compressible and by lemma 3.2 $f(\partial H)$ has distance at most $B < D_3$ from Γ_N or $f(\alpha)$ is incompressible and α^*, the geodesic representative of α, has distance $\leq D_1$ from Γ_N. Then by lemma 2.10,

$$\begin{aligned} d_N(f(\partial H), \Gamma_N) &\leq d_N(f(\alpha), \alpha^*) + B + D_1 \\ &\leq \cosh^{-1}\left(\frac{l_{\sigma_f}(\alpha)}{l_N(\alpha^*)}\right) + B + D_1 \\ &\leq \cosh^{-1}\left(\frac{B}{\eta}\right) + B + D_1 \\ &\leq D_3. \end{aligned}$$

In either case, $d_N(f(\partial H), \Gamma_N) \leq D_3$ and therefore

$$\alpha \in \mathbf{short}(f, B) \subset A_{D_3, B} = \Pi_N(\alpha, B).$$

For part (ii), first suppose that either α or β, say α, has a geodesic representative with distance $> D_2$ of Γ_N. By theorem 7.5 and our assumption that $d = D_1$, we know that β has a geodesic representative with distance more than D_1 from Γ_N. Therefore we have used the first definition for projection of both α and β. Statements (b) and (d) of theorem 7.5 imply that

$$\mathbf{short}(f, B) \cup \mathbf{short}(g, B)$$

has diameter bounded by $2A$ in $\mathcal{C}(\partial H)$ for every $f \in \overline{\mathbf{pleat}}_N(\alpha)$ and $g \in \overline{\mathbf{pleat}}_N(\beta)$. Hence

$$\operatorname{diam}_{\mathcal{C}}(\Pi_N(\alpha, B) \cup \Pi_N(\beta, B)) \leq 2A.$$

On the other hand suppose neither α nor β have geodesic representatives with distance $> D_2$ of Γ_N. We claim that $\Pi_N(\alpha)$ and $\Pi_N(\beta)$ are both included in $A_{D_3,B}$ and therefore their union has diameter bounded by diameter of $A_{D_3,B}$.

If α does not have a geodesic representative with distance $\geq D_1$ of Γ_N then the claim for α follows by definition of $\Pi_N(\alpha, B)$. If not then α has a geodesic representative α^* with

$$D_1 \leq d_N(\alpha^*, \Gamma_N) \leq D_2.$$

In particular, every $f \in \overline{\mathbf{pleat}}_N(\alpha)$ has distance $\leq D_2$ from Γ_N. Then

$$\mathbf{short}(f, B) \subset A_{D_2,B} \subset A_{D_3,B}$$

by corollary 7.7 and therefore

$$\Pi_N(\alpha, B) \subset A_{D_3,B},$$

and we have proved our claim for α. The same argument proves the claim for β and finishes proof of part (ii) by setting

$$c = \max\{\mathrm{diam}_{\mathcal{C}}(A_{D_3,B}), 2A\}.$$

□

Lemma 7.10. (Minsky [Min01, Lem. 3.3]) *Let X be a δ-hyperbolic geodesic metric space and $Y \subset X$ a subset admitting a map $\Pi : X \to Y$ which is coarse-Lipschitz and coarse-idempotent. That is, there exists $C' > 0$ such that*

- *If $d(x, x') \leq 1$ then $d(\Pi(x), \Pi(x')) \leq C'$, and*
- *If $y \in Y$ then $d(y, \Pi(y)) \leq C'$. Then Y is K-quasi-convex, and furthermore if g is a geodesic in X whose endpoints are within distance a of Y then*

  $$d(x, \Pi(x)) \leq b$$

  for some $b = b(a, \delta, C')$, and every $x \in g$.

Similar to Minsky's [Min01], this proves:

Theorem 7.11. (Quasi-convexity Theorem) *There exists L depending only on R and $\chi(\partial H)$ such that for every B bigger than the Bers' constant B_0 and $N \in \mathcal{B}_0(R)$, the set*

$$\mathcal{C}(B, N) := \bigcup_{f \in \overline{\mathbf{pleat}}_N} \mathbf{short}(f, B),$$

is L-quasi-convex. Moreover, if β is a geodesic in $\mathcal{C}(\partial H)$ with endpoints in $\mathcal{C}(B, N)$ then $d_{\mathcal{C}}(x, \Pi_N(x, B)) \leq L$ for each $x \in \beta$.

8 Bounded geometry

Here in this section, we prove the following theorem:

Theorem 8.1. (Bounded geometry) *There exists $\eta > 0$ depending only on R and $\chi(\partial H)$ such that the injectivity radius of every hyperbolic structure in $\mathcal{B}_0(R)$ is bounded below by η.*

The proof is the same as Minsky's proof of the main theorem in [Min01]. We will discuss the differences in our setting. We can use lemma 7.2 and translate lemmas 4.4 (Homotopy bound) and 4.5 (Halfway surface) and corollary 4.6 (Interpolation) into our setting and in particular we have:

Corollary 8.2. (Interpolation) *Given $\epsilon > 0$ there exists $D > 0$ and $K > 0$ depending on ϵ, R and $\chi(\partial H)$ such that for a hyperbolic structure $N \in \mathcal{B}_0(R)$ and a useful compact core C the following holds. Let $P_0 \to P_1 \to \cdots \to P_n$ be an elementary-move sequence of pants decompositions on ∂H and let $f_0 \in \overline{\mathbf{pleat}}_N(P_0)$ with*

$$d_N(f_0(\partial H), \Gamma_N) \geq D$$

then either

1. *there exists $F : \partial H \times [0, n] \to N \setminus C$ such that*
   - $F_0 = f_0$,
   - $F_i = F|_{\partial H \times \{i\}} \in \overline{\mathbf{pleat}}_N(P_i)$,
   - $F_{i-1/2} = F_{\partial H \times \{i-1/2\}} \in \overline{\mathbf{pleat}}_N(P_{i-1}) \cap \overline{\mathbf{pleat}}_N(P_i)$ *and*
   - *F is a (K, ϵ)-good homotopy restricted to $\partial H \times [i-1, i-\frac{1}{2}]$ and $\partial H \times [i - \frac{1}{2}, i]$*

   for every $i = 1, \ldots, n$ or

2. *there exists $F : \partial H \times [0, k] \to N$ for some $0 \leq k \leq n$ such that*
   - $F_0 = f_0$,
   - $F_i = F|_{\partial H \times \{i\}} \in \overline{\mathbf{pleat}}_N(P_i)$,
   - $F_{i-1/2} = F_{\partial H \times \{i-1/2\}} \in \overline{\mathbf{pleat}}_N(P_{i-1}) \cap \overline{\mathbf{pleat}}_N(P_i)$,
   - *F is a (K, ϵ)-good homotopy restricted to $\partial H \times [i-1, i-\frac{1}{2}]$ and $\partial H \times [i - \frac{1}{2}, i]$ and*
   - $d_N(F_k(\partial H), \Gamma_N) < D$

   for every $i = 1, \ldots, k$.

8.1 The resolution sequence

Let $P_0 \to P_1 \to \cdots \to P_n$ be an elementary-move sequence and $\beta \in \mathcal{C}_0(\partial H)$, we denote

$$J_\beta := \{i \in [0, n] : \beta \in P_i\}.$$

Note that if J_β is an interval $[k, l]$, then the elementary move $P_{k-1} \to P_k$ exchanges some α for β and $P_l \to P_{l+1}$ exchanges β for some α', and we call them *predecessor* and *successor* of β, respectively.

We also use the notation

$$J_{[s,t]} := \bigcup_{i=s}^{t} J_{\beta_i},$$

where $\beta_0, \ldots, \beta_m$ is a sequence of vertices in $\mathcal{C}(\partial H)$. The following theorem is a consequence of work of Masur-Minsky [MM00].

Theorem 8.3. (Controlled Resolution Sequences) *[Min01, Thm. 5.1] Let P and Q be pants decompositions in ∂H. There exists a geodesic in $\mathcal{C}_1(\partial H)$ with vertex sequence $\beta_0, \ldots, \beta_m$, and an elementary move sequence $P_0 \to \cdots \to P_n$, with the following properties:*

1. *$\beta_0 \in P_0 = P$ and $\beta_m \in P_n = Q$.*
2. *Each P_j contains some β_i.*
3. *J_β, if nonempty, is always an interval, and if $[s, t] \subset [0, m]$ then*

   $$|J_{[s,t]}| \leq b(t - s) \sup_Y d_Y(P, Q)^a,$$

   where the suprimum is over only those non-annular subsurfaces Y whose boundary curves are components of some P_k with $k \in J_{[s,t]}$.

4. *If β is a curve with non-empty J_β, then its predecessor and successor curves α and α' satisfy*

   $$|d_\beta(\alpha, \alpha') - d_\beta(P, Q)| \leq \delta.$$

The constants a, b, δ depend only on $\chi(\partial H)$. The expression $|J|$ for an interval J denotes its diameter.

Let γ be a closed curve on ∂H such that $l_N(\gamma)$ is very small. We will try to bound the diameter of the Margulis tube $\mathbf{T}_\gamma(\epsilon_0)$ depending only on R and $\chi(\partial H)$ and if we are successful, we have proved theorem 8.1.

8.2 Initial pants

We know that $N \in \mathcal{B}_0(R)$ is associated to some $\alpha \in \mathcal{A}_0(R)$ and there exists some $\beta \in \mathbf{m}_0(\mathrm{H})$ such that α and β have $(R+1)$-bounded combinatorics respect to each other. Let P_- be the pants decomposition of β. Using lemma 7.6, we can immediately conclude that

Fact 8.4. *Given D there exists $B_1 > 0$ such that $l_{\sigma_f}(P_-) \leq B_1$ whenever $f \in \overline{\mathbf{pleat}}_N^{<D}$.*

If N is convex cocompact (equivalently α is a marking), let P_+ be the pants decomposition of α. Then by using theorem 2.11, we know that

$$l_{\sigma_g}(P_+) \leq JB_0$$

for some J depending only on ϵ_0, where $g \in \overline{\mathbf{pleat}}_N$ parametrizes $\partial\mathcal{CH}(N)$. By assuming $d_N(\partial\mathcal{CH}(N), \Gamma_N)$ is large, we can make sure that $\overline{\mathbf{pleat}}_N(P_+) \neq \emptyset$. Choose some $f_+ \in \mathbf{pleat}_N(P_+)$; we have

$$f_+, g \in \mathbf{good}_N(P_+, JB_0).$$

Then because of lemma 4.4 (Homotopy bound), there is a (K, ϵ_0)-good homotopy between f_+ and g which stays away from a useful compact core of N. This is in fact a $(K, 0)$-good homotopy since σ_g is ϵ_0-thick. Therefore the region enclosed between f_+ and g has uniform bounded diameter and the intersection of $\mathbf{T}_\gamma(\epsilon_0)$ and this region will have uniformly bounded diameter too. (Notice that again in all this we assume $\partial\mathcal{CH}(N)$ is uniformly far from Γ_N whenever appropriate; otherwise $\mathcal{CH}(N)$ will have bounded diameter and since we have a compact family of structures the lower bound for the injectivity radius of these cases is obvious.)

If N is geometrically infinite, let $Q_1, Q_2, \ldots$ be a sequence of pants decompositions which converge to a measured lamination supported on α. If i is sufficiently large $\overline{\mathbf{pleat}}_N(Q_i)$ will be nonempty and every element of $\overline{\mathbf{pleat}}_N(Q_i)$ will be far into the end of N. Choose $P_+ = Q_i$ and $f_+ \in \overline{\mathbf{pleat}}_N P_+$ that $f_+(\partial H)$ encloses a compact subset of N that contains $\mathbf{T}_\gamma(\epsilon_0)$.

8.3 The interpolation

We fix $\epsilon_1 = \delta(\epsilon_0, \chi(\partial H))$ to be the constant obtained in lemma 3.3 and in particular

$$f((\partial H)^{\geq \epsilon_0}) \subset N^{\geq \epsilon_1} \qquad (8.1)$$

whenever $f \in \overline{\mathbf{pleat}}_N$ because of our assumption that $d_N(f(\partial H), \Gamma_N) \geq D_0$ for every $f \in \overline{\mathbf{pleat}}_N$.

Now join P_+ and P_- with a resolution sequence $P_+ = P_0 \to \cdots \to P_n = P_-$ as in theorem 8.3. Then we can use corollary 8.2 for ϵ_1 and obtain a continuous family

$$F : \partial H \to [0, k] \to N$$

such that the second possibility in that corollary occurs. Notice that the first possibility cannot happen here, because P_- consists of a set of meridians and they cannot be realized in N.

Therefore there exists $k \geq 0$ such that

- $F_0 = f_+$,
- $F_i = F|_{\partial H \times \{i\}} \in \overline{\mathbf{pleat}}_N(P_i)$,
- $F_{i-1/2} = F|_{\partial H \times \{i-1/2\}} \in \overline{\mathbf{pleat}}_N(P_{i-1}) \cap \overline{\mathbf{pleat}}_N(P_i)$,
- $F|_{\partial H \times [i-1, i-1/2]}$ and $F|_{\partial H \times [i-1/2, i]}$ are (K, ϵ)-good homotopies within $N \setminus \Gamma_N$ and
- $d_N(F_k(\partial H), \Gamma_N) < D$

for every $i = 1, \ldots, k$, where constants K and D depend only on ϵ_1, R and $\chi(\partial H)$.

By fact 8.4, we know that

$$l_{F_k}(P_-) \leq B_1$$

for a constant B_1 that is independent of choice of $N \in \mathcal{B}_0(R)$. Also note that $F_k(\partial H)$ encloses a compact set of bounded diameter independent of our choice of N or the resolution sequence; therefore image of F covers $\mathbf{T}_\gamma(\epsilon_0)$ with degree 1 except for a set of uniformly bounded diameter and it is enough to show that $\mathbf{T}_\gamma(\epsilon_0) \cap F(\partial H \times [0, k])$ has bounded diameter.

Fix $B = \max\{B_1, JB_0\}$ and let ϵ_2 be such that a K-neighborhood of any ϵ_2-Margulis tube is still contained in an ϵ_1-Margulis tube.

Claim 8.1. There is a subinterval $I_\gamma \subset [0, k]$ of diameter at most $2L$, so that $F(\partial H \times [i, i+1])$ can meet $\mathbf{T}_\gamma(\epsilon_2)$, where L is the quasi-convexity constant of theorem 7.11 and depends only on R and $\chi(\partial H)$.

Proof. Suppose β_i is a component of P_i. If $F_i = F(\partial H \times \{i\})$ meets $\mathbf{T}_\gamma(\epsilon_1)$ then because of (8.1), $l_{\sigma_i}(\gamma) \leq \epsilon_0$ where $\sigma_i = \sigma_{F_i}$ and in particular

$$\gamma \in \mathbf{short}(F_i, B) \subset \Pi_N(\beta_i, B).$$

It follows from theorem 7.11 (Quasiconvexity) that

$$d_{\mathcal{C}}(\beta_i, \gamma) \leq L$$

where L depends only on R and $\chi(\partial H)$. Then because $\{\beta_0, \ldots, \beta_k\}$ are the vertices of a geodesic, the possible values of i lie in an interval of diameter at most $2L$, which we call I_γ and we have $i \in I_\gamma$.

Now notice that because of our choice of ϵ_2 and since $F|_{\partial H \times [i,i+1]}$ is a (K, ϵ_1)-good homotopy, if any track of a block $F(\partial H \times [i, i+1])$ meets $\mathbf{T}_\gamma(\epsilon_2)$ then one of the boundaries must meet $\mathbf{T}_\gamma(\epsilon_1)$ and hence i or $i+1$ is in J_{I_γ}. □

Let us restrict our elementary move sequence to

$$P_{s-1} \to \cdots \to P_{t+1}$$

where $[s, t] = J_{I_\gamma}$ and notice that this subsequence must still encase $\mathbf{T}_\gamma(\epsilon_2)$, since we have thrown away the blocks which avoid $\mathbf{T}_\gamma(\epsilon_2)$. Let $M = t - s = |J_{I_\gamma}|$.

Using part (3) of Theorem 8.3, tells us that

$$M \leq b(2L) \sup_Y d_Y(P_+, P_-)^a,$$

where the suprimum is over subsurfaces Y whose boundaries appear among the P_i in our subsequence. Such P_i must lie in a $L + 1$ neighborhood of γ, in the $d_{\mathcal{C}}$ metric.

It follows from our assumption about R-bounded combinatorics of α and β respect to each other that

$$d_Y(P_+, P_-) \leq R + 2u$$

where u depends only on $\chi(\partial H)$ and $Y \subset \partial H$ is any subsurface. (This is obvious for convex cocompact case and for the geometrically infinite case, we can use Klarreich's theorem in the same way as in Minsky [Min01, Lem. 7.3].) This gives a uniform bound on M.

Suppose γ is not a component of any P_i. Then each block $F|_{\partial H \times [i,i+1]}$ has track lengths of at most $2K$ within $\mathbf{T}_\gamma(\epsilon_2)$. There are only $M+2$ blocks in our restricted sequence and they cover all of $\mathbf{T}_\gamma(\epsilon_2)$, so

$$\mathrm{diam}\mathbf{T}_\gamma(\epsilon_2) \leq 2K(M+2)$$

. This bounds $l_N(\gamma)$ from below, and we are done.

When γ does not appear among the $\{P_i\}$, the argument is exactly the same as Minsky's [Min01] that argument finishes the proof of theorem 8.1 (Bounded geometry).

9 The sweep-out

Here by using the bounded geometry result and similar to our constructions in the last section, we want to give an interpolation that covers almost all of the convex core of $N \in \mathcal{B}_0(R)$ and is more efficient.

Note that, the bounded geometry theorem tells us that length of all incompressible curves is at least 2η. By definition, the pleated surfaces in $\overline{\mathbf{pleat}}_N$ have distance at least 1 from Γ_N and therefore by lemma 3.2, the length of image of compressible curves by elements of $\overline{\mathbf{pleat}}_N$ is at least 1. We always assume η is smaller than 1 and therefore the metric induced by any element of $\overline{\mathbf{pleat}}_N$ is η-thick.

Lemma 9.1. *There exists a constant $k_0 > 0$ such that if $N \in \mathcal{B}_0(R)$ and $f, g \in \overline{\mathbf{pleat}}_N$ are given with $d_N(f(\partial H), g(\partial H)) \geq k_0$, then the Teichmüller distance between the induced metrics is at least 1.*

Proof. If the distance between $f(\partial H)$ and $g(\partial H)$ is more than $2B_0 + 2d_0$ then we know that at least one of them, say $f(\partial H)$, has distance more than $B_0 + d_0$ from Γ_N. Now let α be the shortest simple closed curve on σ_f. Bers' observation shows that the σ_f-length of α is at most B_0. Therefore $f(\alpha)$ has length at most B_0 and by lemma 3.2, since its distance from a useful compact core of N is more than B_0, α cannot be a meridian and has a geodesic representative α^* in N.

Let $D = d_{\mathfrak{T}}(\sigma_f, \sigma_g)$; then similar to Minsky [Min93], we can see that

$$\frac{l_{\sigma_g}(\alpha)}{l_{\sigma_f}(\alpha)} \leq ce^D,$$

where c depends only on $\chi(\partial H)$ and η. Therefore $l_{\sigma_g}(\alpha) \leq ce^D B_0$. Suppose α^* is the geodesic representative of α in N then by lemma 2.10 and using the fact that length of α^* is at least η, we have

$$d_N(f(\alpha), \alpha^*) \leq \cosh^{-1}(\frac{B_0}{\eta})$$

and

$$d_N(g(\alpha), \alpha^*) \leq \cosh^{-1}(\frac{ce^D B_0}{\eta}).$$

Since length of α^* is at most B_0, this gives an upper bound for the distance between $f(\partial H)$ and $g(\partial H)$ and we are done. ☐

We fix the constant $k_0 \geq 1$ such that it satisfies the above lemma.

Lemma 9.2. *Given $D > 0$, there exists $K > 0$ such that for every f and $g \in \overline{\textbf{pleat}}_N$, $N \in \mathcal{B}_0(R)$ and a useful compact core C, if*

$$d_N(f(\partial H), g(\partial H)) \leq D$$

then there is a $(K,0)$-good homotopy between f and g within $N \setminus C$: a homeomorphism $\phi : \partial H \to \partial H$ and a homotopy with tracks length bounded by K between f and $g \circ \phi$ within $N \setminus C$. Also, the identity map on ∂H is K-bi-Lipschitz from the metric induced by f to the metric induced by $g \circ \phi$.

Proof. Suppose we have a sequence of counterexamples (N_i, f_i, g_i) which satisfy the hypothesis but fail the conclusion for bigger and bigger constants K_2. There are two cases, which we argue separately. Suppose there is a subsequence, say (N_i, f_i, g_i) itself, such that $f_i(\partial H)$ stays within a bounded distance from Γ_{N_i} independently of i. Then we can assume the sequence (N_i, p_{N_i}) converges strongly to $(N, p_N) \in \mathcal{B}_0(R)$ as marked hyperbolic structures. Since $f_i(\partial H)$ and therefore $g_i(\partial H)$ have bounded distance from Γ_N by using lemma 7.3 (Uniform bounded diameter lemma), we know that they stay in the D_1-neighborhood of Γ_N for some constant D_1 independent of i and they are η-thick; therefore the surfaces $(f_i(\partial H))$ and $(g_i(\partial H))$ converge to two pleated surfaces in N. Let C be a useful compact core in N and assume the marking $j : \partial H \to N$ is isotopic to $\partial H \hookrightarrow H$ outside of C. Also let

$$\kappa_i : N \to N_i$$

be the approximating maps. For i sufficiently large, $C_i = \kappa_i(C)$ is also a useful compact core in N_i and since the convergence $N_i \to N$ is as marked structures, we can assume $j_i = \kappa_i \circ j$ is the marking for N_i. The region between ∂C and $j(\partial H)$ is homeomorphic to $\partial H \times [0,1]$ and we call it M. There exists a deformation retract $r : M \times [0,1] \to M$ such that $r(\cdot, 0)$ is identity, $r(\cdot, t)|_{j(\partial H)}$ is identity for every $t \in [0,1]$ and image of $r(M,1)$ is contained in $j(\partial H)$. For i sufficiently large $M_i = \kappa_i(M)$ is homeomorphic to $\partial H \times [0,1]$ too and r induces a deformation retract $r_i : M_i \times [0,1] \to M_i$ to $j_i(\partial H)$ with the same properties and with tracks $r_i(\{x\} \times [0,1])$ which have bounded length independently of i and $x \in M_i$. The maps f_i and g_i are π_1-isomorphisms into M_i and therefore by using the deformation retract r_i we get a homotopy with bounded tracks between $f_i \circ \phi_i$, $g_i \circ \psi_i$ and j_i within M_i, where ϕ_i and ψ_i are self-homeomorphisms of ∂H homotopic to identity. This immediately show that the sequences $(f_i \circ \phi_i)$ and $(g_i \circ \psi_i)$ converge (as marked pleated surfaces) to f and $g \in \overline{\mathbf{pleat}}_N$ respectively and the bi-Lipschitz constant for $\text{id} : (\partial H, \sigma_f) \to (\partial H, \sigma_g)$ gives an upper-bound for the bi-Lipschitz constant of $\text{id} : (\partial H, \sigma_{f_i \circ \phi_i}) \to (\partial H, \sigma_{g_i \circ \psi_i})$ and we have a contradiction after precomposing with $\phi_i^{-1} \circ \psi_i$.

On the other hand assume $d_{N_i}(f_i(\partial H), \Gamma_{N_i}) \to \infty$ as $i \to \infty$. Choose a new base point for N_i to be a point $x_i \in f_i(\partial H) \subset N_i$. Then because the manifolds and surfaces are all η-thick, we can assume that the sequence of pointed manifolds (N_i, x_i) converges to a hyperbolic 3-manifold (N_∞, x_∞) in the geometric topology and the sequence of pleated surfaces $(f_i(\partial H))$ and $(g_i(\partial H))$ converge to pleated surfaces $f_\infty(\partial H)$ and $g_\infty(\partial H)$ in N_∞.

It is not hard to see that $f_\infty(\partial H)$ and $g_\infty(\partial H)$ are incompressible in N_∞. Using methods of Freedman-Hass-Scott [FHS83] (cf. Canary-Minsky [CM96]), we can choose embedded surfaces Σ_f and Σ_g which are homotopic to $f_\infty(\partial H)$ and $g_\infty(\partial H)$ and are contained in a small neighborhood of those respectiely. Suppose

$$\kappa_i : (N_\infty, x_\infty) \to (N_i, x_i)$$

are the approximating maps. For i sufficiently large, $\kappa_i(\Sigma_f)$ (resp. $\kappa_i(\Sigma_g)$) is embedded and is contained in a small neighborhood of $f_i(\partial H)$ (resp. $g_i(\partial H)$) and is homotopic to $f_i(\partial H)$ (resp. $g_i(\partial H)$) within $N_i \backslash C_i$, where C_i is a useful compact core of N_i. Hence $\kappa_i(\Sigma_f)$ and $\kappa_i(\Sigma_g)$ enclose a subset $M_i \subset N_i \setminus C_i$ homeomorphic to $\partial H \times [0, 1]$ and with bounded diameter independently of i. (This is because $f_i(\partial H)$ and $g_i(\partial H)$ have bounded diameter and bounded distance from each other independently of i.) Hence in the limit Σ_f and Σ_g are homotopic and therefore $f_\infty(\partial H)$ and $g_\infty(\partial H)$ are homotopic. After precomposing g_∞ with a self-homeomorphism of ∂H, we get a homotopy between f_∞ and g_∞. The κ_i-image of this homotopy is close to a homotopy between f_i and $g_i \circ \phi_i$ with tracks length bounded independently of i, for a homeomorphism $\phi_i : \partial H \to \partial H$. Since this homotopy stays within $N_i \backslash C_i$ for $i \gg 0$, ϕ_i has to be isotopic to the identity. Also note that the bi-Lipschitz constant for

$$\text{id} : \sigma_{f_\infty} \to \sigma_{g_\infty}$$

is close to the bi-Lipschitz constant of

$$\text{id} : \sigma_{f_i} \to \sigma_{g \circ \phi_i},$$

for $i \gg 0$ and therefore it is bounded independently of i and we have a contradiction. □

Definition 9.1. Given a useful commpact core $C \subset N$, a map $G : \partial H \times I \to N$ for $N \in \mathcal{B}_0(R)$ is a *K-sweep-out*, if G maps into $\mathcal{CH}(N) \setminus C$ and has the following properties:

(a) $I = [0, n]$ for some integer $n \geq 0$ when N is convex cocompact and $I = [0, \infty)$ if N is geometrically infinite,

(b) for each integer $i \in I$

$$G_i = G|_{\partial H \times \{i\}} \in \overline{\mathbf{pleat}}_N,$$

(c) the block $G|_{\partial H \times [i-1,i]}$ has tracks $G(\{x\} \times [i-1, i])$ with length bounded by K for every $x \in \partial H$ and integer $0 < i \in I$,

(d) G covers every point in the convex core of N except a set of diameter bounded by K with degree 1,

(e) G_0 has distance at most K from p_N and when N is convex cocompact and $I = [0, n]$, G_n gives a parametrization of the boundary of the convex core,

(f) $d_N(G_{i-1}(\partial H), G_i(\partial H)) \geq k_0$ for every positive integer $i \in I$,

(g) $G_i(\partial H)$ separates $G_{i-1}(\partial H)$ from the end of N and

(h) the identity map on ∂H is K-bi-Lipschitz from the metric induced by G_{i-1} to the metric induced by G_i and the Teichmüller distance between these metrics is bounded by K for every positive integer $i \in I$.

In this section, we want to prove the following:

Proposition 9.3. *There exists $K > 0$ depending only on R and $\chi(\partial H)$ such that every $N \in \mathcal{B}_0(R)$ with any useful compact core $C \subset N$ admits a K-sweep-out.*

Proof. We fix a useful compact core C for N. First, we show how to construct a map $G' : \partial H \times I' \to N$ that has prperties (a), (b), (c), (d) and (e) for a constant K'.

When N is convex cocompact, consider pants decompositions P_- and P_+ introduced in 8.2. Similar to our construction there, this time with $\epsilon = \eta$, we obtain an interpolation

$$F : \partial H \times [0, k] \to N$$

such that

- F_0 realizes P_+,
- F_i and $F_{i-1/2} \in \overline{\mathbf{pleat}}_N$
- $F|_{\partial H \times [i-1, i-1/2]}$ and $F|_{\partial H \times [i-1/2, i]}$ are (K_1, η)-good homotopies within $N \setminus C$ and
- $d_N(F_k(\partial H), \Gamma_N) < D$

for every $i = 1, \dots k$ and constants K_1 and D depending only on η, R and $\chi(\partial H)$. As we explained there, since P_+ has bounded length in σ_g, where $g \in \overline{\mathbf{pleat}}_N$ parametrizes $\partial \mathcal{CH}(N)$, we can assume that there is a $(K_2, 0)$-good homotopy between F_0 and g, where K_2 depends only on $\chi(\partial H)$. Also note that since η is smaller than the injectivity radii of the pleated surfaces and N, we can replace (K_1, η)-good homotopies with $(K_1, 0)$-good homotopy. Now define $G' : \partial H \times [0, k] \to N$ to be $G'(x, t) = F(x, k-t)$ for every $x \in \partial H$ and $t \in [0, k]$ and concatenate this with the $(K_2, 0)$-good homotopy between $F_0 = G'_k$ and g to have

$$G' : \partial H \times [0, k+1] \to N \setminus C$$

that satisfies properties (a), (b), (c), (d) and (e) for $I' = [0, k+1]$ and the constant

$$\max\{2K_1, K_2, D\}.$$

When N is geometrically infinite, suppose $P_- = P_0 \to P_1 \to \cdots$ is any elementary-move sequence of pants decompositions such that $P_n \to \mathcal{E}(N)$ (the ending lamination of N) in $\partial \mathcal{C}(\partial H)$ as $n \to \infty$. We construct an interpolatin associated to this sequence as follows:

Choose a fixed element $f \in \overline{\mathbf{pleat}}_N^{<D_1}$, which is possible for a uniform D_1 depending only on R and $\chi(\partial H)$ by lemma 7.8. For every $i \geq 0$, let f_i be an element of $\overline{\mathbf{pleat}}_N(P_i)$ if nonempty otherwise define it to be equal to f. Using lemmas 4.4 (Homotopy bound) and 4.5 (Halfway surface), we know that there exists D_2, which we can assume is $> D_1$, and K_3 depending only on R and $\chi(\partial H)$ such that for $i \geq 0$ if

$$d_N(f_i(\partial H), C) \geq D_2$$

then $\overline{\mathbf{pleat}}_N(P_{i+1})$ is nonempty and there is a $(K_3, 0)$-good homotopy between f_i and every element of $\overline{\mathbf{pleat}}_N(P_{i+1})$ within $N \setminus C$. We are using the fact that because N is η-thick, we have $(K_3, 0)$-good homotopy instead of (K_3, η)-good homotopy and $d_N^{\geq \eta}(\cdot,\cdot) = d_N(\cdot,\cdot)$. Hence, $f_{i+1} \in \overline{\mathbf{pleat}}_N(P_{i+1})$ and there is a $(K_3, 0)$-good homotopy between f_i and f_{i+1} and in particular

$$d_N(f_{i+1}(\partial H), C) \geq d_N(f_i(\partial H), C) - K_3.$$

On the other hand, if

$$d_N(f_i(\partial H), C) < D_2,$$

the same argument shows that

$$d_N(f_{i+1}(\partial H), C) \leq D_2 + K_3$$

and by using lemma 9.2, we know that there is a $(K_4, 0)$-good homotopy between f_i and f_{i+1}. We can concatenate these homotopies with possibly precomposing each f_i with a self-homeomorphism of ∂H isotopic to identity, to obtain a map

$$G' : \partial H \times [0, \infty) \to N,$$

where $G'|_{\partial H \times \{i\}} = f_i$ for $i \geq 0$ and $G'|_{\partial H \times [i,i+1]}$ is a $(K_5, 0)$-good homotopy for a constant K_5 that depends only on R and $\chi(\partial H)$. As in the geometrically finite case, we get G' that satisfies properties (a), (b), (c), (d) and (e) for $I' = [0, \infty)$ and the constant

$$\max\{K_5, D_2\}.$$

Next, we will try to modify this map to get another one which is a K-sweep-out. First we choose an increasing subset $(k_i) \subset I'$ of nonnegative integers inductively. Define $k_0 = 0$ and suppose we have chosen k_i. Consider the pleated surfaces $G'_{k_i}, G'_{k_i+1}, \ldots$ and define k_{i+1} to be the smallest index bigger than k_i, if there exists one, such that $G'_{k_{i+1}}$ separates G'_{k_i} from the end of N and

$$d_N(G'_i(\partial H), G'_{i+1}(\partial H)) \geq k_0.$$

If N is geometrically infinite, then this process never stops, since the sequence of pleated surfaces (G'_i) exits the end of N; in this case define $I = [0, \infty)$. But if N is convex cocompact and we couldn't choose k_{i+1} as above, then simply replace k_i with the last index in I' and stop; in this case define $I = [0, i]$, where k_i is the last chosen index. It should be obvious that

$$d_N(G'_{k_{i-1}}, G'_{k_i}) \leq K' + k_0. \tag{9.1}$$

Now we define G inductively. Define $G_0 = G'_{k_0}$ and assume $G|_{\partial H \times [0,i]}$ has been defined such that G_i and G'_{k_i} are the same after precomposing with a self-homeomorphism of ∂H isotopic to identity. Then by using lemma 9.2 and because of (9.1), there exists $\phi_i : \partial H \to \partial H$ isotopic to identity such that there is a $(K_6, 0)$-good homotopy between G_i and $G_{i+1} = G'_{k-i+1} \circ \phi_i$ within $N \setminus C$ and

$$\mathrm{id} : (\partial H, \sigma_{G_i}) \to (\partial H, \sigma_{G_{i+1}})$$

is K_6-bi-Lipschitz. The constant K_6 depends only on R and $\chi(\partial H)$ and we define $G|_{\partial H \times [i,i+1]}$ to be the homotopy described above.

Note that if two hyperbolic metrics on ∂H are η-thick and the identity is K_6-bi-Lipschitz between them, then the Teichmüller distance between the metrics is bounded by some K_7, where K_7 depends only on $\chi(\partial H)$, η and K_6 (cf. Minsky [Min92]). Let $K = \max\{K', K_6, K_7\}$ and the claim follows. $\square$

10 The Model Manifold

In this section, we want to use the sweep-out constructed in the last section to construct a bi-Lipschitz model for the geometry of every $N \in \mathcal{B}_0(R)$, which is determined by $\mathcal{E}(N)$ and the bi-Lipschitz constant depends only on R and $\chi(\partial H)$.

The models are similar to Minsky's models in [Min94]. But we use the description of the model in a way similar to Mosher [Mo03]. Mosher's work is set up to give a model for the degenerate hyperbolic structures with bounded geometry on interval bundles over a surface. But the same construction gives a uniform model for the convex cocompact case as well. Here, we use this to get a uniform model for both convex cocompact and geometrically infinite structures in $\mathcal{B}_0(R)$.

Theorem 10.1. (The model manifold) *Suppose the handlebody H of genus > 1 is given. Given R, there exists constants L and c, for which the following holds. Let $N \in \mathcal{B}_0(R)$ be a hyperbolic structure on H. For a choice of a useful compact core $C \subset N$, there exists a cobounded geodesic ray or segment g in $\mathfrak{T}(\partial H)$, such that:*

(1) *There is a map $\Phi : \mathcal{S}_g^{\mathrm{SOLV}} \to N_e$, properly homotopic to a homeomorphism and in the homotopy class determined by j, which lifts to a (L, c)-quasi-isometry of universal covers $\mathcal{H}_g^{\mathrm{SOLV}} \to \widetilde{N}_e$, where $N_e = \mathcal{CH}(N) \setminus C$.*

(2) *The initial point of g is a fixed point $\tau_H \in \mathfrak{T}$ and the "terminal" point is $\tau(\mathcal{E}(N))$. (The "terminal" point is a finite endpoint that corresponds to the conformal structure at infinity, in case N is convex cocompact and an ideal endpoint in Thurston's compactification of Teichmüller space, that corresponds to the ending lamination of N, when N is geometrically infinite.)*

Suppose $G : \partial H \times I \to N$ is the K-sweep-out constructed in proposition 9.3 and let $\sigma_i = \sigma_{G_i}$ be the metric induced by the pleated surface G_i for every integer $i \in I$. We know that the Teichmüller distance between σ_i and σ_{i+1} is at most K. Hence, there is a $\mathbb{Z}$-piecewise affine, K-Lipschitz path $\gamma : I \to \mathfrak{T}$ with $\gamma(n) = \sigma_n$. Since $\mathrm{inj}(\sigma_n)$ is at most η, it follows that the path γ is $\mathcal{K}$-cobounded, where $\mathcal{K}$ depends only on η and K.

Now consider the canonical hyperbolic surface bundle $\mathcal{S}_\gamma \to I$ and its universal cover, the canonical hyperbolic plane bundle $\mathcal{H}_\gamma \to I$. Note that

$\mathcal{S}_n \approx \sigma_n$ for every integer $n \in I$. We can identify $\partial H \times I$ with $\mathcal{S}_\gamma$ and from now on we consider the sweep-out as a map

$$G : \mathcal{S}_\gamma \to N.$$

Note that the restriction of G to $\mathcal{S}_n$ is length preserving for every integer $n \in I$ and because of the property (c) of the sweep-out, the restriction of G to any $x \times [n, n+1]$ is a K-Lipschitz map. Also note that property (f) of the sweep-out and lemma 9.1 show that the distance between σ_n and σ_{n+1} is at least 1 and therefore the Hausdorff distance between $\mathcal{S}_n$ and $\mathcal{S}_{n+1}$ is between 1 and K.

We know that image of the sweep-out is contained inside $\mathcal{CH}(N) \setminus C$. Let $N_e = \mathcal{CH}(N) \setminus C$, which is homeomorphic to $\partial H \times [0, \infty)$ or $\partial H \times [0, 1]$ depending on whether N is geometrically infinite or convex cocompact.

Proposition 10.2. *The map $G : \mathcal{S}_\gamma \to N_e$ lifts to a quasi-isometry of universal covers $\widetilde{G} : \mathcal{H}_\gamma \to \widetilde{N}_e$, with constants depending only on R and $\chi(\partial H)$.*

Proof. First we show that $\widetilde{G}$ is coarse surjective. By property (e) of the sweep-out, we actually know that G is surjective to N_e except possibly for a neighborhood of bounded diameter about ∂C. But we know that G_0 is a pleated surface contained in a bounded neighborhood of Γ_N. Therefore by lemma 9.2, there is a homotopy with bounded tracks between ∂C and $G_0(\partial H)$. This homotopy covers every point in the complement of $G(\mathcal{S}_\gamma)$ in N_e. Therefore, by lifting it to the universal cover, one can conclude that every point has bounded distance from image of $\widetilde{G}$.

Using fact 2.1, it is enough to show that $\widetilde{G}$ is uniformly proper with constants and properness gauge independent of N. We know that $\widetilde{G}|_{\mathcal{H}_n}$ is lift of a pleated surface and is distance non-increasing. On the other hand, length of a connection line $x \times [n, n+1]$ is at least 1 in $\mathcal{H}_\gamma$ and its image $\widetilde{G}(x \times [n, n+1])$ has length at most K and therefore $\widetilde{G}$ is Lipschitz along the connection lines as well and these two easily prove that it is Lipschitz everywhere.

Now let's prove that $\widetilde{G}$ is uniformly proper along $\mathcal{H}_n$ for an integer n.

Lemma 10.3. (Pleated surfaces are proper) *Given R there exists a properness gauge $\rho : [0, \infty) \to [0, \infty)$, such that if $N \in \mathcal{B}_0(R)$ and $f \in \overline{\mathbf{pleat}}_N$ are given then the lift to the universal covers $\widetilde{f} : \widetilde{\partial H} \to \widetilde{N}_e$ is ρ-uniformly proper.*

Proof. First note that $\widetilde{f}$ is distance non-increasing and therefore $\widetilde{f}$ is actually 1-Lipschitz. Hence, it is enough to show that given $a > 0$ there exists $b > 0$ such that if $l : I \to N_e$ is an arc of length at most a and $l(\partial I) \subset f(\partial H)$ then l is homotopic relative to the endpoints and within N_e to an arc of length at most b contained in $f(\partial H)$.

The idea of proof is by taking geometric limits and very similar to the proof of lemma 9.2. Assume (N_i, f_i, l_i) are a sequence of counter examples. Take a base point $x_i \in N_i$ to be on $f_i(\partial H)$ and assume the sequence of pointed manifolds (N_i, x_i) converges in the geometric topology to (N_∞, x_∞). The sequence of pleated surfaces (f_i) also converge to a pleated image of ∂H in N_∞ (because $f_i(\partial H)$ is η-thick for every i).

If $d_{N_i}(x_i, \Gamma_{N_i})$ stays bounded then the limit N_∞ is in $\mathcal{B}_0(R)$. Similar to lemmas 7.6 and 9.2, one can see that the pleated surfaces (f_i) converge to an element f_∞ of $\overline{\mathbf{pleat}}_{N_\infty}$. We can replace each l_i with another arc l'_i homotopic to l_i relative endpoints and with length bounded depending on a and distance more than d_0 from Γ_{N_i}. Then the arcs l'_i converge to an arc l'_∞ with bounded length which has distance more than d_0 from Γ_{N_∞} and has endpoints on $f_\infty(\partial H)$.

Let C_∞ be a useful compact core for N_∞. Since f_∞ is a homotopy equivalence between ∂H and $N_\infty \setminus C_\infty$, there exists an arc in $f_\infty(\partial H)$ which is homotopic to l'_∞ relative to endpoints and within $N_\infty \setminus C_\infty$. If we map this arc and the homotopy between these two arcs to the approximates, we get a bounded length arc in $f_i(\partial H)$ homotopic relative to endpoints and outside of a useful compact core to l. This contradiction proves the claim in this case.

On the other hand, if $d_{N_i}(x_i, \Gamma_{N_i}) \to \infty$ as $i \to \infty$ then we can see that limit of the pleated surfaces and the arcs is contained in a subset $M \subset N_\infty$ that is homeomorphic to $\partial H \times [0, 1]$ and Σ the limit of pleated surfaces gives a homotopy equivalence to M. Therefore, there is a homotopy fixing the endpoints that takes limit of (l_i) to an arc in Σ. Again by using the approximating maps and mapping this homotopy to the approximates we get a contradiction. Notice that in this case, the distance between the image of this homotopy by the approximating maps and every useful compact core goes to infinity and everything stays outside this compact core for $i \gg 0$. □

Once we have the above lemma, we can argue similar to Mosher [Mo03, Claim 4.7]. Using property (f) of the sweep-out, it follows that if $x \in \mathcal{H}_t$ and $y \in \mathcal{H}_s$ are given then

$$d_{\widetilde{N}_e}(\widetilde{G}(x), \widetilde{G}(y)) \geq k_0 \lfloor s - t \rfloor, \tag{10.1}$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$. This fact together with the above lemma prove that $\widetilde{G}$ is uniformly proper with constants and properness gauge independent of N and we are done. ☐

Lemma 10.4. ($\mathcal{H}_\gamma$ is hyperbolic) *The space $\mathcal{H}_\gamma$, or equivalently $\widetilde{N}_e$, is a hyperbolic metric space in sense of Gromov with constants depending only on R and $\chi(\partial H)$.*

Proof. In the proof of the above lemma, we use an idea which was briefly described in Mosher [Mo03, Sec. 4.4] and was partly based on Farb-Mosher [FM02]. The proof is similar to Farb-Mosher's proof of [FM02, Lem. 5.2] with some modifications in our situation and a part which was missing in their proof.

Given $\kappa > 1$, an integer $n \geq 1$ and $A \geq 0$, we say that a sequence of nonnegative integers $(r_i)_{i \in J}$ indexed by a subinterval $J \subset \mathbb{Z}$ satisfies the (κ, n, A)-*flaring property* if, whenever the three intergers $i-n, i, i+n$ are all in J, we have:

$$r_i > A \quad \Longrightarrow \quad \max\{r_{i-n}, r_{i+n}\} \geq \kappa \cdot r_i.$$

The number A is called the *flaring threshold* and notice that by making n larger, we can make κ as large as we want.

Let $\mathcal{H}_\gamma$ be given for a $\mathbb{Z}$-piecewise affine, K-Lipschitz, $\mathcal{K}$-cobounded path $\gamma : I \to \mathfrak{T}$. We say $\mathcal{H}_\gamma$ satisfies the *horizontal flaring property* if there exists $\kappa > 1$, an interger $n \geq 1$, and a function $A(\lambda) : [1, \infty) \to (0, \infty)$, such that if $\alpha, \beta : I' \to \mathcal{H}_\gamma$ are two λ-quasihorizontal paths with the same domain I', then setting $J = I' \cap \mathbb{Z}$ the sequence

$$d_i(\alpha(i), \beta(i)), \quad i \in J$$

satisfies the $(\kappa, n, A(\lambda))$ flaring property, where d_i is the distance function on $\mathcal{H}_i$, $i \in I$.

Farb-Mosher [FM02, Lem. 5.4] used Bestvina-Feighn Combination Theorem [BF92] to prove that if $\mathcal{H}_\gamma$, for a $\mathcal{K}$-cobounded, K-Lipschitz $\mathbb{Z}$-piecewise affine path γ, satisfies $(\kappa, n, A(\lambda))$-horizontal flaring then $\mathcal{H}_\gamma$ is δ-hyperbolic in sense of Gromov, where δ depends on K, $\mathcal{K}$ and the flaring data $\kappa, n, A(\lambda)$. Therefore it will be enough to show that $\mathcal{H}_\gamma$ satisfies $(\kappa, n, A(\lambda))$ horizontal flaring.

Let $\Sigma_i = G_i(\mathcal{S}_i)$ for every integer $i \in I$ and let $\widetilde{\Sigma}_i$ be its lift in $\widetilde{N}_e$. Recall that it follows from properties of the sweep-out that the Hausdorff distance between Σ_i and Σ_{i+1} (or between $\widetilde{\Sigma}_i$ and $\widetilde{\Sigma}_{i+1}$) is at least k_0 and at most K. Also recall that $G_i : \mathcal{S}_i \to \Sigma_i$ and its lift $\widetilde{G}_i : \mathcal{H}_i \to \widetilde{\Sigma}_i$ are length preserving and we denote the distance in $\widetilde{\Sigma}_i$ by d_i and the distance in $\widetilde{N}_e$ by d.

Notice that $\widetilde{G}$-image of a λ'-quasihorizontal path in $\mathcal{H}_\gamma$ is a λ-Lipschitz path $\alpha : I' \to \widetilde{N}_e$, $I' \subset I$, such that $\alpha(i) \in \widetilde{\Sigma}_i$ for every integer $i \in I'$ and λ depends on λ' and K. By abuse, we call these λ-quasihorizontal in $\widetilde{N}_e$.

It follows that it is enough to show the existance of flaring data $\kappa, n, A(\lambda)$ such that every two λ-quasihorizontal paths $\alpha, \beta : I' \to \widetilde{N}_e$, $I' \subset I$, the sequence

$$d_i(\alpha(i), \beta(i)) \quad i \in I' \cap \mathbb{Z}$$

satisfies $(\kappa, n, A(\lambda))$ flaring.

Fix a number $\lambda_0 \geq K$; first we obtain constants κ_0, n_0, A_0 such that the above sequence satisfies (κ_0, n_0, A_0)-flaring for λ_0-quasihorizontal paths α, β. Then we use this and prove the existance of uniform flaring data $\kappa, n, A(\lambda)$.

Suppose $\alpha, \beta : I' \to \widetilde{N}_e$, $I' \to I$, are λ_0-quasihorizontal. Let $J = I' \cap \mathbb{Z} = \{i_-, \ldots, i_+\}$ and assume $i_+ - i_-$ is even and $i_0 = \frac{i_+ + i_-}{2} \in J$. Also define $D_i = d_i(\alpha(i), \beta(i))$.

The lift of $G(\partial H \times [i,j])$ to $\widetilde{N}_e$ gives a map $h_{ij} : \widetilde{\Sigma}_i \to \widetilde{\Sigma}_j$ which is $K^{|i-j|}$-bi-Lipschitz for every pair of integers $i, j \in I$. This is the image of the connection map in $\mathcal{H}_\gamma$. For each $i \in J$, let $\rho_i : [0, D_i] \to \widetilde{\Sigma}_i$ be a $\widetilde{\Sigma}_i$-geodesic with endpoints $\alpha(i)$ and $\beta(i)$. Similar to [FM02, Claim 5.3], we have:

Claim 10.1. there is a family of quasihorizontal paths v described as follows:

- For each $i \in J$ and each $t \in [0, D_i]$ the family contains a unique quasi-horizontal path $v_{it} : [i_-, i_+] \to \widetilde{N}_e$ that passes through the point $\rho_i(t)$. If we fix $i \in J$, we thus obtain a parametrization of the family v_{it} by points $t \in [0, D_i]$.

- The ordering of the family v_{it} induced by the order on $t \in [0, D_i]$ is independent of i. The first path v_{i0} in the family is identified with α, and the last path v_{iD_i} is identified with β.

- Each v_{it} is λ'_0-quasihorizontal, where λ'_0 depends only on λ_0 and K.

It is easy to see that a λ'_0-quasihorizontal path in $\widetilde{N}_e$ is a (λ''_0, a) quasi-geodesic for constants λ''_0 and a depending on λ, K and k_0. We already know that it is Lipschitz and on the other hand using property (f) of the sweep-out, we know that $d(\alpha(i), \alpha(j)) \geq k_0|i - j|$ for integers i, j in the domain of α. In fact, the same argument shows that its images in N and its universal cover $\widetilde{N} = \mathbb{H}^3$ are also (λ_0'', a)-quasigeodesic. In $\mathbb{H}^3$, there exists a constant δ_1 depending only on λ_0'' and a such that for any rectangle of the form $v * \sigma * w * \sigma'$ where σ, σ' are geodesics and v, w are (λ_0'', a)-quasigeodesics, any point on v is within distance δ_1 of $\sigma \cup w \cup \sigma'$.

By lemma 10.3 (Pleated surfaces are proper), there exists a constant δ_2 such that:

for all $i \in J$, $x, y \in \widetilde{\Sigma}_i$, if $d(x,y) \leq \lambda_0''(\delta_1 + 1) + \delta_1$ then $d_i(x,y) \leq \delta_2$.
Now consider the flaring parameters κ', n', A' defined below:

$$\begin{aligned} \kappa' &= \frac{3}{2} \\ n' &= \lfloor \delta_1 + 3\delta_2 \rfloor + 1 \\ A' &= \delta_2 \end{aligned}$$

First, we show that if all the indexes in J are bigger than $6\delta_2$ then the sequence $\{D_i\}_{i \in J}$ has κ', n', A' flaring property. Suppose $i\pm = i_0 \pm n_0$, we must prove that

- if $D_{i_0} > A'$ then $\max\{D_{i-}, D_{i+}\} \geq \kappa' D_{i_0}$.

Case 1. $\max\{D_{i-}, D_{i+}\} \leq 6\delta_2$. We claim that we can take geodesics $\sigma_\pm$ in the interior of $\widetilde{N}_e$ with the same endpoints as $\rho_{i\pm}$ and length $\leq 6\delta_2$. The reason why $\sigma_\pm$ will be in the interior of $\widetilde{N}_e$ is that all the indexes are bigger than $6\delta_2$, therefore the endpoints of $\rho_{i\pm}$ have distance at least $6\delta_2$ from $\widetilde{\Sigma}_0$ and $\partial\widetilde{N}_e$. Therefore the geodesic representative of these arcs (relative endpoints) is inside $\widetilde{N}_e$.

Notice that a geodesic in the interior of $\widetilde{N}_e$ projects to a geodesic in N. This shows that the rectangle $\alpha * \sigma_- * \beta * \sigma_+$ projects to a homotopically trivial rectangle in N_e and its lift to $\widetilde{N} = \mathbb{H}^3$ is a rectangle $\alpha' * \sigma'_- * \beta' * \sigma'_+$ where $\sigma'_\pm$ are geodesics and α' and β' are (λ_0'', a)-quasigeodesics. Hence every point of α' has distance at most δ_1 from $\sigma'_- \cup \beta' \cup \sigma'_+$. This rectangle lifts isometrically to $\widetilde{N}_e$ and therefore, every point of α has distance at most δ_1 from $\sigma_- \cup \beta \cup \sigma_+$. Consider now the point $\alpha(i_0)$ and suppose it has distance at most δ_1 from $z \in \sigma_- \cup \beta \cup \sigma_+$.

If $z \in \sigma_+$ or σ_-, say σ_+ then it follows that

$$d(\alpha(i_0), \widetilde{\Sigma}_{i+}) \leq \delta_1 + \frac{6\delta_2}{2} < n'$$

which implies that $d(\widetilde{\Sigma}_{i_0}, \widetilde{\Sigma}_{i+}) < n'$ but we know that the distance between these two is at least $n_0 \cdot k_0 > n_0$ and we have a contradiction. Therefore $z = \beta(s) \in \beta$.

By (10.1), we know that the distance from $\beta(s)$ to $\widetilde{\Sigma}_{i_0}$ is at least $k_0 \lfloor |s - i_0| \rfloor > |s - i_0| - 1$ and therefore

$$|s - i_0| - 1 < \delta_1 \;\Rightarrow\; |s - i_0| < \delta_1 + 1 \;\Rightarrow\; d(\beta(s), \beta(i_0)) < \lambda_0''(\delta_1 + 1).$$

Hence

$$d(\alpha(i_0), \beta(i_0)) \leq d(\alpha(i_0), \beta(s)) + d(\beta(s), \beta(i_0)) < \delta_1 + \lambda_0''(\delta_1 + 1),$$

and this implies that $D_{i_0} = d_{i_0}(\alpha(i_0), \beta(i_0)) < A'$.

Case 2. $\max\{D_{i_-}, D_{i_+}\} \geq 3\delta_2$ Suppose v is the family of quasihorizontals constructed in claim 10.1 and assume we consider the parametrization at $i = i_0$. We can see that there is a discrete subfamily $\alpha = v_{t_0}, v_{t_1}, \ldots, v_{t_k} = \beta$, with $t_0 < t_1 < \cdots < t_k$, such that the following is satisfied: for each $l = 1, \ldots, k$, letting

$$\Delta_{l\pm} = d_{i\pm}(v_{t_{l-1}}(i\pm), v_{t_l}(i\pm))$$

then we have

$$\max\{\Delta_{l-}, \Delta_{l+}\} \in [3\delta_2, 6\delta_2].$$

The proof is exactly the same as in [FM02], which we do not repeat. Since

$$\max\{\Delta_{l-}, \Delta_{l+}\} \leq 6\delta_1,$$

the argument in **Case 1** shows that

$$\Delta_{l0} = d_{i_0}(v_{t_{l-1}}(i_0), v_{t_l}(i_0)) \leq \delta_2$$

for all $l = 1, \ldots, k$. Hence

$$D_{i_0} = \sum_{l=1}^{k} \Delta_{l0} \leq k\delta_2$$

$$\begin{aligned} D_{i-} + D_{i+} &= \sum_{l=1}^{k} \Delta_{l-} + \Delta_{l+} \geq \sum_{l=1}^{k} \max\{\Delta_{l-}, \Delta_{l+}\} \\ &\geq k \cdot 3\delta_2. \end{aligned}$$

Then

$$\max\{D_{i-}, D_{i+}\} \geq \frac{3}{2}k\delta_2 \geq \frac{3}{2}D_{i_0}.$$

This proves that the sequence (D_i) has κ', n', A' flaring when we restrict it to indexes bigger than $6\delta_2$.

Lemma 10.5. *There exists a constant $b > 0$ depending on K, λ_0 and the properness guage ρ in lemma 10.3 such that the sequence (D_i) satifies a* (K, b)-coarse Lipschitz growth condition*: $D_i \leq KD_j + b$ given $i, j \in J$ with $|i - j| = 1$.*

Proof. First recall that the map $h_{ij} : \widetilde{\Sigma}_i \to \widetilde{\Sigma}_j$ is $K^{|i-j|}$-bi-Lipschitz. Given $i, j \in J$ with $|i - j| = 1$, let $a = h_{ij}(\alpha(i))$ and $b = h_{ij}(\beta(i))$. The points a and $\alpha(j)$ are connected by a path of length at most $K\lambda_0 + K$, consisting of a segment of α from $\alpha(j)$ to $\alpha(i)$ and a path of length at most K from $\alpha(i)$ to a, and similarly the distance between b and $\beta(j)$ is at most $K\lambda_0 + K$. Suppose $b = 2\rho(K\lambda_0 + K)$ where ρ is the properness gauge in lemma 10.3. Then

$$\begin{aligned} d_j(a, \alpha(j)) &\leq b/2 \\ d_j(b, \beta(j)) &\leq b/2 \\ d_j(\alpha(j), \beta(j)) &\leq d_j(\alpha(j), a) + d_j(a, b) + d_j(b, \beta(j)) \\ &\leq b + Kd_i(\alpha(i), \beta(i)) \end{aligned}$$

This finishes the proof of the lemma. □

From the above, we can easily see that there exists b' depending only on K, b and δ_2 such that if $i, j \in J$ and $i - j \leq 6\delta_2$ then

$$D_i \geq K^{-m}D_j - b' \tag{10.2}$$

where $m = \lfloor 6\delta_2 \rfloor + 1$.

We knew that the sequence (D_i) has (κ', n', A')-flaring property when restricted to the indexes $\geq 6\delta_2$. By choosing n'' to be a multiple of n', we can assume that the sequence $(D_i)_{i \geq m}$ also has (κ'', n'', A') flaring, where $\kappa'' > 4K^m$. Now consider

$$\begin{aligned} \kappa_0 &= 3 \\ n_0 &= n'' + m \\ A_0 &= \max\{A', b'\} \end{aligned}$$

and we claim that the entire sequence $(D_i)_{i\in J}$ has (κ_0, n_0, A_0)-flaring. Suppose $i \pm n_0$ and i are in J. If $i - n_0 \geq m$ then it is obvious. Suppose $D_i > A_0 \geq A'$; we already know that

$$\max\{D_{i-n''}, D_{i+n''}\} \geq \kappa'' D_i,$$

since indexes $i \pm n''$ and i are bigger than $6\delta_2$. Suppose $D_{i+n''} \geq \kappa'' D_i$, then by (10.2)

$$\begin{aligned} D_{i+n_0} &\geq K^{-m} D_{i+n''} - b' \\ &\geq K^{-m}\kappa'' D_i - b' \\ &> 4D_i - b' \\ &> 3D_i + (D_i - b') \\ &> 3D_i. \end{aligned}$$

The same argument works in the other case and we have proved that the sequence $(D_i = d_i(\alpha(i), \beta(i)))_{i\in J}$ has (κ_0, n_0, A_0)-flaring property for every pair of λ_0-quasihorizontals $\alpha, \beta : I' \to \widetilde{N}_e$, $I' \subset I$.

Using the above, we want to prove that for arbitrary $\lambda > 0$ and λ-quasihorizontals $\alpha, \beta : I' \to \widetilde{N}_e$, $I' \subset I$, the sequence

$$d_i(\alpha(i), \beta(i)) \quad i \in J = I' \cap \mathbb{Z}$$

has $(\kappa, n, A(\lambda))$ property, where

$$\begin{aligned} \kappa &= 2 \\ n &= n_0 \\ A(\lambda) &= \max\{A_0, 2\rho(K + K\cdot\lambda)\} \end{aligned}$$

and ρ is the properness gauge in lemma 10.3 (Pleated surfaces are proper).

Suppose i_0 and $i_0 \pm n_0$ are in J and we define $i\pm = i_0 \pm n_0$,

$$D_0 = d_{i_0}(\alpha(i_0), \beta(i_0)) \text{ and } D_\pm = d_{i\pm}(\alpha(i\pm), \beta(i\pm)).$$

We want to prove that

$$\max\{D_+, D_-\} \geq 3D_0 \quad \text{if } D_0 > A(\lambda).$$

If $\lambda \leq \lambda_0$ then the statement easily follows since $A(\lambda) \geq A_0$; therefore we can assume $\lambda > \lambda_0 \geq K$.

Let

$$\alpha' = \widetilde{G}|_{\{x\}\times[i-,i+]} : [i-,i+] \to \widetilde{N}_e$$

such that $\alpha'(i_0) = \alpha(i_0)$. In the same way define $\beta' : [i-,i+] \to \widetilde{N}_e$. We know that α' and β' are K-quasihorizontals and since $K \leq \lambda_0$

$$\max\{d_{i-}(\alpha'(i-),\beta'(i-)), d_{i+}(\alpha'(i+),\beta'(i+))\} \geq 3d_{i_0}(\alpha'(i_0),\beta'(i_0)) = 3D_0$$

whenever $D_0 \geq A_0$. Similar to proof of lemma 10.5 there is a path of length at most $K + K\cdot\lambda$ connecting $\alpha(i\pm)$ and $\alpha'(i\pm)$ obtained by moving along α from $\alpha(i\pm)$ to $\alpha(i_0)$ and then by moving along α' from $\alpha'(i_0)$ to $\alpha'(i\pm)$. By lemma 10.3 (Pleated surfaces are proper), we have

$$d_{i\pm}(\alpha(i\pm),\alpha'(i\pm)) \leq \rho(K + K\cdot\lambda) \leq \frac{A(\lambda)}{2}.$$

The same argument shows that

$$d_{i\pm}(\beta(i\pm),\beta'(i\pm)) \leq \frac{A(\lambda)}{2},$$

and we have

$$\begin{aligned} D_{\pm} &= d_{i\pm}(\alpha(i\pm),\beta(i\pm)) \\ &\geq d_{i\pm}(\alpha'(i\pm),\beta'(i\pm)) - d_{i\pm}(\alpha(i\pm),\alpha'(i\pm)) - d_{i\pm}(\beta(i\pm),\beta'(i\pm)) \\ &\geq d_{i\pm}(\alpha'(i\pm),\beta'(i\pm)) - A(\lambda) \end{aligned}$$

Hence

$$\begin{aligned} \max\{D_+, D_-\} &\geq \max\{d_{i-}(\alpha'(i-),\beta'(i-)), d_{i+}(\alpha'(i+),\beta'(i+))\} - A(\lambda) \\ &\geq 3D_0 - A(\lambda) \\ &= 2D_0 + (D_0 - A(\lambda)) \\ &\geq 2D_0 \end{aligned}$$

whenever $D_0 \geq A(\lambda)$ and this finishes our proof.

$\square$

Once we know that $\mathcal{H}_\gamma$ is hyperbolic, it follows from work of Mosher [Mo03, Thm 1.1, Prop. 2.3] that γ fellow travels a cobounded geodesic segment or ray g' in $\mathfrak{T}$ with $\gamma(0) = g'(0)$ and we obtain $\pi_1(\partial H)$-equivariant quasi-isometries

$$\mathcal{H}_{g'}^{\mathrm{SOLV}} \to \mathcal{H}_\gamma \xrightarrow{\widetilde{G}} \widetilde{N}_e,$$

with constants independent of N.

Going back to the steps of our construction, we notice that the initial point $\gamma(0)$ was the metric induced by a pleated surface in $\mathbf{pleat}_{N\setminus\Gamma_N}$ with uniformly bounded distance from the useful compact core C independently of N. An immediate, application of lemmas 7.3 and 7.6 will be that such initial points are all contained in a compact subset of $\mathfrak{T}(\partial H)$ and therefore they have bounded distance from the base point $\tau_H \in \mathfrak{T}(\partial H)$.

On the other hand, when N is convex cocompact, the terminal point of γ was the metric induced by the boundary of the convex core, in the homotopy class represented by j. Then since the conformal structure at infinity $\tau(\mathcal{E}(N))$ is ϵ_0-thick, we can use Bridgeman-Canary's result, theorem 2.11, and conclude that this terminal point has bounded distance from $\tau(\mathcal{E}(N))$ independently of N.

Finally, when N is geometrically infinite, we know that the sequence $(\gamma(i))$ represents the metrics induced by a sequence of elements of $\overline{\mathbf{pleat}}_N$ which exit the end of N. Then it follows from Canary's [Can93b] description of the ending laminations for these structures that every limit of the sequence $(\gamma(i))$ in Thurston's compactification of the Teichmüller space, is an element of $\mathcal{PML}$ supported on $\mathcal{E}(N)$, the ending lamination of N. Since g' has bounded Hausdorff distance from γ and they are cobounded, one can see that the ideal endpoint of g in $\mathcal{PML}$ is the ending lamination of N as well. (Cf. Minsky [Min96].)

Then it follows from the above that if g is a geodesic ray or segment with $g(0) = \tau_H$ and in case N is convex cocompact, the terminal point of g is the metric induced by the conformal structure at infinity and in case N is geometrically infinite, its ideal end point in $\mathcal{PML}$ is the same as the ideal end point of g' and supported on the ending lamination of N, then g has bounded Hausdorff distance from g' independently of N (cf. [FM02]). This finishes the proof of theorem 10.1 (The model manifold).

11 Gluing

Now suppose a Heegaard splitting $H^+ \cup_S H^-$ with R-bounded combinatorics is given. Let $P^\pm \subset \Delta(H^\pm)$ be the pants decompositions which realize the curve complex distance between $\Delta(H^+)$ and $\Delta(H^-)$ and have R-bounded combinatorics. One can easily extend each $P^\pm$ to a full marking $\alpha^\pm$ such that α^+ and α^- still have R-bounded combinatorics.

Now use a homeomorphism $\phi^+ : H^\pm \to H$ to identify H^+ with H and S with ∂H. With an abuse of notation, we denote the induced maps on the corresponding complex of curves, marking spaces and Teichmüller spaces by ϕ^+ as well. After possibly postcomposing ϕ^+ with an element of $\mathrm{Mod}_0(H)$, we assume that $\phi^+(\alpha^+)$ is in $\mathbf{m}_0$(H). Then $\phi^+(\alpha^-)$ has R-bounded combinatorics respect to H and in fact belongs to $A_0(R)$.

In section 6, we see that $\phi^+(\alpha^-)$ corresponds to a marked hyperbolic structure $N \in B_0(R)$. Let g be a Teichmuller geodesic segment that connects τ_0 to $\tau(\phi^+(\alpha^-))$. Then theorem 10.1 (The model manifold), proves that there is a map

$$\Phi : \mathcal{S}_g^{\mathrm{SOLV}} \to N_e$$

that lifts to an (L, c)-quasi-isometry $\mathcal{H}_g^{\mathrm{SOLV}} \to \widetilde{N}_e$, where $N_e = \mathcal{CH}(N) \setminus C$ for a useful compact core $C \subset N$ and also that g is $\mathcal{K}$-cobounded, where $\mathcal{K}$ depends only on R and $\chi(\partial H)$. In other terms, Φ and $\mathcal{S}_g^{\mathrm{SOLV}}$ give a model description of the convex core of N, outside of a small compact core.

Now use ϕ^+ and pull back all these structures to H^+. We get a hyperbolic structure N^+ on H^+, a geodesic segment $g^+ \subset \mathfrak{T}(S)$ and a map

$$\Phi^+ : \mathcal{S}_{g^+}^{\mathrm{SOLV}} \to N_e^+$$

which lifts to an (L, c)-quasi-isometry and is in the homotopy class determined by $S \hookrightarrow H^+$. The set $N_e^\pm$ is $\mathcal{CH}(N^+) \setminus (\phi^+)^{-1}(C)$ the complement of a small compact core of N^+.

Obviously the end point of g^+ is $\tau(\alpha^-)$; we claim that the initial point is uniformly close to $\tau(\alpha^+)$. We know that the initial point is $(\phi^+)^{-1}(\tau_H)$. Now it is enough to notice that the upper-bound for the total length of elements of $\mathbf{m}_0$(H) in τ_H gives an upper bound for the total length of α^+ in $(\phi^+)^{-1}(\tau_H)$. But support of α^+ is a *binding collection of curves* (every essential closed curve intersects at least one element of this collection), and a standard fact (cf. Minsky [Min92, Lem. 4.7]) shows that diameter of a set of points of $\mathfrak{T}(S)$ where a binding collection has bounded length is bounded depending only on $\chi(S)$ and the upper-bound for the length of the binding collection. This immediately implies that $(\phi^+)^{-1}(\tau_H)$ and $\tau(\alpha^+)$ are uniformly close.

Thus, if we let h to be a geodesic segment in $\mathfrak{T}(S)$ whose initial and terminal endpoints are $\tau(\alpha^+)$ and $\tau(\alpha^-)$, then g^+ and h have bounded Hausdorff distance and they both are cobounded. From here, a standard argument (cf. Farb-Mosher [FM02, Prop. 4.2]) proves that there is a map $\mathcal{S}_h \to \mathcal{S}_{g^+}$ taking fibers to fibers that lifts to a quasi-isometry $\mathcal{H}_h \to \mathcal{H}_{g^+}$ with constants depending on the Hausdorff distance of h and g^+ and the set $\mathcal{K}$ and $\chi(\partial H)$. Hence, we can replace g^+ with h and the map Φ^+ with

$$\Psi^+ : \mathcal{S}_h^{\mathrm{SOLV}} \to N_e^+$$

which satisfies all the properties of Φ^+ except possibly with bigger quasi-isometry constants (L_1, c_1).

We can do the same construction for H^-. This gives a a convex cocompact structure N^- on H^- and $N_e^- = \mathcal{CH}(N^-) \setminus C^-$, where C^- is a small compact core. We also have a map

$$\Psi^- : \mathcal{S}_h^{\mathrm{SOLV}} \to N_e^-$$

in the homotopy class determined by $S \hookrightarrow H^-$ that lifts to an (L_1, c_1)-quasi-isometry. Notice that by considering the direction of h to be pointing from $\tau(\alpha_1)$ to $\tau(\alpha_2)$ and use the orientation of $\mathcal{S}_h$ that corresponds to the product of this orientation on h and the orientation of S, then Ψ^+ is orientation preserving but Ψ^- is orientation reversing and this is because the orientation of S in ∂H^+ (∂H^-) matches (does not match) with the orientation induced by orientation of H^+ (H^-).

Also recall that the Teichmüller distance between $\tau(\alpha^+)$ and $\tau(\alpha^-)$ (or equivalently length of h) tends to infinity as the curve complex distance between α^+ and α^-, or equivalently the handlebody distance for the Heegaard splitting, goes to infinity. This is because, as it is shown for example in Minsky [Min92], when two points τ_1 and τ_2 of the Teichmuller space are $\mathcal{K}$-cobounded and have distance at most D, then there exists a K-bi-Lipschitz map isotopic to identity from $(\partial H, \tau_1)$ to $(\partial H, \tau_2)$, where K depends only on $\mathcal{K}$, D and $\chi(\partial H)$. In particular, length of α in τ_2 is bounded depending on length of α in τ_1 and D and $\mathcal{K}$. As a result, the shortest marking on τ_1 and the shortest marking on τ_2 have bounded $\mathcal{C}$-distance depending on D and $\mathcal{K}$.

Therefore, by assuming that the handlebody distance is large, we can make sure that the Teichmüller distance between $\tau(\alpha^+)$ and $\tau(\alpha^-)$ is large and equivalently the diameter of the convex cores of N^+ and N^- is large.

Proposition 11.1. *Given $\epsilon' > 0$ and $D > 0$ there exists a number d such that the following holds. Suppose $H^+ \cup_S H^-$ is a Heegaard splitting with handlebody distance at least d, $\alpha^\pm$ are handlebody markings for $H^\pm$ which have R-bounded combinatorics and realize the handlebody distance and $N^\pm$ are hyperbolic structures on $H^\pm$ that correspond to $\tau(\alpha^\mp)$.*
Then there exists a doubly degenerate surface group $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ and maps

$$T^\pm : N_\rho \to N_e^\pm$$

which are ϵ'-close to an isometry on the ball of radius D about p_0 in the C^∞-topology and $T^\pm$ is in the homotopy class determined by $S \hookrightarrow H^\pm$, where $N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ has $\mathrm{inj}(N_\rho) \geq \eta$, $p_0 \in N_\rho$ is the image of $0 \in \mathbb{H}^3$ and $N_e^\pm$ is the complement of a useful compact core in the convex core of $N^\pm$ as usual.

Proof. We can prove this by means of taking a geometric limit. As described above, if we have a sequence of R-bounded Heegaard splittings whose handlebody distance goes to infinity, these give a sequence of Teichmüller geodesic segments $h_n : [-n, n] \to \mathfrak{T}(S)$ whose length goes to infinity. We also have the corresponding hyperbolic structures $N_n^\pm$ on the handlebodies $H^\pm$ and uniform approximations of neighborhoods of their end by

$$\Psi_n^\pm : \mathcal{S}_{h_n} \to N_n^\pm \setminus C_n^\pm,$$

which lift to (L_1, c_1)-quasi-isometries of the universal covers and $C_n^\pm$ are useful compact cores of $N_n^\pm$.

Notice that actions of mapping class group of S on the Teichmüller space and the geodesic h_n corresponds to precompositions of the embedding $S \hookrightarrow H^+ \cup_S H^-$ with self-homeomorphisms of S. Therefore, these give the same Heegaard splittings and we consider them equivalent. The geodesic segments (h_n) are all uniformly cobounded and therefore up to actions of $\mathcal{MCG}(S)$, we can assume that they converge in the Hausdorff topology to a cobounded biinfinite geodesic $h_\infty : (-\infty, \infty) \to \mathfrak{T}(S)$ in a way that the points $h_n(0)$ converge to $h_\infty(0)$ and the convergence preserves the orientation of the geodesics h_n. Take a point $w_\infty \in \mathcal{S}_{h_\infty(0)}$ and let $w_n \in \mathcal{S}_{h_n(0)}$ be the point obtained by moving along the connection lines between $\mathcal{S}_{h_\infty(0)}$ and $\mathcal{S}_{h_n(0)}$. Then let $x_n^\pm = \Psi_n^\pm(w_n)$ be the base point of $N_n^\pm$. The sequence of pointed manifolds $(N_n^\pm, x_n^\pm)$ converges in the geometric topology to a hyperbolic manifold $(N_\infty^\pm, x_\infty^\pm)$ with $\mathrm{inj}(N_\infty^\pm) \geq \eta$. In fact, it is not hard to see that this limit is a doubly degenerate hyperbolic structure on $S \times \mathbb{R}$. Let

$$\kappa_n^\pm : N_\infty^\pm \to N_n^\pm$$

be the approximating maps; we claim that there is a map

$$\Psi_\infty^\pm : \mathcal{S}_{h_\infty} \to N_\infty^\pm$$

that lifts to a quasi-isometry and the map $\kappa_n^\pm \circ \Psi_\infty^\pm$ is in the homotopy class of $S \hookrightarrow H^\pm$ for every n.

Using Sullivan's rigidity theorem, we can see that the map

$$\Psi_\infty^- \circ \Psi_\infty^+$$

is homotopic to an isometry. Even more, N_∞^+ and N_∞^- both represent a doubly degenerate surface group $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ whose ending laminations are determined by the ideal endpoints of h_∞.

Therefore, we can consider the approximating maps $\kappa_n^\pm$ as maps defined on $N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ and for n sufficiently large, they will satisfy the required properties for $T^\pm$ and this proves our claim. $\square$

It is not hard to prove the next lemma, using a geometric limit argument.

Lemma 11.2. *There exists a constant D depending only on η and $\chi(S)$ such that the ball of radius D about any point in the convex core of a doubly degenerate hyperbolic structure N_ρ on $S \times \mathbb{R}$ with $\mathrm{inj}(N_\rho) \geq \eta$ contains a subset $V \subset N_\rho$ homeomorphic to $S \times [0,1]$ with $V \hookrightarrow N_\rho$ a homotopy equivalence and the distance at least 1 between the boundary components of V. In addition, there is a smooth bump function $\theta : V \to [0,1]$ where $\theta|_{\partial_- V} \equiv 0$ and $\theta|_{\partial_+ V} \equiv 1$, where $\partial_- V$ and $\partial_+ V$ are the boundary components of V, and there is an upper-bound for all the first and second derivatives of θ depending only on η and $\chi(S)$.*

Suppose the handlebody distance for $M = H^+ \cup_S H^-$ is larger than d in the statement of proposition 11.1 for D obtained in the above lemma and $\epsilon' > 0$ small which will be determined soon. Suppose N_ρ is the associated doubly degenerate hyperbolic structure on $S \times \mathbb{R}$ and

$$T^\pm : N_\rho \to N_e^\pm$$

are the maps described in proposition 11.1. Choose a subset $V \subset N_\rho$ contained in the D-neighborhood of the ball about p_0 and a bump function

$$\theta : V \to [0,1]$$

that satisfy lemma 11.2.

We consider $T^\pm$ restricted to V and the idea is to use these and construct a nearly hyperbolic metric on M. We know that $T^\pm : V \to N_e^\pm$ gives a homotopy equivalence in the homotopy class determined by $S \hookrightarrow N_e^\pm$. Without loss of generality, we can assume that $T^+(\partial_2 V)$ separates $T^+(\partial_1 V)$ from the end of N^+. Notice that from this it follows that in N^-, $T^-(\partial_1 V)$ separates $T^-(\partial_2 V)$ from the end and this is because the map T^+ is orientation preserving, whereas T^- is orientation reversing.

The complement of $T^\pm(V)$ in $N^\pm$ has two components; one is a bounded diameter set homeomorphic to the interior $H^\pm$ and the other one which we call $Y^\pm$ is homeomorphic to $S \times \mathbb{R}$ and gives a neighborhood of the end of $N^\pm$. Observe that

$$(N^+ \setminus Y^+) \cup_{T^+ \circ (T^-)^{-1}} (N^- \setminus Y^-)$$

is homeomorphic to $M = H^+ \cup_S H^-$. We denote the image of the collar V by V and the two components of $M \setminus V$ by X^+ and X^- which are respectively contained in $N^+ \setminus Y^+$ and $N^- \setminus Y^-$. The hyperbolic metric of $N^\pm$ induces a hyperbolic metric $\nu^\pm$ on $M \setminus X^\mp$. These metrics do not coincide but they are $2\epsilon'$-close in the C^∞ topology.

Now we can define the metric ν on M to be

$$\nu(x) = \theta(x) \cdot \nu^+(x) + (1 - \theta(x)) \cdot \nu^-(x),$$

for any $x \in M$. This metric is smooth and of course hyperbolic on $M \setminus V$. Moreover on V, the metrics $\nu^\pm$ are ϵ'-close to the metric induced by N_ρ which we call ν_ρ. In particular we have

$$\nu^\pm = \nu_\rho + \xi^\pm,$$

where $\xi^\pm$ is a 2-tensors which is C^2-close to zero. This implies that on V, we have

$$\nu = \nu_\rho + \theta\xi^+ + (1-\theta)\xi^-.$$

Since the first and second derivatives of θ are bounded from above independently of the Heegaard splitting, by making the handlebody distance large, we can make sure that ν and ν_ρ are as C^2-close as we want. The sectional curvatures of ν depend only on the first and second derivatives of the metric and therefore all sectional curvatures of ν stay in the interval $[-1-\epsilon, -1+\epsilon]$ if the handlebody distance is large.

In addition to this, it is obvious that the injectivity radius of ν at every point is at least $\eta/2$ and we have proved our main theorem.

In fact, our construction immediately shows the following:

Theorem 11.3. *There are constants K, L_1, c_1 and η depending only on $\chi(S)$ and R such that the following holds.*

Let $M = H^+ \cup_s H^-$ be an R-bounded Heegaard splitting and $\alpha^\pm$ is a handlebody marking for $H^\pm$ such that α^+ and α^- realize the handlebody distance of the splitting and have R-bounded combinatorics. Then there exists a Riemannian metric ν on M and an η-cobounded geodesic segment g connecting $\tau(\alpha_1)$ and $\tau(\alpha_2)$ such that there is a map

$$\Psi : \mathcal{S}_g \to M \setminus (C^+ \cup C^-)$$

which lifts to an (L_1, c_1)-quasi-isometry of the universal covers, where $C^\pm \subset H^\pm$ is a compact core of $H^\pm$ with ν-diameter bounded by K.

12 Tian's theorem and hyperbolicity

In [Ti90], Tian claims the following theorem:

Theorem 12.1. *Let (M, ν) be a negatively curved Riemannian three manifold and η a Margulis number for negatively curved three manifolds. Denote by M_η the η-thin piece of M. Then, there is a universal constant ϵ' such that if M satisfies*

1. *M_η is a disjoint union of convex neighborhoods $\{C_\alpha\}$ of closed geodesics γ_α with length $\leq 2\eta$ such that the normal injectivity radius of γ_α in C_α is greater than 1.*

2. *let P_α be a smooth function such that P_α is equal to η near the boundary of C_α and $P_\alpha(y)$ is equal to the injectivity radius at y whenever this is less than $\eta/2$ (such P_α always exists). We require that for some choice of P_α,*

$$\int_{C_\alpha} \frac{1}{P_\alpha} |\mathrm{Ric}(\nu) + 2\nu|_\nu^2 \, dV_\nu \leq \epsilon' \qquad \text{for each } \alpha.$$

3. *all sectional curvatures of M lie between $-1-\epsilon'$ and $-1+\epsilon'$.*

4. *$\int_M |\mathrm{Ric}(\nu) + 2\nu|_\nu^2 dV_\nu \leq (\epsilon')^2$*

then M admits an Einstein metric which is close to ν up to third derivatives.

Here $\mathrm{Ric}(\nu) + 2\nu$ is the trace-free Ricci curvature of M. in fact Tian's result is stronger than this and allows dimensions other than 3 and norms other than L^2 norm. However this is more than enough for our application. Note that in dimension three, Einstein manifolds have constant sectional curvature.

Therefore, Tian's theorem implies the following:

Corollary 12.2. *Suppose (M, ν) is a Riemannian three manifold with η-bounded geometry. Also assume (M, ν) is hyperbolic outside a set of volume bounded by some d' and everywhere else the sectional curvatures are between $-1-\epsilon$ and $-1+\epsilon$ for ϵ sufficiently small. Then M admits a hyperbolic metric ν' which is close to ν up to third derivatives.*

Proof. To apply Tian's theorem, we need to verify that (M, ν) satisfies the assumptions. We know that (M, ν) has η-bounded geometry, therefore the η-thin part of the manifold is empty and the first and second assumptions are vacuous. The third assumption is satisfied by the hypothesis too when $\epsilon \leq \epsilon'$. For the last one, note that relative to an orthonormal frame, the entries in the 3×3 matrix for $\mathrm{Ric}(\nu) + 2\nu$ are all between -4ϵ and 4ϵ if all sectional curvatures are pinched between $-1-\epsilon$ and $-1+\epsilon$. This follows from the fact that the Ricci tensor may be recovered by polarization from its associated quadratic form $Q(u) = \mathrm{Ric}(u, u)$ and that $\mathrm{Ric}(u, u)$ is simply $< u, u >$ multiplied by the sum of the sectional curvatures of any 2 orthogonal planes containing u. Therefore, the function in the integral is zero outside a set of volume bound by d' and is small when ϵ is small inside that set. So by making sure that ϵ is small enough, we also have the last assumption and the Tian's theorem proves the claim. □

In particular, putting our main theorem 1 and the last corollary together we have:

Theorem 12.3. *If $M = H^+ \cup_S H^-$ is a Heegaard splitting with R-bounded combinatorics and suficiently large handlebody distance then M admits a hyperbolic metric ν'. Also similar to theorem 11.3, there is geodesic segment g in $\mathfrak{T}(S)$ determined by combinatorics of the splitting and a map*

$$\Psi : \mathcal{S}_g \to (M \setminus (C_1 \cup C_2), \nu')$$

that lifts to an (L_1', c_1')-quasi-isometry of the universal covers, where $C^\pm \subset H^\pm$ is a compact core with ν'-diameter bounded by K' and constants L_1', c_1' and K' depend only on R and $\chi(S)$.

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