Automorphisms of complexes of curves and of Teichmüller spaces

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To every compact orientable surface one can associate, following Harvey [Ha1], [Ha2], a combinatorial object, the so-called *complex of curves*, which is analogous to Tits buildings associated to semisimple Lie groups. The basic result of the present paper is an analogue of a fundamental theorem of Tits for these complexes. It asserts that every automorphism of the complex of curves of a surface is induced by some element of the Teichmüller modular group of this surface, or, what is the same, by some diffeomorphism of the surface in question. This theorem allows us to give a completely new proof of a famous theorem of Royden [R] about isometries of the Teichmüller space. In contrast with Royden's proof, which is local and analytic, this new proof is a global and geometric one and reveals a deep analogy between Royden's theorem and the Mostow's rigidity theorem [Mo1], [Mo2]. Another application of our basic theorem is a complete description of isomorphisms between subgroups of finite index of a Teichmüller modular group. This result, in its turn, has some further applications to modular groups.

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1. Statement of the main results.

Let S be a compact orientable surface, possibly with non-empty boundary. The *complex of curves* $C(S)$ of S is a simplicial complex in the sense given to this term in [S], Chapter 3, for example. Thus, it consists of a set of *vertices* and a set of *simplexes*, which are non-empty sets of vertices. The vertices of $C(S)$ are isotopy classes $\langle C \rangle$ of simple closed curves (also called *circles*) C on S, which are *nontrivial*, i.e. are not contractible in S to a point or to ∂S. A set of vertices is declared to be a simplex if and only if these vertices can be represented by (pairwise) disjoint circles. Every diffeomorphism $S \to S$ takes nontrivial circles to nontrivial circles and obviously preserves the disjointness of circles. Thus it defines an automorphism $C(S) \to C(S)$. Clearly, this automorphism depends only on the isotopy class of the diffeomorphism $S \to S$. Hence we get an action of the group of isotopy classes of diffeomorphisms of S on $C(S)$. This group is known as the *mapping class group* of S or as the *Teichmüller modular group* of S. We denote this group by Mod_S. Note that we include the isotopy classes of orientation-reversing diffeomorphisms in Mod_S. (Often this version of the mapping class group is called the *extended* mapping class group.)

Theorem 1. *If the genus of S is at least 2, then all automorphisms of $C(S)$ are given by elements of Mod_S. That is,* $\mathrm{Aut}\,(C(S)) = \mathrm{Mod}_S$.

If S is either a sphere with four holes, or a torus, or a torus with one hole, then $C(S)$ is an (infinite) set of vertices without any edges (i.e. $\dim C(S) = 0$) and the conclusion of this theorem is obviously false. If S is a sphere with at most tree holes, then $C(S)$ is empty and the conclusion of the theorem is vacuous. In the remaining cases of genus 0 or 1 surfaces the question about validity of the conclusion of the theorem was open till recently. Cf. Section 5 for further details.

The role of the complexes of curves in the theory of Teichmüller spaces is similar to the role of Tits buildings in the theory of symmetric spaces of non-compact type. Originally only cohomological aspects of this analogy were discovered; cf., for example, [Ha2], [H] or [I3]. Theorem 1 together with other results of this paper exhibits new sides of this analogy. It is similar to a well-known theorem of Tits [T] asserting that all automorphisms of Tits buildings stem from automorphisms of corresponding algebraic groups. In its turn this theorem of Tits extends the "basic theorem of projective geometry", according to which all maps of a projective space to itself preserving lines, planes, etc. are (projectively) linear.

Theorem 2. *Let Γ_1, Γ_2 be subgroups of finite index of Mod_S. If the genus of S is at least 2 and S is not closed surface of genus 2, then all isomorphisms $\Gamma_1 \to \Gamma_2$ have the form $x \mapsto gxg^{-1}$, $g \in \mathrm{Mod}_S$. If Γ is a subgroup of finite index in Mod_S and if the genus of S is at least 2, then the group of outer automorphisms $\mathrm{Out}\,(\Gamma)$ is finite.*

The second assertion of this theorem obviously follows from the first one, except when S is a closed surface of genus 2. In the case of a closed surface of genus 2 some additional automorphisms can appear, exactly as in [McC], [I4]. This theorem extends the author's theorem [I1], [I4] (cf. also [McC]) to the effect that all automorphisms of Mod_S are inner (except when S is closed surface of genus 2). The assertion about finiteness of $\mathrm{Out}\,(\Gamma)$ proves a conjecture stated in [I2] in connection with this theorem. It is analogous to the Mostow's theorem about finiteness of outer automorphisms groups of lattices in semisimple Lie groups [Mo2].

Theorem 2 is a simple corollary of Theorem 1 given some ideas and results of [I4]. Another application of Theorem 1 is concerned with the Teichmüller space T_S of the surface S. We define the Teichmüller space T_S as the space of isotopy classes of conformal structures on $S \setminus \partial S$ without ideal boundary curves (only with punctures) and consider T_S together with its Teichmüller metric. The modular group Mod_S naturally acts on T_S as a group of isometries.

Theorem 3. *If the genus of S is at least 2, then all isometries of T_S belong to the group Mod_S.*

This theorem is due to Royden [R] for closed surfaces S and to Earle and Kra [EK] for surfaces with non-empty boundary. Theorem 1 allows us to give a completely new proof of this theorem. This new proof follows the same general outline as Mostow's proof [Mo2] of the rigidity theorem for symmetric spaces of rank at least 2. In particular, Theorem 1 plays a role similar to the role of the above mentioned theorem of Tits about automorphisms of buildings in Mostow's proof. The analogy between Royden's theorem and the Mostow rigidity theorem is quite unexpected and was not anticipated before. Some recent remarks by Kra (cf. [Kr], p. 268, footnote [2]) suggest that this new proof may be in some sense the right one.

Note that the conclusion of Theorem 3 is also true for almost all surfaces of genus 0 and 1 and the proof of Earle and Kra [EK] works uniformly well in all cases. Since Theorem 1 was recently extended to most of these surfaces (cf. Section 5), our proof applies to most of the surfaces of genus 0 and 1 also.

Further results along these lines are discussed in Section 5.

2. Sketch of the proof of Theorem 1.

The starting point of the proof is Fig. 1.

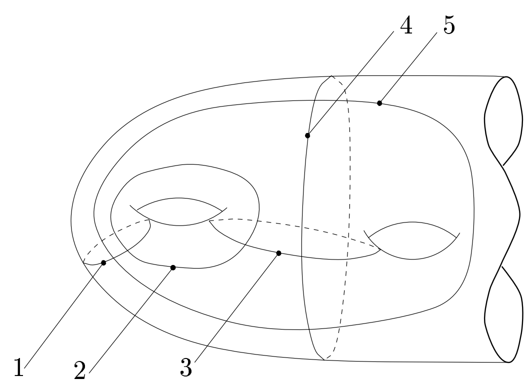

Figure 1.

Lemma 1. *Let α_1, α_2 be isotopy classes of two nontrivial circles on S. The geometric intersection number (in Thurston's sense)* $\mathrm{i}(\alpha_1, \alpha_2)$ *is equal to* 1 *if and only if there exist isotopy classes α_3, α_4, α_5 of nontrivial circles having*

the following two properties:
(i) $\mathrm{i}(\alpha_i,\, \alpha_j) = 0$ *if and only if i-th and j-th circles on Fig. 1 are disjoint;*
(ii) *if α_4 is the isotopy class of a circle C_4, then C_4 divides S into two parts and one of these parts is a torus with one hole containing some representatives of the isotopy classes α_1, α_2.*

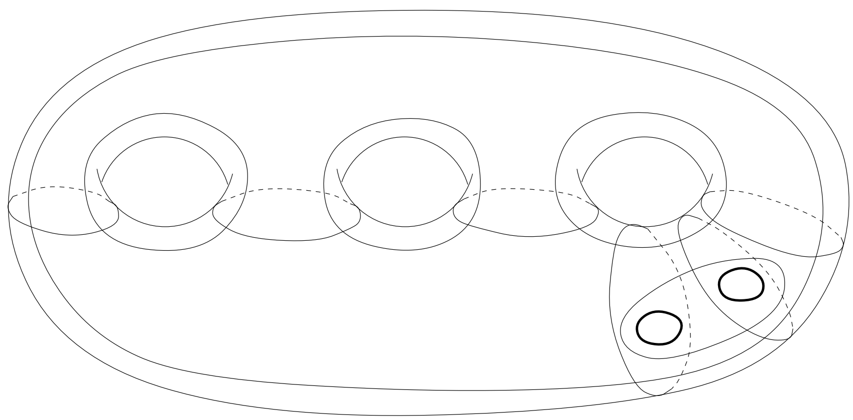

Figure 2.

Note that $\mathrm{i}(\alpha_i,\, \alpha_j) = 0$ if and only if the vertices α_i, α_j are connected by an edge in the complex $C(S)$. It follows that the property (i) can be recognized in $C(S)$ and, hence, is preserved by all automorphisms of $C(S)$. It turns out that the property (ii) also can be recognized in $C(S)$. To see this, start with a vertex $\alpha = \langle C \rangle$ of $C(S)$. Let L_α be the link of α in $C(S)$. Let us consider the graph L^*_α having the same vertices as L_α and having as edges exactly that pairs of vertices that are not connected by an edge in L_α (or, what is the same, in $C(S)$). It is clear that the connected components of L^*_α correspond to the connected components of the result S_C of cutting S along C. After recognizing the components, we can return to the structure of the complex of curves on corresponding sets of vertices. If it is known beforehand that the boundary of a surface R is nonempty, one can recognize the topological type of R using only the structure of a simplicial complex of $C(R)$ in the following way: it is sufficient to use the fact that if $\partial R \neq \emptyset$, then $\dim C(R) = 3g - 4 + b$ and $C(R)$ is homotopy equivalent to a wedge of spheres of dimension $2g - 3 + b$, where g is the genus and b is the number of boundary components of R (at least if $g \geq 1$). The latter result is due to Harer [H]; cf. [I3] for an alternative proof. By applying this remark to $R = S_C$, we see that the property (ii) also can be recognized in $C(S)$ and is preserved by automorphisms of $C(S)$. Hence Lemma 1 implies that the property of two isotopy classes to have the geometric intersection number 1 can be recognized in $C(S)$ and so is preserved by all automorphisms of $C(S)$.

Lemma 1 immediately implies that some useful geometric configurations such as chains $\gamma_1, \gamma_2, \ldots, \gamma_n$ with $\mathrm{i}(\gamma_i, \gamma_{i+1}) = 1$, $\mathrm{i}(\gamma_i, \gamma_j) = 0$ for $\mid i - j \mid > 1$ are mapped by automorphisms of $C(S)$ into similar configurations. More precisely, every automorphisms of $C(S)$ agrees on the set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ with some element of Mod_S. Especially important are the configuration of circles presented in Fig. 2 in the case of 2 boundary components and similar configurations for other surfaces. Lemma 1 implies that every automorphism of $C(S)$ agrees on the set of the isotopy classes of circles in Fig. 2 with some element of Mod_S.

For surfaces R with nonempty boundary we consider in addition to $C(R)$ another complex $B(R)$. Its vertices are the isotopy classes $\langle I \rangle$ of arcs I properly embedded in R (i.e., such that $\partial I \subset \partial R$ and I is transversal to ∂R); it is allowed to move the ends of I during an isotopy, but they are required to remain in the boundary. As in the definition of complexes of curves, a set of vertices is declared to be a simplex if these vertices can be represented by disjoint arcs. It is easy to see that every codimension 1 simplex of $B(R)$ is a face of one or two codimension 0 simplices. Moreover, every two top dimensional simplices Δ, Δ' of $B(R)$ can be connected by a chain of simplices $\Delta = \Delta_1, \ldots, \Delta_m = \Delta'$ such that any two consecutive simplices Δ_i, Δ_{i+1} have a common codimension 1 face. This follows from a well-known theorem about ideal triangulations of Teichmüller spaces (cf., for example, [H]). Apparently, the idea of this theorem is due to Thurston; Mumford, Harer, Penner, Bowditch and Epstein contributed to various proofs of it. A more elementary approach to the existence of such chains was suggested by Hatcher [Hat]. This chain-connectedness property of $B(R)$ immediately implies the following lemma.

Lemma 2. *If an automorphism of $B(R)$ agrees with an element of Mod_R (note that Mod_R obviously acts on $B(R)$) on some simplex of codimension 0, then this automorphism agrees with this element of Mod_R on the whole $B(R)$.*

The vertices of $B(R)$ can be encoded by vertices or pairs of vertices of $C(R)$. For example, let us consider an arc I in R connecting two different components D_1 and D_2 of ∂R. Then the vertex $\langle I \rangle$ is encoded by $\langle C \rangle$, where C is the boundary of some regular neighborhood of $D_1 \cup I \cup D_2$ in R. If I connects a boundary component D of R with itself, then in the most cases $\langle I \rangle$ is encoded by the pair $\{\langle C_1 \rangle, \langle C_2 \rangle\}$, where C_1, C_2 are two components of the boundary of a regular neighborhood of $D \cup I$ in R. In the exceptional cases one of these components is trivial in R and is omitted from the pair, but we keep a record of this in order to be always able to distinguish arcs connecting two different boundary components from arcs connecting a component with itself. This coding allows us to assign an automorphism of $B(R)$ to every automorphism of $C(R)$ preserving the property of having the geometric intersection number 1 (as we saw, every automorphism has this property if the genus ≥ 2).

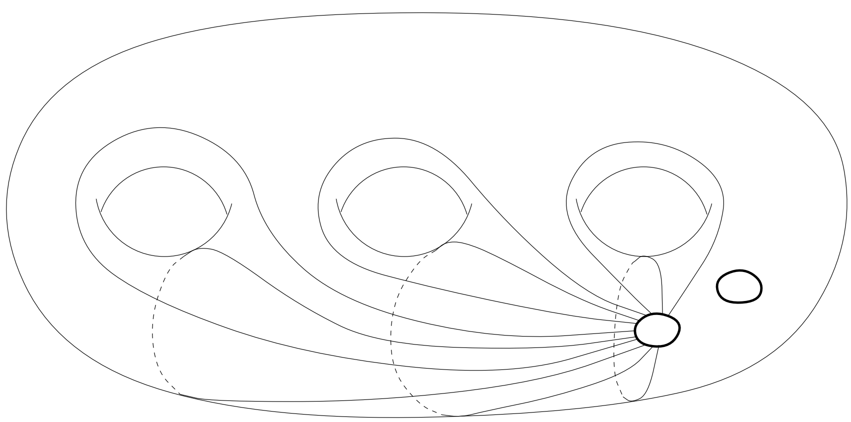

Figure 3.

Suppose now that R has 2 boundary components, as in Fig. 2. The case of surfaces with > 2 boundary is similar, but requires more complicated pictures; the case of surfaces with < 2 boundary components requires additional arguments outlined in the next paragraph. Let us consider an automorphism $G : C(R) \to C(R)$. As we noticed above, it agrees with some element $g \in \mathrm{Mod}_S$ on the set of the isotopy classes of circles in Fig. 2. After replacing G by $g^{-1} \circ G$ we may assume that G fixes all these isotopy classes.

Let us consider the arcs in Fig. 3. The codings of the isotopy classes of all these arcs consist of two isotopy classes of circles. Some of these isotopy classes coincide with the isotopy classes of some circles on Fig. 2. Others are determined by knowing which of their geometric intersection numbers with the isotopy classes of circles on Fig. 2 are equal to 0, 1, or are ≥ 2. It follows that G_* fixes the isotopy classes of all arcs in Fig. 3. Now we need to complete the set of these isotopy classes to a simplex of maximal dimension in $B(R)$. Let us cut R along all arcs in Fig. 3. We get a polygon (with vertices coming from the endpoints of arcs) with one hole. Among the sides of this polygon, $4g$ sides, where g is the genus of R, arise from the boundary component of R containing the endpoints of the arcs. Let us connect the hole in this polygon with these $4g$ sides by disjoint arcs. These arcs obviously define $4g$ arcs in R, and the isotopy classes of these arcs together with the isotopy classes of arcs in Fig. 3 form a simplex of maximal dimension in $B(R)$. It is easy to see that G_* fixes the isotopy classes of the additional $4g$ arcs, and hence fixes a simplex of maximal dimension in $B(R)$. By Lemma 2 G_* is equal to the identity. It follows easily that the original automorphism $G : C(R) \to C(R)$ is also equal to the identity.

This proves Theorem 1 for surfaces with at least two boundary components. The cases of closed surfaces and of surfaces with one boundary component can be reduced to that of surfaces with at least two boundary components by the following arguments. First, if the number of boundary components is ≤ 1, then a circle C is nonseparating if and only if the dual graph L^*_γ is connected, where $\gamma = \langle C \rangle$. It follows that if the number of boundary components is ≤ 1, then every automorphism takes the isotopy classes of nonseparating circles to the isotopy classes of nonseparating circles (this is true for an arbitrary number of boundary components, but the general case is more complicated). Since all nonseparating circles on S are in the same orbit of the group of diffeomorphisms of S, we can assume that our automorphism of $C(S)$ fixes some vertex γ represented by a nonseparating circle C. Such an automorphisms induces an automorphism of the link L_γ, and hence of the complex $C(S_C)$, where S_C is, as above, the result of cutting S along C. This automorphism preserves the property of having the geometric intersection number 1, even if the genus of S_C is less than 2, because this automorphism is equal to the restriction of an automorphism of $C(S)$. Since S_C has at least two boundary components, one can apply previous results to this automorphism of $C(S_C)$ and conclude that it is equal to an element of Mod_{S_C}. Considering different nonseparating circles C (in fact, all such circles), one can deduce that the original automorphism of $C(S)$ agrees with some element of Mod_S.

3. Sketch of the proof of Theorem 2.

Using the technique of [I4], it is not difficult to prove that every isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ takes sufficiently high powers of Dehn twists to powers of Dehn twists. Taking into account the fact that powers of Dehn twists commute if and only if the corresponding circles have the geometric intersection number 0 (i.e., their isotopy classes are connected by an edge in $C(S)$), we see that every isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ induces an automorphism $C(S) \to C(S)$. By Theorem 1 this automorphism is induced by some element $g \in \mathrm{Mod}_S$. This means that for some sufficiently high N we have

$$\varphi(t_\alpha^N) = t_{g(\alpha)}^{M_\alpha}$$

for some $M_\alpha \neq 0$, for all vertices α of $C(S)$. The potential dependence of M_α on α is irrelevant in what follows, and we write simply M for M_α.

Now, let $f \in \Gamma_1$. Then, for any vertex α,

$$\varphi(f t_\alpha^N f^{-1}) = \varphi(t_{f(\alpha)}^N) = t_{g(f(\alpha))}^M.$$

On the other hand,

$$\varphi(f t_\alpha^N f^{-1}) = \varphi(f)\varphi(t_\alpha^N)\varphi(f)^{-1} = \varphi(f) t_{g(\alpha)}^M \varphi(f)^{-1} = t_{\varphi(f)(g(\alpha))}^M.$$

Comparing the results of these two computations, we conclude that

$$\varphi(f)(g(\alpha)) = g(f(\alpha))$$

for all α and (after putting $\alpha = g^{-1}(\beta)$) that $\varphi(f)(\beta) = g \circ f \circ g^{-1}(\beta)$ for all vertices β of $C(S)$. If S is not a closed surface of genus 2, this implies that $\varphi(f) = g \circ f \circ g^{-1}$, i.e. φ has the required form. If S is a closed surface of genus 2, then $\varphi(f)$ can differ from $g \circ f \circ g^{-1}$ by the hyperelliptic involution.

4. Sketch of a geometric proof of Theorem 3.

Let $x \in T_S$. Consider the set R_x of all geodesic rays in T_S starting at the point x. Let M be a Riemann surface with the underlying topological surface S representing the point x. By results of Teichmüller R_x is in a natural bijective correspondence with the set of straight rays starting at 0 in the space Q_M of quadratic differentials on M. By results of Hubbard and Masur [HM] (see also Kerckhoff's paper [K]) the set of rays in Q_M is, in its turn, in a natural bijective correspondence with the space PF_S of projective equivalence classes of measured foliations on S. We denote by $r_{\mu,x}$ or simply by r_μ the ray in T_S starting at x and corresponding to the projective class $[\mu]$ of a foliation $\mu \neq 0$. We consider r_μ as an isometric embedding $\mathbf{R}_{\geq 0} \to T_S$. By $\mathrm{i}(\mu, \nu)$ we denote the geometric intersection number (in Thurston's sense) of two foliations μ, ν.

Lemma 3. *Let $x, y \in T_S$. If $\mathrm{i}(\mu,\nu) \neq 0$, then two rays $r_{\mu,x}$, $r_{\nu,y}$ are* divergent*, i.e. $\lim_{t\to\infty} d(r_{\mu,x}(t), r_{\nu,y}(t)) = \infty$, where $d(\,\cdot\,,\,\cdot\,)$ denotes the Teichmüller distance.*

The proof of this lemma is based on the technique of Kerckhoff [K]. According to Kerckhoff [K], the Teichmüller distance $d(m', m'')$ between two points $m', m'' \in T_S$ is given by the formula

$$d(m', m'') = \frac{1}{2} \log(\sup_\lambda (E_{m'}(\lambda)/E_{m''}(\lambda))),$$

where $E_{m'}(\lambda)$, $E_{m''}(\lambda)$ are the extremal lengths of the foliation λ with respect to the conformal structures representing m', m'', and where λ ranges over all non-zero measured foliations. It turns out that the extremal length of μ tends to 0 along the ray $r_{\mu,x}$, and to ∞ along the ray $r_{\nu,y}$ if $\mathrm{i}(\mu,\nu) \neq 0$. Therefore, the lemma follows from the Kerckhoff's formula.

Let us introduce a relation $\bowtie$ on the space MF_S of (Whitehead equivalence classes of) measured foliations. By the definition, $\mu \bowtie \nu$ if and only if there exist two sequences $\{\mu_i\}_{i=1,2,\ldots}$, $\{\nu_i\}_{i=1,2,\ldots}$ of measured foliations on S such that $\lim_{i\to\infty} \mu_i = \mu$, $\lim_{i\to\infty} \nu_i = \nu$ and for every i two rays $r_{\mu_i,x}$, $r_{\nu_i,x}$ are *not* divergent. Note that $\bowtie$ is not an equivalence relation. Since the intersection number $\mathrm{i}(\,\cdot\,,\,\cdot\,)$ is continuous, Lemma 3 implies that $\mathrm{i}(\mu,\nu) = 0$

if $\mu \bowtie \nu$. We set $\Delta(\mu) = \{\nu : \nu \bowtie \mu\}$ and $\overline{\Delta}(\mu) = \{\nu : \mathrm{i}(\nu, \mu) = 0\}$ for a foliation $\mu \in MF_S$. Clearly, $\Delta(\mu) \subset \overline{\Delta}(\mu)$.

Recall that any nontrivial circle on S (more precisely, its isotopy class, which is a vertex of $C(S)$) gives rise to a measured foliation on S, constructed by the *thickening* of this circle (of course, this foliation is defined only up to the Whitehead equivalence); cf. [FLP].

Lemma 4. *If μ is defined by a circle, then $i(\mu, \nu) = 0$ implies $\mu \bowtie \nu$. Hence, $\Delta(\mu) = \overline{\Delta}(\mu)$ in this case.*

Clearly, to prove this lemma one needs be able to construct non-divergent pairs of rays. This can be done by adapting some ideas of Masur [Ma2] (actually Masur had solved a more subtle problem of constructing *asymptotic* rays starting at different points).

Lemma 5. *A foliation μ is defined by a circle if and only if* $\operatorname{codim} \Delta(\mu) = 1$. *(The codimension is understood to be the codimension in the space MF_S of measured foliations.)*

The proof of this lemma is based on the following ideas. First, one can prove by purely topological arguments that μ is defined by a circle if and only if $\operatorname{codim} \overline{\Delta}(\mu) = 1$. In addition, if μ is defined by a circle, then $\Delta(\mu) = \overline{\Delta}(\mu)$ by Lemma 4, and hence $\operatorname{codim} \Delta(\mu) = 1$. On the other side, if $\operatorname{codim} \Delta(\mu) = 1$, then $\operatorname{codim} \overline{\Delta}(\mu) \leq 1$, because $\Delta(\mu) \subset \overline{\Delta}(\mu)$, and hence $\operatorname{codim} \overline{\Delta}(\mu) = 1$ (the codimension cannot be equal to 0). It follows that μ is defined by a circle.

Consider now an isometry $F : T_S \to T_S$. Take an arbitrary point $x \in T_S$. The isometry F maps the set of rays in T_S starting at x into the set of rays in T_S starting at $F(x)$. Since both these sets are in a natural bijective correspondence with PF_S, we get a map $F_* : PF_S \to PS_S$. Obviously, F_* is a homeomorphism, and, in particular, takes the sets of codimension 1 into sets of codimension 1. In addition, $F_*(\Delta(\mu)) = \Delta(F_*(\mu))$ (because $\Delta(\,\cdot\,)$ is defined in terms of the geometry of rays). By combining these remarks with Lemma 5, we see that F_* preserves the set $V(S) = \{[\mu] : \mu \text{ is defined by a circle}\} \subset PF_S$. Now notice that $V(S)$ is essentially the set of vertices of $C(S)$ and that the induced map $F_{**} : V(S) \to V(S)$ takes pairs of vertices connected by an edge to pairs of vertices connected by an edge. Indeed, two vertices $[\mu]$, $[\nu]$ are connected by an edge if and only if $\mathrm{i}(\mu, \nu) = 0$ and by Lemma 4 this condition is equivalent to $\mu \bowtie \nu$. The last condition $\mu \bowtie \nu$ is defined in terms of the geometry of rays and hence is preserved by isometries. It follows that F_{**} is an automorphism of the complex of curves $C(S)$ (it is well known that a set of vertices is a simplex of $C(S)$ if and only if every two vertices from this set are connected by an edge). Now Theorem 1 implies that F_{**} acts on $C(S)$ as an element f of the modular group Mod_S. Replacing F by $f^{-1} \circ F$, we can assume that $F_{**} = \mathrm{id}$. It remains to prove that in this case $F = \mathrm{id}$.

Let γ_1, γ_2 be two circles on S such that the pair $\{\gamma_1, \gamma_2\}$ *fills* S (this means that there is no nontrivial circle γ on S such that $\mathrm{i}(\gamma, \gamma_1) = \mathrm{i}(\gamma, \gamma_2) = 0$). Such circles can be thickened to a pair μ_1, μ_2 of transverse foliations. Together these two foliations define a conformal structure and a quadratic differential on S (cf. [FLP], Exp. 13 and [Ma3]). In its turn, this quadratic differential defines a geodesic g in T_S, passing through the point x corresponding to this conformal structure. The point x divides g into two rays and these rays, by the construction, correspond to the foliations μ_1, μ_2 (i.e., they are the rays $r_{\mu_1,x}$, $r_{\mu_2,x}$). Since, as we now assuming, $F_{**} = \mathrm{id}$, the isometry F takes g to another geodesic $F(g)$ such that $F(x)$ divides $F(g)$ into two rays corresponding also to μ_1, μ_2. Such a geodesic is necessarily equal to g (cf., for example, the description of the geodesic flow on T_S given in [Ma3]).

Let us consider now one more circle γ_2', filling S together with γ_1. One can choose γ_2' in such a way that $\mathrm{i}(\gamma_2', \gamma_2) \neq 0$. In addition to g let us consider the geodesic g' defined by γ_1, γ_2'. By the previous paragraph $F(g) = g$ and $F(g') = g'$. Clearly F acts on each of these geodesics as a translation. Since these geodesics are not divergent in one direction (the direction corresponding to γ_1; this is an easy application of the ideas of Masur [Ma1], [Ma2]) and are divergent in the other direction (corresponding to γ_2, γ_2'; this follows from Lemma 3), the translation distances are both equal to 0. It follows that F is equal to the identity on g.

Since the union of all such geodesics is dense (in fact, the set of all such geodesics is dense in the space of all geodesic), it follows that $F = \mathrm{id}$. This completes the proof.

5. Further results.

Recently, M. Korkmaz [Ko] extended Theorem 1 to all surfaces of genus 0 and 1 with the exception of spheres with ≤ 4 holes and tori with ≤ 2 holes. Since the conclusion of Theorem 1 is obviously false for spheres with ≤ 4 holes and for tori with ≤ 1 holes, his work left open the question about the computation of $\mathrm{Aut}\,(C(S))$ only in one case, namely, in the case of a torus with 2 holes. His results allow to extend Theorem 2 and the geometric proof of the Royden-Earle-Kra theorem (Theorem 1 is sufficient to prove the Royden's result, concerned only with closed surfaces) to all surfaces with the exception of spheres with ≤ 4 holes and tori with ≤ 2 holes (note that Teichmüller spaces of spheres with 4 holes and of tori with ≤ 1 holes are isometric to the hyperbolic plane, and hence have a continuous group of isometries much bigger than the modular group).

The key point of the Korkmaz's work is an analogue of Lemma 1 for genus 0 and 1 surfaces. Given such an analogue, the rest of the proof generalizes fairly straightforwardly. Note that there is no circles with the geometric intersection number 1 on genus 0 surfaces. They are replaced by the simplest possible pairs of intersecting circles; such circles bound discs with two holes in the surface and have the geometric intersection number 2. It turns out that both the pairs of circles with the geometric intersection number 1 on surfaces of genus 1 and the simplest possible pairs of intersecting circles on surfaces of genus 0 admit a characterization parallel to the characterization of Lemma 1. Amazingly, in all cases a configuration of five circles forming a pentagon in $C(S)$ appears. We refer to [Ko] for further details.

Very recently, F. Luo [L2] suggested a different proof of Theorem 1, still based on the ideas outlined in Section 2 and also on a multiplicative structure on the set of vertices of $C(S)$ introduced in [L1]. His approach allows also to deal with the genus 0 and 1 cases (giving another proof of the results of M. Korkmaz). Also, he observed that $\mathrm{Aut}\,(C(S))$ is not equal to Mod_S if S is a torus with 2 holes. The reason is very simple: If $S_{1,2}$ is a torus with 2 holes, and $S_{0,5}$ is a sphere with 5 holes, then $C(S_{1,2})$ is isomorphic to $C(S_{0,5})$, but $\mathrm{Mod}_{S_{1,2}}$ is not isomorphic to $\mathrm{Mod}_{S_{0,5}}$. Note that the torus with 2 holes is an exceptional case in the Royden-Earle-Kra theorem also, by a similar reason: $T_{S_{1,2}}$ is isometric to $T_{S_{0,5}}$ (cf. [EK], for example).

The following corollary of Theorem 2 is motivated by a conjecture of of Gromov about hyperbolic groups (cf. [G], Section 0.3 (C)). Note that

Teichmüller modular groups are far from being hyperbolic, but often exhibit a hyperbolic behavior.

Theorem 4. *Let* Γ *be a subgroup of finite index in* Mod_S *and let* Γ' *be a torsionless group containing* Γ *as a subgroup of finite index. Then* Γ' *is naturally contained in* Mod_S.

In the proof, we may assume that Γ is normal in Γ' and centerless, replacing, if necessary, Γ by a smaller subgroup. Then the action of Γ' on Γ by conjugation induces a map $\Gamma' \to \mathrm{Aut}\,(\Gamma)$. The facts that Γ' is torsionless, Γ is of finite index in Γ', and Γ is centerless, imply that this map is injective. On the other hand, it follows from Theorem 2 that $\mathrm{Aut}\,(\Gamma)$ is naturally contained in Mod_S.

Another nice application of Theorem 2 is a computation of the abstract commensurators of Teichmüller modular groups. It leads a new proof of the non-arithmeticity of the latter (the question about arithmeticity of Teichmüller modular groups was posed by Harvey [Ha1] and first answered in [I1]). Before stating the results, let us recall the definition of the abstract commensurator.

Let Γ be a group. Let us consider all possible isomorphisms $\varphi : \Gamma_1 \to \Gamma_2$ between subgroups Γ_1, Γ_2 of finite index of Γ. Let us identify two such isomorphisms φ, φ' defined on Γ_1, Γ_1' respectively, if they agree on a subgroup of finite index in the intersection $\Gamma_1 \cap \Gamma_1'$. We can compose them in an obvious manner; the composition $\varphi' \circ \varphi$ of $\varphi : \Gamma_1 \to \Gamma_2$ with $\varphi' : \Gamma_1' \to \Gamma_2'$ is defined on $\varphi^{-1}(\Gamma_2 \cap \Gamma_1')$. Under this composition, the classes of such isomorphisms form a group, which is called the *abstract commensurator* of Γ and is denoted by $\mathrm{Comm}\,(\Gamma)$. There is a natural map $i : \Gamma \to \mathrm{Comm}\,(\Gamma)$, sending an element γ of Γ to the (class of the) inner automorphism $g \mapsto \gamma g \gamma^{-1}$. This map is injective if the centralizers of subgroups of finite index in Γ are trivial, as is the case for arithmetic groups and for Teichmüller modular groups. As a good example, let us mention that $\mathrm{Comm}\,(\mathbf{Z}^n) = SL_n(\mathbf{Q})$.

Theorem 5. *The natural map* $i : \mathrm{Mod}_S \to \mathrm{Comm}\,(\mathrm{Mod}_S)$ *is an isomorphism if* S *is not a sphere with* ≤ 4 *holes or a torus with* ≤ 2 *holes.*

This theorem follows easily from Theorem 2 (and its extension based on the results of Korkmaz [Ko]).

Theorem 6. *If if S is not a sphere with ≤ 4 holes or a torus with ≤ 2 holes, then* Mod_S *is not arithmetic.*

In fact, if Γ is an arithmetic group, then $i(\Gamma)$ is of infinite index in $\mathrm{Comm}\,(\Gamma)$. Cf. [Z], Chapter 6 for a proof. A converse to the latter theorem is also true: if Γ is a *lattice* in a semisimple Lie group G (i.e. if G/Γ has finite invariant volume) and if $i(\Gamma)$ is of infinite index in $\mathrm{Comm}\,(\Gamma)$, then Γ is arithmetic. This result is an immediate corollary of an arithmeticity theorem of Margulis (cf. [M] and also [Z], Chapter 6) and Mostow's rigidity theorem [Mo1], [Mo2]. This converse is much more deep and difficult than the result we are using. While it is not needed to prove Theorem 6, it served as a motivation for the present proof of Theorem 6. In contrast with all previous proofs of the non-arithmeticity of Mod_S, which were based on deep properties of arithmetic groups, this new proof is based only on some (relatively) elementary properties of them.

Our new proof of Theorem 3 leads to an extension of it to the so-called almost isometries. A map $f : X \to Y$ between two metric spaces X, Y with metrics $d_X(\cdot,\cdot)$, $d_Y(\cdot,\cdot)$ respectively is called an *almost isometry* if for all $x, y \in X$ we have

$$d_X(x,y) - C \leq d_Y(f(x), f(y)) \leq d_X(x,y) + C,$$

and if , in addition, the distance of every point of Y from $f(X)$ is $\leq C$, where C is some constant depending only on f. Note that an almost isometry does not have to be even continuous.

Theorem 7. *Suppose that S is not a sphere with ≤ 4 punctures and not a torus with ≤ 2 punctures. Then for any almost isometry $f : T_S \to T_S$ of the Teichmüller metric d_T there exist an isometry (induced by an element of* Mod_S*) $g : T_S \to T_S$ such that for all $x \in T_S$ the distance*

$$d_T(f(x), g(x)) \leq C,$$

where C is some constant depending only on f.

In contrast with the Royden's theorem itself, this result is purely *global*: under the assumptions of the theorem, we don't have any local information about f whatsoever. A first result of this sort, for almost isometries of Hilbert spaces, was proved by Hyers and Ulam in the forties [HU]. The key new ingredient in the proof of Theorem 7, compared with the proof of Section 4, is the fact that the image of any geodesic ray under an almost-isometry converges to a set of points in the Thurston's boundary PF_S of T_S pairwise related by $\bowtie$. Note that in view of recent results of Minsky [Mi], a direct extension of Theorem 7 to the more wide class of *quasi-isometries* seems to be unlikely.

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