GEOMETRIC STRUCTURES AND VARIETIES OF REPRESENTATIONS

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Abstract. Many interesting geometric structures on manifolds can be interpreted as structures locally modelled on homogeneous spaces. Given a homogeneous space (X, G) and a manifold M, there is a *deformation space* of structures on M locally modelled on the geometry of X invariant under G. Such a geometric structure on a manifold M determines a representation (unique up to inner automorphism) of the fundamental group π of M in G. The deformation space for such structures is "locally modelled" on the space $\mathrm{Hom}(\pi, G)/G$ of equivalence classes of representations of $\pi \to G$. A strong interplay exists between the local and global structure of the variety of representations and the corresponding geometric structures. The lecture in Boulder surveyed some aspects of this correspondence, focusing on: (1) the "Deformation Theorem" relating deformation spaces of geometric structures to the space of representations; (2) representations of surface groups in $\mathrm{SL}(2;\mathbb{R})$, hyperbolic structures on surfaces (with singularities), Fenchel-Nielsen coordinates on Teichmüller space; (3) convex real projective structures on surfaces; (4) representations of Schwarz triangle groups in $\mathrm{SL}(3;\mathbb{C})$. This paper represents an expanded version of the lecture.

Introduction

According to Felix Klein's Erlanger program of 1872, geometry is the study of the properties of a space which are invariant under a group of transformations. A geometry in Klein's sense is thus a pair (X, G) where X is a manifold and G is a Lie group acting (transitively) on X. If (X, G) is Euclidean geometry (so $X = \mathbb{R}^n$ and G is its group of isometries), then one might "infinitesimally model" a manifold M on Euclidean space by giving each tangent space a Euclidean geometry, which varies from point to point; the resulting *infinitesimally Euclidean* geometry is a Riemannian metric. Infinitesimally modelling a space on other homogeneous spaces gives rise to other G-structures and connections. But to model a space *locally* on a geometry (and not just

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infinitesimally) requires certain integrability conditions to be satisfied; these can be expressed in terms of vanishing curvature. For example, a flat Riemannian metric is a *locally* Euclidean structure since every point has a neighborhood isometric to an open subset of Euclidean space. The study of such geometric structures was initiated in general by Charles Ehresmann [13] under the name of "locally homogeneous structures." By assuming a standard local structure, the existence and classification of such structures becomes a global topological problem, involving the fundamental group of the underlying space and in particular its representations in the structure group of the geometry. One constructs a "space of geometric structures" on a given manifold, the points of which correspond to equivalence classes of geometric structures; this *deformation space* is itself "locally modelled" on the space of equivalence classes of representations of the fundamental group. This paper surveys several aspects of the deformation theory of geometric structures on a manifold and its relation to the variety of representations of the fundamental group in a Lie group.

Let G be a Lie group acting transitively on a manifold X and let M be a manifold of the same dimension as X. An (X, G)-*atlas* on M is a pair $(\mathcal{U}, \Phi)$ where $\mathcal{U}$ is an open covering of M and $\Phi = \{\phi_\alpha : U_\alpha \longrightarrow X\}_{U_\alpha \in \mathcal{U}}$ is a collection of coordinate charts such that for each pair $(U_\alpha, U_\beta) \in \mathcal{U} \times \mathcal{U}$ and connected component C of $U_\alpha \cap U_\beta$ there exists $g_{C,\alpha,\beta} \in G$ such that $g_{C,\alpha,\beta} \circ \phi_\alpha = \phi_\beta$. An (X, G)-*structure* on M is a maximal (X, G)-atlas and an (X, G)-*manifold* is a manifold together with an (X, G)-structure on it. It is clear that an (X, G)-manifold has an underlying real analytic structure, since the action of G on X is real analytic.

Suppose that M and N are two (X, G)-manifolds and $f : M \longrightarrow N$ is a map. Then f is an (X, G)-*map* if for each pair of charts $\phi_\alpha : U_\alpha \longrightarrow X$ and $\psi_\beta : V_\beta \longrightarrow X$ (for M and N respectively) and a component C of $U_\alpha \cap f^{-1}(V_\beta))$ there exists $g = g(C, \alpha, \beta) \in G$ such that the restriction of f to C equals $\psi_\beta^{-1} \circ g \circ \phi_\alpha$. In particular we only consider (X, G)-maps which are local diffeomorphisms. It is easy to see that if $f : M \longrightarrow N$ is a local diffeomorphism where M and N are smooth manifolds, then for every (X, G)-structure on N, there is a unique (X, G)-structure for which f is an (X, G)-map. In particular every covering space of an (X, G)-manifold has a canonical (X, G)-structure. In the converse direction, if $\Gamma \subset G$ is a discrete subgroup which acts properly and freely on X then X/Γ is an (X, G)-manifold. Such a structure is said to be *complete.*

1. Examples of Geometric Structures

Many familiar geometric structures are (X, G)-structures, for various choices of X and G. Here is a list of a few of them:

1.1. **Euclidean structures.**

Here $X = \mathbb{R}^n$ and G is the group of all isometries. An (X, G)-structure is then identical to a flat Riemannian metric. Completeness of the Euclidean structure is equivalent to geodesic completeness of the metric which is in turn equivalent to completeness as a metric space. Thus a closed Euclidean manifold is covered by $\mathbb{R}^n$ and by Bieberbach's theorem is finitely covered by a flat torus. Furthermore every homotopy equivalence between closed Euclidean manifolds is homotopic to an affine diffeomorphism.

1.2. **Locally homogeneous Riemannian manifolds.**

A natural generalization of Euclidean structures are constant (nonzero) curvature manifolds — a manifold with a Riemannian metric of constant sectional curvature is locally isometric either to Euclidean space, the sphere, or hyperbolic space, depending on whether the curvature is zero, positive, or negative respectively. If X is one of these model spaces and G is its isometry group, then a constant curvature metric on a manifold M is exactly an (X, G)-structure on M.

This is part of the following more general construction. Suppose that (X, g_X) is a Riemannian manifold upon which a Lie group G acts transitively by isometries. (This is basically equivalent to requiring that the homogeneous space $X = G/H$ have the property that the isotropy group H is compact.) Then every (X, G)-manifold M inherits a Riemannian metric g_M locally isometric to g_X. Each coordinate chart $\phi_\alpha : U_\alpha \longrightarrow X$ determines a Riemannian metric $\phi_\alpha^*(g_X)$ on U_α; since G acts isometrically on X with respect to g_X, any two of these local metrics agree on the intersection of two coordinate patches. Thus there exists a metric g_M on M whose restriction to U_α equals $\phi_\alpha^*(g_X)$. It is easily seen that any Riemannian metric on M which is everywhere locally isometric to X arises from an (X, G)-structure as above. This collection of structures includes all the geometric structures used to uniformize 2- and 3-dimensional manifolds in the sense of the geometrization conjecture of Thurston [67] (see also Scott [61]). Completeness of the (X, G)-structure as above is equivalent to geodesic completeness of the Riemannian metric and hence metric completeness; see Thurston [66] for discussion. Since every such (X, G)-structure on a closed manifold M is complete and hence determines a subgroup $\Gamma \subset G$ which will be discrete and cocompact, the subject of (X, G)-structures on closed manifolds is essentially equivalent to the study of discrete cocompact subgroups of G which act freely on X.

A basic example is the following. If S is a closed orientable surface of genus $g > 1$, then the deformation space of hyperbolic structures on S is a manifold, the *Teichmüller space* $\mathfrak{T}(S)$ of S, and is homeomorphic to a cell of dimension $6g - 6$. One can alternatively describe $\mathfrak{T}(S)$ as the quotient of the space of discrete embeddings $\pi_1(S) \hookrightarrow \mathrm{PGL}(2;\mathbb{R})$ by the action of $\mathrm{PGL}(2;\mathbb{R})$ by conjugation.

1.3. **Locally homogeneous indefinite metrics.** Other interesting geometric structures are modelled on homogeneous pseudo-Riemannian manifolds, where the invariant metric tensor is allowed to be indefinite. The arguments that compact implies complete completely break down; however no example of a closed manifold with an incomplete locally homogeneous indefinite metric seems to be known. Complete flat pseudo-Riemannian metrics on closed manifolds have been classified in dimension 3 (Fried-Goldman [18]), dimension 4 and signature (1,3) (Fried [17]), dimension 4 and signature (2,2) (Wang [71]), and a general structure theorem has been obtained for flat Lorentz metrics (Goldman-Kamishima [28]).

As observed and analyzed by Kulkarni and Raymond [49], a large class of Seifert-fibered 3-manifolds admit *constant negative curvature* Lorentz metrics. In particular every Seifert 3-manifold over hyperbolic base having nonzero (rational) Euler class has such a structure. Such structures are modelled on the Lie group $X = \mathrm{PSL}(2;\mathbb{R})$ with automorphism group generated by left-multiplications and automorphisms of the group $\mathrm{PSL}(2;\mathbb{R})$. The most tractable class of such structures are those which Kulkarni and Raymond call "standard" — these are characterized by having a timelike Killing vector field on the time-orientable double covering. Equivalently, such structures have a Riemannian metric locally isometric to a left-invariant Riemannian metric on $\mathrm{PSL}(2;\mathbb{R})$. Clearly every standard structure is complete and Kulkarni-Raymond [49] prove that a closed 3-manifold which admits a standard structure must be a Seifert manifold of the above type (the deformation theory of standard structures is worked out in Kulkarni-Lee-Raymond [47]). However nonstandard complete structures exist, although every closed 3-manifold which admits a complete structure also admits a standard structure and therefore is a Seifert 3-manifold over hyperbolic base with nonzero Euler class ([22]).

1.4. **Flat affine structures.** Here $X = \mathbb{R}^n$ and $G = \mathrm{Aff}(n;\mathbb{R})$ is the group of affine transformations. An (X, G)-manifold here is called an *affinely flat* or just an *affine* manifold and an *affine structure* on manifold M is just a flat torsionfree affine connection on M. In the absence of a compatible metric, an affine manifold does not possess a canonical sense of distance; yet there is a well-defined notion of *parallel*. Thus while it makes no sense to speak of a particle moving at "unit speed" on an affine manifold (as there is on a Riemannian manifold) there is nonetheless a notion of moving at "constant speed" along a straight line. Geodesic completeness can then be expressed as the property that a particle moving at constant speed in a straight line will continue forever. Geodesic completeness in this sense is equivalent to completeness of the (X, G)-structure above. Complete affine manifolds are discussed in Milnor [55] and the classification of complete affine structures on closed 3-manifolds is given in Fried-Goldman [18]. It is conjectured that a *compact* complete affine manifold is finitely covered by quotients of solvable Lie groups with left-invariant complete affine structures; this conjecture is equivalent to the assertion that the fundamental group of a compact complete affine manifold contains a solvable subgroup of finite index. Such structures can be classified in terms of representations of solvable Lie algebras. A simple case occurs for complete affine structures on the two-torus, where there are two types of left-invariant complete affine structures; the deformation space of complete affine structures on the two-torus may be identified with the real plane $\mathbb{R}^2$ with the topology whose only proper open sets are the open subsets of $\mathbb{R}^2 - \{0\}$; in particular in even such a simple example as this the deformation space is non-Hausdorff. A homotopy equivalence between two compact complete affine manifolds with virtually solvable fundamental group is generally not necessary homotopic to an affine diffeomorphism, but is homotopic to a *polynomial* diffeomorphism (reminiscent of the rigidity in Bieberbach's theorem); see [18] for details. Boyom [3] has recently shown that every nilmanifold admits a homogeneous complete affine structure. Recently Margulis [52,53] has constructed a very surprising example of a *noncompact* complete flat affine 3-manifold (in fact a flat Lorentz manifold) whose fundamental group is a nonabelian free group.

Kostant and Sullivan [44] proved that the Euler characteristic of a compact complete affine manifold must vanish, affirming a conjecture first stated by Chern. Benzécri [2] proved that a closed surface which admits a (possibly incomplete) affine structure must have zero Euler characteristic. It is unknown in higher dimension whether the Euler characteristic of a compact affine manifold in higher dimensions is zero; for more information on affine structures on compact manifolds, see

Goldman-Hirsch [27], Kobayashi [42], Shima [62] and the references cited there.

1.5. **Flat projective structures.** Let X be a projective space (either real or complex) and G its group of projective automorphisms. In traditional terminology, a *(flat) projective structure* modelled on this projective geometry is a flat normal projective connection. Projective structures on Riemann surfaces (called in [24] "$\mathbb{C}\mathbf{P}^01$-structures") arose classically in the study of second order differential equations on Riemann surfaces and in connection with the uniformization problem. For a survey of the analytic theory of $\mathbb{C}\mathbf{P}^01$-structures, the reader is referred to Gunning [32] (see also Earle [11], Hejhal [34], Hubbard [37], Kra-Maskit [45], and the references cited there). The deformation space $\mathbb{C}\mathbf{P}^01(S)$ of $\mathbb{C}\mathbf{P}^01$-structures on a closed orientable surface S of genus $g > 1$ can be naturally identified with the cotangent bundle of the Teichmüller space of S. An alternative description of this space is due to Thurston (unpublished), who identifies $\mathbb{C}\mathbf{P}^01(S)$ with the (trivial) bundle over Teichmüller space $\mathfrak{T}(S)$ having for fiber the the space $\mathfrak{ML}(S)$ of measured geodesic laminations on S. This identification

$$\Theta : \mathbb{C}\mathbf{P}^01(S) \longrightarrow \mathfrak{T}(S) \times \mathfrak{ML}(S)$$

is based on the fundamental insight that a $\mathbb{C}\mathbf{P}^01$-structure can be identified with a locally convex pleated map of the universal covering surface into hyperbolic 3-space (compare Epstein-Marden [14] and Canary-Epstein-Green [7]); the parameters in $\mathfrak{T}(S) \times \mathfrak{ML}(S)$ describe the intrinsic geometry and the extrinsic geometry (the bending parameters) of the pleated surface respectively.

Projective structures modelled on the real projective plane (iė˙ $\mathbb{R}\mathbf{P}^02$-structures) are not nearly as well understood. In 1976, J. Smillie [63] and W. Thurston [65] independently discovered an $\mathbb{R}\mathbf{P}^02$-structure on the 2-torus whose developing map (see below) fails to be a covering onto its image (the complement of 3 points in $\mathbb{R}\mathbf{P}^02$). Moreover, every closed surface of genus $g > 1$ enjoys an $\mathbb{R}\mathbf{P}^02$-structure whose developing map is a surjection onto $\mathbb{R}\mathbf{P}^02$. Even so, the deformation space of $\mathbb{R}\mathbf{P}^02$-structures on a compact surface S with $\chi(S) < 0$ is a Hausdorff real analytic manifold of dimension $-8\chi(S)$ (Goldman [26]). This is a consequence of the fact that the holonomy group of an $\mathbb{R}\mathbf{P}^02$-structure on a closed surface with $\chi(S) < 0$ cannot fix a point in $\mathbb{R}\mathbf{P}^02$; however it seems to be unknown whether the holonomy group can leave invariant a projective line in $\mathbb{R}\mathbf{P}^02$ (such a line would necessarily intersect the developing image, by the result of Benzècri above). It seems that a tractable class of $\mathbb{R}\mathbf{P}^02$-structures are the *convex* $\mathbb{R}\mathbf{P}^02$*-manifolds,* iė˙ $\mathbb{R}\mathbf{P}^0 2$-manifolds of the form Ω/Γ, where $\Omega \subset \mathbb{R}\mathbf{P}^0 2$ is a convex domain and $\Gamma \subset \mathrm{PGL}(3;\mathbb{R})$ is a discrete group acting properly and freely on Ω. As a special case, the hyperbolic structures on S determine convex $\mathbb{R}\mathbf{P}^0 2$-structures on S and thus the Teichmüller space of S lies inside the deformation space $\mathcal{P}(S)$ of convex $\mathbb{R}\mathbf{P}^0 2$-structures on S. Projective structures modelled on $\mathbb{C}\mathbf{P}^0 n$ for $n \geq 2$ are discussed in Kobayashi-Ochiai [41] as well as Gunning [31] and Yoshida [74].

1.6. **Flat conformal structures.** Here $X = \mathrm{S}^n$ and $G = \mathrm{SO}(n+1,1)$ is the group of conformal transformations of X where $n > 2$. An (X,G)-structure on M is the same as a conformal class of Riemannian metrics, each locally conformal to a flat metric; hence such structures are called *flat conformal structures.* A large class of closed 3-manifolds admit flat conformal structures, although relatively simple ones might not — in particular an nontrivial oriented S^1-bundle over a 2-torus admits no such structure [21] (an analytic proof of this is contained in the theory developed in the recent paper of Schoen-Yau [60]). However, Gromov, Lawson and Thurston [30] have produced startling examples of flat conformal structures on certain nontrivial S^1-bundles over closed hyperbolic surfaces. Hyperbolic structures determine flat conformal structures; although hyperbolic structures on closed manifolds of dimension ≥ 3 satisfy Mostow rigidity (every homotopy equivalence is homotopic to a unique isometry), there is often a rich supply of deformations of the flat conformal structure; see Apanosov [1], Gusevskii-Kapovich [76], Johnson-Millson [38], Kamishima [75], Kapovich [40,77], Korouniotis [43], Kulkarni [46], Kulkarni-Pinkall [48], Millson [56] and Schoen-Yau [60] for further discussion.

1.7. **Spherical CR structures.** Here $X = \mathrm{S}^{2n-1}$ the boundary of the unit ball in $\mathbb{C}^n$ and $G = \mathrm{PU}(n,1)$ is the group of biholomorphisms of the unit ball acting on its boundary by CR-automorphisms. An (X,G)-structure on a real manifold M^{2n-1} is then a CR-structure all of whose local invariants (the analogue of curvature) vanish. Every Seifert 3-manifold with nonzero Euler class admits such a structure; however, the 3-torus is not sufficiently complicated to admit such a structure [21]. No example seems to be currently known of a closed hyperbolic 3-manifold which admits such a structure, although one expects them to exist in abundance. For a general discussion of these geometric structures, see Burns-Shnider [5] and Ehlers-Neumann-Scherk [12].

2. The Graph of a Geometric Structure

We seek a concrete geometric object which so to speak is the "graph" of a geometric structure. We begin by graphing the coordinate charts. For each coordinate chart (U_α, ϕ_α) we consider the following information:

(1) The product $E_\alpha = U_\alpha \times X$ is an X-bundle over the coordinate patch U_α with projection $\pi_\alpha : E_\alpha \longrightarrow U_\alpha$;
(2) The sets $U_\alpha \times \{x\}$, parametrized by $x \in X$, are the leaves of a foliation $\mathcal{F}_\alpha$ of U_α transverse to the fibers of $\pi_\alpha : E_\alpha \longrightarrow U_\alpha$;
(3) Graphing the coordinate chart $\phi_\alpha : U_\alpha \longrightarrow X$ defines a section $\Phi_\alpha : U_\alpha \longrightarrow E_\alpha$ of the bundle E_α which is transverse to $\mathcal{F}_\alpha$ (because ϕ_α is nonsingular).

This local data pieces together globally as follows. Since the coordinate charts on overlapping coordinate patches differ by elements of G, the bundles E_α glue together to form a fiber bundle $\pi : E \longrightarrow M$ with fiber X and structure group G in the sense of Steenrod [64]. Since the action of G on X is *strongly effective,* (two elements of G which agree on a nonempty open set are identical), there is a foliation $\mathcal{F}$ of the total space E whose restriction to each E_α equals $\mathcal{F}_\alpha$. Since the foliation is defined by submersions to X in local foliation boxes E_α and the coordinate changes lie in G, the foliation has a *transverse* (X, G)*-structure* and the pair $(\pi : E \longrightarrow M, \mathcal{F})$ is a *flat* (X, G)*-bundle* over M. (In the terminology of Steenrod [64] a flat (X, G)-bundle is a bundle whose fiber is X and whose structure group is the group G^{discrete}, iė˙ the group G endowed with the discrete topology. A flat structure on an (X, G)-bundle $\pi : E \longrightarrow M$, iė˙ a reduction of the structure group of E to G^{discrete} is a foliation $\mathcal{F}$ as above transverse to the fibers of E whose holonomy lies in G such that for each leaf $L \subset E$, the restriction $\pi|_L : L \longrightarrow M$ is a covering space.)

Thus the "graph" of an (X, G)-structure on M consists of the following: a flat (X, G)-bundle $\pi : E \longrightarrow M$, called the *tangent flat* (X, G)*-bundle to* M, or the (X, G)*-holonomy bundle of* M, and a section $f : M \longrightarrow E$ which is transverse to the flat structure $\mathcal{F}$, called the *developing section.* We call the triple $(E, \mathcal{F}, f)$ the *graph* of the (X, G)-manifold and denote it by graph(M).

Suppose that $(E, \mathcal{F})$ is a flat (X, G)-bundle over M and let $f : M \longrightarrow E$ be a section which is transverse to $\mathcal{F}$. Choose a covering of the total space E by foliation boxes — open sets $U \subset E$ equipped with submersions $\psi_U : U \longrightarrow X$ defining the flat (X, G)structure $\mathcal{F}|_U$ on U. The inverse images $f^{-1}(U)$ define an open covering of M for which the compositions $\psi_U \circ f : f^{-1}(U) \longrightarrow X$ define coordinate charts for an (X, G)-structure on M. Thus we obtain:

Lemma 2.1. *Let $(E, \mathcal{F})$ be a flat (X, G)-bundle over M. Then a section $M \longrightarrow E$ is a developing section for an (X, G)-structure on M if and only if it is transverse to $\mathcal{F}$.*

A flat (X, G)-bundle as above arises from a homomorphism of the fundamental group (the *holonomy homomorphism*) $\Gamma = \pi_1(M)$ into G as follows. Let $\rho : \Gamma \longrightarrow G$ be a homomorphism. Let $\mathbf{p} : \tilde{M} \longrightarrow M$ be a universal covering space of M with covering group Γ. Then Γ acts on the trivial (X, G)-bundle $\tilde{E} = \tilde{M} \times X$ over $\tilde{M}$ by

$$\gamma : (\tilde{m}, x) \mapsto (\gamma\tilde{m}, \rho(\gamma) \cdot x)$$

where $\gamma \in \Gamma, \tilde{m} \in \tilde{M}, x \in X$. Since the action of Γ on $\tilde{M}$ is properly discontinous and free, so is the action on $\tilde{E}$ defined by (1) and it follows that the projection $\tilde{E} \longrightarrow \tilde{M}$ makes the quotient $E_\rho = \tilde{E}/\Gamma$ the total space of an (X, G)-bundle over $M = \tilde{M}/\Gamma$. The foliation of $\tilde{E}$ by leaves $\{\tilde{M} \times x\}_{x \in X}$ is a flat (X, G)-structure invariant under the Γ-action (1) and hence defines a flat structure $\mathcal{F}_\rho$ on E_ρ.

If $(\pi : E \longrightarrow M, \mathcal{F})$ is a flat (X, G)-bundle over M, then it is isomorphic to a flat (X, G)-bundle $(E_{\rho, \mathcal{F}_\rho})$ for a representation ρ as above. To see this, let $m_0 \in M$ be a basepoint and consider the fiber $E_0 = \pi^{-1}(m_0)$ over m_0. Let $\mathbf{p} : \tilde{M} \longrightarrow M$ be the universal covering space corresponding to m_0: a point of $\tilde{M}$ corresponds to a relative homotopy class of paths $\sigma : [0, 1] \longrightarrow M$ with $\sigma(0) = m_0$ as above and the covering projection $\mathbf{p}$ is the evaluation map $\sigma \mapsto \sigma(1)$ at $t = 1$. We shall trivialize the pullback $\mathbf{p}^*E$ by the flat structure. Let $\sigma : [0, 1] \longrightarrow M$ be a path with $\sigma(0) = m_0$. For each point $e_0 \in E_0$ there exists a unique path $\tilde{\sigma}^{e_0} : [0, 1] \longrightarrow \tilde{M}$ with $\tilde{\sigma}^{e_0}(0) = e_0$ which is "$\mathcal{F}$-horizontal:" $\tilde{\sigma}^{e_0}(t)$ lies in a single leaf of $\mathcal{F}$. The correspondence

$$\begin{aligned} E_0 \times \tilde{M} &\longrightarrow \mathbf{p}^*E \\ (e_0, \sigma) &\mapsto \tilde{\sigma}^{e_0}(1) \end{aligned}$$

defines a trivialization of $\mathbf{p}^*E$ as a flat (X, G)-bundle. Choose an identification $E_0 \longrightarrow X$; in other words, trivialize the bundle E over m_0 as an (X, G)-bundle. Then E is recovered from $\mathbf{p}^*E$ as a quotient of $\mathbf{p}^*E \cong \tilde{M} \times X$ by an action of $\pi_1(M; m_0)$ which extends the action of $\pi_1(M; m_0)$ on $\tilde{M}$ and comes from the action of G on the fibers. Such an action is given by the above construction applied to a representation $\rho : \pi_1(M; m_0) \longrightarrow G$. Now G acts simply transitively on the trivializations of E_0; applying $g \in G$ to the trivialization $E_0 \longrightarrow X$ composes the holonomy representation ρ with the inner automorphism $G \longrightarrow G$ determined by g.

We may now interpret a section of the flat (X, G)-bundle $(E, \mathcal{F})$ associated to a representation ρ of the fundamental group as the graph of an equivariant map of the universal covering $\tilde{M}$ as follows. Since $\mathbf{p}^*E$ is the trivial (X, G)-bundle $\tilde{M} \times X$ over $\tilde{M}$, a section $\tilde{M} \longrightarrow \mathbf{p}^*E$ is just the graph of a map $\tilde{M} \longrightarrow X$. Suppose that $f : M \longrightarrow E$ is a section; then f pulls back to a section $\tilde{M} \longrightarrow \mathbf{p}^*E \cong \tilde{M} \times X$ which must be the graph of a map $\tilde{f} : \tilde{M} \longrightarrow X$ satisfying the equivariance condition

$$\tilde{f} \circ \gamma = \rho(\gamma) \circ \tilde{f}.$$

If f is the developing section for an (X, G)-structure, then the corresponding equivariant map $\tilde{f}$ is called a *developing map* for the (X, G)-structure. The condition that f is $\mathcal{F}$-transverse is just the condition that the developing map $\tilde{f}$ is a nonsingular smooth map (iė˙ a local diffeomorphism).

A pair $(\tilde{f}, \rho)$ is called a *development pair* and is a useful globalization of an (X, G)-structure defined by local coordinates. The developing map pulls back the (X, G)-structure from X to $\tilde{M}$ and thus defines the (X, G)-structure on $\tilde{M}$. The holonomy homomorphism determines the action of $\pi_1(M)$ on $\tilde{M}$ by (X, G)-automorphisms. Thus a development pair completely determines the (X, G)-structure.

We have already seen that isomorphism classes of flat (X, G)-bundles over M correspond bijectively to G-orbits of representations $\pi_1(M) \longrightarrow G$; a well-defined holonomy representation is obtained once one chooses a basepoint $m_0 \in M$ and a trivialization τ_0 of the bundle $E_0 = E|_{m_0}$ over the base-point. Suppose that $f : M \longrightarrow E$ is a section; then the corresponding equivariant map $\tilde{f} : \tilde{M} \longrightarrow X$ is determined by the trivialization of $\mathbf{p}^*E$ corresponding to m_0 and τ_0. A given $g \in G$ changes τ_0 to $g \cdot \tau_0$ and the corresponding developing map is the composition $g \circ \tilde{f}$. Thus a development pair $(\tilde{f}, \rho)$ is determined up to the diagonal action of G obtained by composing $\tilde{f} : \tilde{M} \longrightarrow X$ with an element $g \in G$ and $\rho : \pi_1(M; m_0) \longrightarrow G$ with conjugation by g.

3. Deformation spaces

We wish to construct "a space of (X, G)-structures" on a compact smooth manifold S. To this end we consider pairs (ϕ, M) where $\phi : S \longrightarrow M$ is a diffeomorphism and M is an (X, G)-manifold. We call such a pair an *(X, G)-structure on S*; two (X, G)-structures (ϕ, M) and (ϕ', M') on S are *isotopic* if there is an (X, G)-map $h : M \longrightarrow M'$

(necessarily an (X, G)-isomorphism) such that ϕ' is isotopic to $h \circ \phi$. In other words, we consider the orbit of $\phi : S \longrightarrow M$ under the group $\mathrm{Diff}_0(S)$ acting by right-composition.

Before defining a topology on the space of isotopy classes of (X, G)-structures on S, we construct a larger space of "based" structures on S. Choose a base-point $s_0 \in S$ and let $\pi = \pi_1(S, s_0)$ and $\tilde{S} \to S$ be the corresponding fundamental group and universal covering spaces. We topologize the set $\mathrm{Hom}(\pi, G)$ with the compact-open topology, which (since π is finitely generated) equals the topology given by pointwise convergence of homomorphisms. If M is a (X, G)-manifold and $m_0 \in M$, we define an *(X, G)-germ at m_0* to be a germ of an (X, G)-structure at m_0, iė˙ the germ of an (X, G)-map from a neighborhood N of m_0 into X. Clearly G acts simply transitively on the set of (X, G)-germs at m_0. Consider the set $\mathcal{D}'_{(X,G)}(S)$ of triples (ϕ, M, Ψ) where $\phi : S \longrightarrow M$ is a diffeomorphism to an (X, G)-manifold M and Ψ is an (X, G)-germ at $\phi(s_0)$. Let $\mathrm{graph}(M) = (E, \mathcal{F}, f)$ be the graph of the (X, G)-manifold M as above. Now the (X, G)-germ Ψ at $m_0 = f(s_0)$ defines an identification of the fiber E_0 over $m_0 = \phi(s_0)$ with X; thus there is a well-defined development pair $(\tilde{f}, \rho)$. The preceding discussion allows us to interpret elements $\mathcal{D}'_{(X,G)}(S)$ alternatively in terms of their graphs as well as their development pairs:

$$\begin{aligned}\mathcal{D}'_{(X,G)}(S) &= \Big\{(\phi, M, \Psi) \mid \phi : S \to M \text{ is a diffeomorphism,} M \text{ is an } (X, G) - \text{manifold, and} \Psi \text{ is a} \\ &= \Big\{(E, \mathcal{F}, f, \bar{\Psi}) \mid (E, \mathcal{F}) \text{ is a flat } (X, G) - \text{bundle over } S, f : S \to E \text{ is a } \mathcal{F}\text{-transvers} \\ &= \Big\{(\tilde{f}, \rho) \mid \rho \in \mathrm{Hom}(\pi, G), \tilde{f} : \tilde{S} \to X \text{ is a } \rho\text{-equivariant nonsingular smooth map}\Big\}\end{aligned}$$

We can topologize $\mathcal{D}'_{(X,G)}(S)$ using the C^∞ topology either on the developing maps $\tilde{f}$, or in terms of the graphs $(E, \mathcal{F}, f)$ as follows. A neighborhood of $(E_1, \mathcal{F}_1, f_1)$ consists of all flat (X, G)-bundles $(E, \mathcal{F})$ over S such that the tangent spaces to the flat structure $\mathcal{F}$ are close in the C^∞ topology to those of $\mathcal{F}_1$ and $\mathcal{F}$-transverse sections f which are C^∞-close to f_1. It is clear these two topologies agree and that the map

$$\mathrm{hol}' : \mathcal{D}'_{(X,G)}(S) \longrightarrow \mathrm{Hom}(\pi, G)$$

associating to $(\tilde{f}, \rho)$ the holonomy representation ρ is continuous. Let $\mathrm{Diff}_0(\mathrm{S}, \mathrm{s}_0)$ denote the identity component in the group of all diffeomorphisms $S \to S$ which fix s_0. Moreover $\mathrm{Diff}_0(\mathrm{S}, \mathrm{s}_0)$ acts properly and freely on $\mathcal{D}'_{(X,G)}(S)$ and hol' is invariant under this action. Furthermore the group G acts on $\mathcal{D}'_{(X,G)}(S)$ by composition with Ψ and on

$\mathrm{Hom}(\pi, G)$ by left-composition with innner automorphisms and hol' is equivariant with respect to these G-actions.

Theorem 3.1 (Deformation Theorem)**.** *The map*

$$\mathrm{hol}' : \mathcal{D}'_{(X,G)}(S) \longrightarrow \mathrm{Hom}(\pi, G)$$

is open and for each $u \in \mathcal{D}'_{(X,G)}(S)$, there exists a neighborhood $U \subset \mathcal{D}'_{(X,G)}(S)$ such that for each $\rho \in \mathrm{hol}'(U)$ and $u_i = (E_i, \mathcal{F}_i, f_i, \Psi_i) \in U \cap \mathrm{hol}'^{-1}(\rho)$ (where $i = 1, 2$), there exists $h \in \mathrm{Diff}_0(\mathrm{S}, \mathrm{s}_0)$ and $g \in G$ such that $(h, g) \cdot u_1 = u_2$.

Proof. Suppose that $u_0 = (E_0, \mathcal{F}_0, f_0, \bar{\Psi}_0) \in \mathcal{D}'_{(X,G)}(S)$ and let $\rho_0 = \mathrm{hol}'(u_0)$. We claim there exists a neighborhood V of $\rho_0 \in \mathrm{Hom}(\pi, G)$ such that the associated flat (X, G)-bundles $(E_v, \mathcal{F}_v)$ over M for $v \in V$ satisfy the following properties. First, by the covering homotopy property there are bundle isomorphisms $E_v \cong E$; indeed a smooth trivialization of the corresponding smooth family of (X, G)-bundles over V exists. Thus we assume that $E_v = E$; with respect to this identification the foliations $\mathcal{F}_v$ vary continuously in the C^∞ topology. By choosing V small enough, the original developing section f_0 remains transverse to $\mathcal{F}_v$ for all $v \in V$. By the previous lemma, this section is a developing section for a nearby (X, G)-structure on M. Thus there is a nearby structure on S with holonomy ρ_v; it follows that hol' is an open map.

To prove that $\mathrm{Diff}_0(\mathrm{S}, \mathrm{s}_0)$ acts simply transitively on the nearby fibers of hol', consider a tubular neighborhood W of the developing section $f_0(S) \subset E_0$ fibered by the intersections of leaves of $\mathcal{F}_0$ with W. Let U be a neighborhood of $u_0 \in \mathcal{D}'_{(X,G)}(S)$ such that $f(S) \subset W$ for $(E, \mathcal{F}, f, \bar{\Psi}) \in U$. For $u = (E, \mathcal{F}, f, \bar{\Psi}) \in U \cap \mathrm{hol}'^{-1}(\rho_0)$ we identify E with E_0 and since the holonomy $\mathrm{hol}'(u) = \rho_0$ of the flat (X, G)-bundle is constant, we identify $\mathcal{F}$ with $\mathcal{F}_0$. Then $u \in U \cap \mathrm{hol}'^{-1}(\rho_0)$ is completely determined by the section f_u and since $f(S) \subset W$, for each $s \in S$ let $h_u(s) \in S$ equal $f_0^{-1}(L_u(s))$ where $L_u(s)$ is the leaf of $\mathcal{F}$ containing $f_u(s)$. This defines the required isotopy joining f_u to f_0. □

This seems to be the most complete general statement one can make concerning the relationship of deformation spaces of geometric structures and spaces of homomorphisms of the fundamental group. This result seems to have been first observed in this generality by Thurston [66,§5. 3. 1] although numerous special cases seem to have been known previously (Weil [72], Hejhal [34], Earle [11], Hubbard [37]); a more detailed proof of this result, based on Weil [72], is given in Lok[50]. Proofs have been given by Dennis Johnson (unpublished) and

Canary-Epstein-Green [7]. The proof given above was worked out with Moe Hirsch in 1978 and was also known to André Haefliger.

Let $\mathcal{D}_{(X,G)}(S)$ denote the quotient of $\mathcal{D}'_{(X,G)}(S)$ by $\mathrm{Diff}_0(\mathrm{S}, \mathrm{s}_0)$ and let $\mathrm{Hom}(\pi, G)/G$ denote the space of G-orbits of $\mathrm{Hom}(\pi, G)$ with the quotient topology. The map hol' defines a local homeomorphism

$$\mathrm{hol} : \mathcal{D}_{(X,G)}(S) \longrightarrow \mathrm{Hom}(\pi, G).$$

We define the *deformation space* of (X, G)-structures on S to be the quotient

$$\mathfrak{T}_{(X,G)}(S) = (\mathcal{D}_{(X,G)}(S))/G$$

and there is a map

$$\mathrm{hol} : \mathfrak{T}_{(X,G)}(S) \longrightarrow \mathrm{Hom}(\pi, G)/G$$

which one would like to say is a local homeomorphism. In general this seems to be difficult, since one has little control over the orbits of the action of G.

Suppose that G is the group of $\mathbb{R}$-points of an algebraic group $\bar{G}$ defined over $\mathbb{R}$; then $\mathrm{Hom}(\pi, G)$ is the set of $\mathbb{R}$-points of an algebraic variety $\mathfrak{R}(\pi, \bar{G})$ defined over $\mathbb{R}$ (compare Johnson-Millson [38], Lubotzky-Magid [51]). Furthermore the action of G by conjugation on $\mathrm{Hom}(\pi, G)$ arises from an algebraic action of $\bar{G}$ on $\mathfrak{R}(\pi, \bar{G})$ and thus the orbits are locally closed. If $\bar{G}$ is reductive, then there is a distinguished open subset $\mathrm{Hom}(\pi, G)^{\mathrm{st}}$ of $\mathrm{Hom}(\pi, G)$ consisting of stable orbits, upon which the action of G is proper. (The algebraic-geometric quotient — denoted $X(\pi, G)$ and called the *character variety,* following Culler-Shalen [9] — is the closure of the space of stable orbits. When G is a linear $\mathbb{C}$-algebraic group, $X(\pi, G)$ may be identified with the space of all characters of representations $\pi \to G$.) Since G acts properly on any space from which there exists a G-equivariant map to a proper G-space, we obtain the following:

Corollary 3.2. *The restriction of* hol *to the subset*

$$\mathrm{hol}^{-1}(\mathrm{Hom}(\pi, G)^{\mathrm{st}}/G) \subset \mathcal{D}_{(X,G)}(S)$$

is a local homeomorphism.

Thus it follows that the local properties of $\mathrm{Hom}(\pi, G)/G$ translate into local properties of the deformation space $\mathfrak{T}_{(X,G)}(S)$ of (X, G)-structures on S. In the case considered above, $\mathrm{Hom}(\pi, G)^{\mathrm{st}}/G$ has the structure of an $\mathbb{R}$-algebraic space and therefore so does $\mathfrak{T}_{(X,G)}(S)$. If π is the fundamental group of a compact Kähler manifold, then by Goldman-Millson [29] in many cases the singularities of $\mathrm{Hom}(\pi, G)$ are described by systems of homogeneous quadratic equations in the

Zariski tangent space. The corresponding quotient space has singularities which are quotients of such quadratic cones and one obtains similar information on the singularites of $\mathcal{D}_{(X,G)}(S)$ in this case. See Johnson-Millson [38] for more examples.

If π is the fundamental group of a compact Kähler manifold,

$$\mathrm{Hom}(\pi, G)/G$$

often inherits a Kähler structure of its own (see Corlette [8]) and thus $\mathfrak{T}_{(X,G)}(S)$ also inherits such structure. Although the Kähler structure on $\mathfrak{T}_{(X,G)}(S)$ will depend on the choice of complex structure on the underlying manifold S, often the *symplectic geometry* is independent of the complex structure. In particular, if S is a closed surface, then the deformation space $\mathfrak{T}_{(X,G)}(S)$ admits a symplectic structure which will be invariant under the natural action of the mapping class group of S. This general construction includes as a special case the symplectic geometry arising from the Weil-Petersson Kähler form on Teichmüller space.

For constant negative curvature Lorentzian metrics on twisted S^1-bundles over closed hyperbolic surfaces, the deformation theorem was used in [22] to construct nonstandard complete examples. Indeed, the examples given in [22] demonstrate that the deformation space $\mathfrak{T}_{(X,G)}(S)$ of such structures is non-Hausdorff (even though G is semisimple, its isotropy group on X is semisimple and S is a compact quotient of a semisimple Lie group).

In [24] global properties of the holonomy map

$$\mathrm{hol} : \mathbb{C}\mathbf{P}^0 1(S) \longrightarrow \mathrm{Hom}(\pi, \mathrm{PSL}(2, \mathbb{C}))/\,\mathrm{PSL}(2, \mathbb{C})$$

are investigated. In particular the fiber$\mathrm{hol}^{-1}(\mathfrak{T}(S))$ over the set of equivalence classes of discrete embeddings $\pi \longrightarrow \mathrm{PGL}(2, \mathbb{R})$ is identified as the set $\Theta^{-1}(\mathfrak{T}(S) \times \mathfrak{ML}_{\mathbf{Z}}(S)$ corresponding to integral measured laminations (iė˙ disjoint families of nontrivial simple closed curves) in Thurston's parametrization of complex projective structures. There is a similar statement for real projective structures on surfaces (although no analogue of the Thurston parametrization is currently known); $\mathbb{R}\mathbf{P}^0 2$-structures on a compact surface S whose holonomy is Fuchsian can similarly classified in terms of disjoint families of simple closed curves weighted by elements of a certain elementary discrete semigroup (see [24] for details).

Using similar methods, one can show that the class of $\mathbb{C}\mathbf{P}^0 1$-structures with holonomy representation $\pi_1(S) \longrightarrow \mathrm{PGL}(2, \mathbb{R})$ equals

$$\Theta^{-1}(\mathfrak{T}(S) \times \mathfrak{ML}_{\frac{1}{2}\mathbf{Z}}(S)$$

where $\mathfrak{ML}_{\frac{1}{2}\mathbf{Z}}(S)$ is the space of *half-integral* points in $\mathfrak{ML}(S)$. (Laminations in $\mathfrak{ML}_{\frac{1}{2}\mathbf{Z}}(S)$ are disjoint families of nontrivial simple closed curves weighted by half-integers.) There is a map

$$H : \mathfrak{ML}_{\frac{1}{2}\mathbf{Z}}(S) \longrightarrow H_1(M; \mathbb{Z}/2)$$

obtained by summing the homology classes of these curves with twice the given weights. In [24] it is incorrectly stated the half-integral points correspond to representations into $\mathrm{PSL}(2, \mathbb{R})$. The correct statement is the following. Let M be a $\mathbb{C}\mathbf{P}^0 1$-manifold whose holonomy lies in $\mathrm{Hom}(\pi, \mathrm{PGL}(2; \mathbb{R}))$ and suppose that its Thurston parameters are

$$\Theta(M) = (g, \lambda) \in \mathfrak{T}(S) \times \mathfrak{ML}(S)$$

where $\lambda \in \mathfrak{ML}_{\frac{1}{2}\mathbf{Z}}(S)$ is half-integral. Then the holonomy of M lies in $\mathrm{PSL}(2; \mathbb{R})$ if and only if $H(\lambda) = 0$. Furthermore the representations $\pi \longrightarrow \mathrm{PGL}(2; \mathbb{R})$ which occur as holonomy representations of $\mathbb{C}\mathbf{P}^0 1$-structures correspond (under the isomorphism $\mathrm{PGL}(2; \mathbb{R}) \cong \mathrm{SO}(2, 1)$) exactly to the representations $\pi_1(S) \longrightarrow \mathrm{SO}(2, 1)$ which occur as holonomy representations of $\mathbb{R}\mathbf{P}^0 2$-structures. In [19] it is shown that such representations are characterized as those whose image is not solvable.

4. Global properties of deformation spaces

In the above applications, the geometry of the variety of representations was used to deduce information on the deformation space of geometric structures. In the converse direction, geometric structures may be used to shed light on the geometry and topology of the spaces of homomorphisms. For example, if M is an (X, G)-manifold with graph $(E, \mathcal{F}, f)$, then a tubular neighborhood of the developing section f inside E is isomorphic to the tangent bundle of M. This idea can be used to investigate the space of representations of a surface group inside $G = \mathrm{PSL}(2; \mathbb{R})$. Let X be the hyperbolic plane; then $\mathfrak{T}_{(X,G)}(S)$ equals the Teichmüller space of M and it is shown in [20] that a flat (X, G)-bundle $(E, \mathcal{F})$ over M admits a transverse section if and only if the Euler class of E equals $\pm\chi(M)$. Indeed as shown in [25] the Euler class of E characterizes the connected components of $\mathrm{Hom}(\pi, G)$. To understand the other components of $\mathrm{Hom}(\pi, G)$ one introduces hyperbolic structures with singularities as follows, (as first explained to me by W. Neumann). Instead of requiring all of the coordinate charts to be local diffeomorphisms, one allows charts which at isolated points look like the map

$$\mathbb{C} \longrightarrow \mathbb{C} \tag{4.1}$$

$$z \mapsto z^k \tag{4.2}$$

(We say that such a chart defines a singularity of “cone angle” $\theta = 2k\pi$.) Such a singular hyperbolic structure may be alternatively described as a singular Riemannian metric whose curvature equals -1 plus Dirac distributions weighted by $2\pi - \theta$ at each singular point of cone angle θ. In particular a hyperbolic structure on M with isolated singularities at points $x_i, (i = 1, \dots, k)$ of cone angle $\theta_i > 0$ each of which is a multiple of 2π has a graph $(E, \mathcal{F}, f)$ such that the Euler class is given by

$$e(E) = \chi(M) + \frac{1}{2\pi} \sum_{i=1}^{k} (\theta_i - 2\pi)$$

(It is convenient to assume that each $\theta_i = 4\pi$ and the points x_i are not necessarily distinct — a cone point of cone angle 4π with multiplicity m is then a cone point with cone angle $2(m+1)\pi$.) There is a uniformization theorem for such singular hyperbolic structures, due independently to McOwen [54], Troyanov [70], and Hitchin [35], that given a Riemann surface M, there exists a unique singular hyperbolic structure in the conformal class of M with cone angle θ_i at x_i for $i = 1, \dots, k$ as long as

$$\chi(M) + \frac{1}{2\pi} \sum_{i=1}^{k} (\theta_i - 2\pi) \leq 0.$$

(Hitchin only considers the case when θ_i are multiples of 2π, while McOwen and Troyanov deal with arbitrary positive angles.) The resulting “uniformization map” then assigns to the collection of points $\{x_1, \dots, x_k\}$ (where $0 \leq k \leq |\chi(S)|$) the singular hyperbolic structure with cone angles 4π (counted with multiplicity) at the x_i. The equivalence class of the holonomy representation in the component

$$e^{-1}(\chi(M) + k) \subset \mathrm{Hom}(\pi, G)/G$$

defines a map from the symmetric product $\Sigma^k(M)$ to $e^{-1}(\chi(M) + k)$. The following remarkable result is due to Hitchin [35]:

Theorem 4.1. *The above map*

$$\Sigma^k(M) \longrightarrow e^{-1}(\chi(M) + k)$$

is a homotopy equivalence.

Indeed, Hitchin obtains much more information; for a particular choice of complex structure on M, the component $e^{-1}(\chi(M)+k)$ has the structure of a holomorphic vector bundle over $\Sigma^k(M)$. See [35] for more details as well as a discussion of the rich geometry of these spaces.

Another striking use of geometric structures on surfaces with cone singularities has recently been given by Thurston [68]. For a certain class of singularities, Thurston has shown that the deformation space of Euclidean structures on the 2-sphere is a complex hyperbolic manifold, itself with cone singularities. The developing map for these complex hyperbolic manifolds is the holonomy map taking the space of developments of Euclidean structures on S^2 to a certain space of representations of the fundamental group of S^2 punctured at the various singular points. In certain cases these moduli spaces are complex hyperbolic orbifolds and some of the recent examples of complex hyperbolic manifolds discovered by Deligne-Mostow [10] are recovered in this way as moduli spaces of singular Euclidean structures on S^2. Geometric structures on 3-manifolds with cone singularities play a crucial role in Thurston's proof of the generalized Smith conjecture; see Hodgson [36] for details. Spherical structures with cone singularities on closed surfaces of higher genus have been considered recently by Calabi [6].

5. Building deformation spaces

One approach for understanding the global structure of spaces of representations and deformation spaces in general is to build them up via a decomposition of the manifold M. Since every closed surface can be decomposed into simpler subsurfaces, one might hope that deformation spaces of geometric structures on surfaces might be tractable from this point of view. This is the spirit of the Fenchel-Nielsen coordinates on Teichmüller space, which gives an explicit homeomorphism of the Teichmüller space of a compact orientable surface with an open cell. We shall outline proofs of the following theorems:

Theorem 5.1. *Let S be a closed orientable surface of genus $g > 1$.*

- *The Teichmüller space $\mathfrak{T}(S)$ is diffeomorphic to a cell of dimension $6g-6$.*
- *The deformation space $\mathcal{P}(S)$ of convex $\mathbb{R}\mathbf{P}^0 2$-structures on S is diffeomorphic to a cell of dimension $16-16$.*

Suppose that S is a compact manifold containing a hypersurface $F \subset S$ such that $S - F$ has two components S_1 and S_2. After appropriate base-points and connecting arcs are chosen, the fundamental group of

S admits an amalgamated free product decomposition

$$\pi_1(S) = \pi_1(S_1) \amalg_{\pi_1(F)} \pi_1(S_2)$$

and we shall assume that the maps

$$\pi_1(F) \longrightarrow \pi_1(S_i) \longrightarrow \pi_1(S)$$

induced by the inclusions

$$F \hookrightarrow S_i \hookrightarrow S$$

are injective. There is a corresponding restriction map

$$\mathrm{Hom}(\pi_1(S), G) \longrightarrow \mathrm{Hom}(\pi_1(S_1), G) \times \mathrm{Hom}(\pi_1(S_2), G)$$

which is injective and whose image consists of pairs (ρ_1, ρ_2) of representations whose restriction to $\pi_1(F)$ coincide.

We shall be interested in the relationship between the character varieties $X(\pi_1(S), G)$ and $X(\pi_1(S_i), G)$, iė˙ the spaces of conjugacy classes of stable representations. For that reason we shall restrict our attention to representations $\rho : \pi_1(S) \to G$ whose restrictions to $\pi_1(S_i)$ are stable. In the case when S is a compact surface and G is a reductive $\mathbb{R}$-algebraic group, it is sufficient to assume that each image $\rho(\pi_1(S_i))$ has discrete centralizer. We henceforth assume this is the case; in all of our applications this condition will be satisfied. Passing to equivalence classes there is a restriction map

$$R : X(\pi_1(S), G) \longrightarrow X(\pi_1(S_1), G) \times X(\pi_1(S_2), G)$$

whose image consists of pairs of characters (χ_1, χ_2) of representations which coincide on $\pi_1(F)$. If (χ_1, χ_2) is such a pair, then its inverse image $R^{-1}(\chi_1, \chi_2)$ can be identified as follows. If ρ is a representation whose restriction ρ_i to $\pi_1(S_i)$ corresponds to χ_i, then the centralizer $\mathcal{Z}(\rho(\pi_1(F)))$ of $\rho(\pi_1(F))$ acts by conjugation on ρ_1 and can be amalgamated with ρ_2 to define a new representation $\pi_1(S) \to G$ as follows. Let $\zeta \in \mathcal{Z}(\pi_1(F))$; then

$$T_\zeta \rho : \gamma \mapsto \begin{cases} \zeta \rho_1(\gamma) \zeta^{-1} & \text{if } \gamma \in \pi_1(S_1) \\ \rho_2(\gamma) & \text{if } \gamma \in \pi_1(S_2) \end{cases}$$

is a representation whose character lies in $R^{-1}(\chi_1, \chi_2)$. One can show that this action is transitive on $R^{-1}(\chi_1, \chi_2)$. (See Goldman [23,25], Johnson-Millson [38] for more details of this construction.)

When $S - F$ is connected, there is a similar discussion. In this case $\pi_1(S)$ decomposes as an HNN-extension of $\pi_1(S - F)$ corresponding to the two embeddings $\iota_i : \pi_1(F) \to \pi_1(S)$, iė˙ $\pi_1(S)$ admits a presentation with generators $\pi_1(S - F) \amalg \langle \eta \rangle$ with relations $\iota_1(\gamma) = \eta \iota_2(\gamma) \eta^{-1}$. Let

$\theta : \pi_1(S) \longrightarrow \mathbb{Z}$ be the homomorphism containing $\pi_1(S - F) \subset \pi_1(S)$ in its kernel and $\theta(\eta) = 1$. The image of the restriction map

$$\mathrm{Hom}(\pi_1(S), G) \longrightarrow \mathrm{Hom}(\pi_1(S - F), G)$$

consists of representations $\rho \in \mathrm{Hom}(\pi_1(S - F), G)$ such that $\rho \circ \iota_1$ is conjugate to $\rho \circ \iota_2$. The centralizer $\mathcal{Z}(\rho(\pi_1(F)))$ acts on the set of representations whose character lies in the fiber of the restriction map

$$X(\pi_1(S), G) \longrightarrow X(\pi_1(S - F), G)$$

as follows. If $\zeta \in \mathcal{Z}(\rho(\pi_1(F)))$, then define

$$T_\zeta \rho : \gamma \mapsto \rho(\gamma)\zeta^{\theta(\gamma)}$$

If we restrict to representations ρ such that the restriction to $\pi_1(S_F)$ is stable, then $\mathcal{Z}(\rho(\pi_1(S_F)))$ acts transitively on the fiber of

$$R : X(\pi_1(S), G) \to X(\pi_1(S - F), G).$$

In the cases we consider below, one can easily prove that this action is actually *simply transitive*.

Of particular interest is the case when S is a compact surface and F is a homotopically nontrivial simple closed curve. We consider two cases, both with $G = \mathrm{PGL}(n; \mathbb{R})$: hyperbolic structures on closed surfaces ($n = 2$) and convex real projective structures on closed surfaces ($n = 3$). In both cases the holonomy representation ρ is a stable point in $\mathrm{Hom}(\pi_1(S - F), G)$ and the above discussion applies. Furthermore for each $\gamma \subset \pi_1(F)$, the image $\rho(\gamma)$ is a *positive hyperbolic* element of $\mathrm{PGL}(n; \mathbb{R})$ — represented by a real matrix with distinct positive real eigenvalues. In both cases conjugacy classes of hyperbolic elements are parametrized in terms of invariants arising from the coefficients of the characteristic polynomial and form a cell of dimension $n - 1$. A hyperbolic element of $\mathrm{PGL}(2; \mathbb{R})$ has a lift to $\mathrm{SL}(2; \mathbb{R})$ whose trace lies in the interval $(-\infty, 2)$; we use this interval to parametrize hyperbolic conjugacy classes. Similarly, the coefficients (x, y) of the characteristic polynomial $t^3 - xt^2 + yt - 1$ of a positive hyperbolic element of $\mathrm{PGL}(3; R) \cong \mathrm{SL}(3; \mathbb{R})$ lies in the set

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2y^2 - 4(x^3 + y^3) + 18xy - 27 > 0, \quad x > 0, y > 0\}$$

and the conjugacy classes of positive hyperbolic elements in $\mathrm{PGL}(3; \mathbb{R})$ are parametrized by the open 2-cell $\mathcal{R}$.

Closely related are the centralizers of positive hyperbolic elements. If $A \in \mathrm{PGL}(n; R)$ is positive hyperbolic, then its centralizer corresponds to the corresponding conjugate of the subgroup of diagonal matrices ($\mathbb{R}$-split Cartan subgroup) containing A. Thus the centralizer of A is an $(n - 1)$-dimensional vector group. In particular the space of positive hyperbolic conjugacy classes in $\mathrm{PGL}(n;\mathbb{R})$ and the centralizer of a positive hyperbolic element are each diffeomorphic to an open $(n-1)$-cell (a manifestation that the rank of $\mathrm{PGL}(n;\mathbb{R})$ equals $n-1$).

We consider the case of hyperbolic structures ($n = 1$) first, deriving the Fenchel-Nielsen coordinates on Teichmüller space. Let $G = \mathrm{PGL}(2;\mathbb{R})$. Let S be a closed orientable surface of genus $g > 1$ and choose a decomposition $\mathfrak{P}$ of S into pairs-of-pants (surfaces diffeomorphic to a sphere minus three discs); since $\chi(S) = 2 - 2g$ and the Euler characteristic of a pair-of-pants equals -1, there are exactly $2g-2$ pants $P_1, \ldots, P_{2g-2}$ in the decomposition. Since ∂P_i consists of 3 decomposition curves, and each decomposition curve abuts two pants, there are exactly $\frac{3}{2}(2g - 2) = 3g - 3$ decomposition curves $C_1, \ldots, C_{3g-3}$. Consider the restriction map

$$R_{\mathfrak{P}} : X(\pi_1(S), G) \longrightarrow \prod_{i=1}^{2g-2} X(\pi_1(P_i), G)$$

By the above discussion, there is an $\mathbb{R}$-action on each fiber corresponding to each C_j for $j = 1, \ldots, 3g-3$; the disjointness of the curves implies that these $\mathbb{R}$-actions commute ([23]) and the resulting $\mathbb{R}^{3g-3}$-action on each fiber

$$R_{\mathfrak{P}}^{-1}(\chi_1, \ldots, \chi_{2g-2})$$

is simply transitive. Thus it suffices to determine the character variety for the fundamental group of a pair-of-pants, which is a free group of rank 2. This is accomplished by the following classical result, which is essentially due to Fricke-Klein [16] (compare Harvey [33], [25]).

Proposition 5.1. *Let $\pi = \langle A, B, C | ABC = I\rangle$ be a presentation of the free group of rank two. Define $\chi : \mathrm{Hom}(\pi, \mathrm{SL}(2;\mathbb{C})) \longrightarrow \mathbb{C}^3$ by $\chi(\rho) = (\operatorname{tr}\rho(A), \operatorname{tr}\rho(B), \operatorname{tr}\rho(C))$. Then χ is invariant under the action of $\mathrm{SL}(2;\mathbb{C})$ by conjugation and if the image of $\rho_0 \in \mathrm{Hom}(\pi, \mathrm{SL}(2;\mathbb{C}))$ is not solvable, then the inverse image $\chi(\chi^{-1}(\rho_0))$ equals the $\mathrm{SL}(2;\mathbb{C})$-orbit of ρ_0. For each $\chi_0 \in (-\infty, -2) \times (-\infty, -2) \times (-\infty, -2)$ there exists a representation $\rho_0 \in \mathrm{Hom}(\pi, \mathrm{SL}(2;\mathbb{R})$ with $\chi(\rho_0) = \chi_0$ and this representation is the holonomy representation of a hyperbolic structure of a pair of pants P with geodesic boundary. Conversely every hyperbolic structure on P with geodesic boundary has holonomy representation in $\chi^{-1}((-\infty, 2)^3)$.*

Geometrically, the last assertion means that a hyperbolic structure on a pair-of-pants P with geodesic boundary is determined up to isometry by the lengths of the boundary components (the length $l(A)$ of a boundary component is related to the holonomy $\rho(A) \in \mathrm{SL}(2;\mathbb{R})$ by

$\operatorname{tr}\rho(A) = \pm 2\cosh(l(A)/2))$. Thus the restriction map is described by the map

$$\Theta_{\mathfrak{P}} : \mathfrak{T}(S) \longrightarrow (-\infty, -2)^{3g-3} \tag{5.1}$$

$$[\rho] \mapsto (\operatorname{tr}\rho(C_1), \ldots, \operatorname{tr}\rho(C_{3g-3})) \tag{5.2}$$

assigning to a representation the conjugacy-invariants of the image of the family of decomposition curves. Thus the Teichmüller space is expressed as a principal $\mathbb{R}^{3g-3}$ bundle over $\mathbb{R}^{3g-3}$. We note that the character map $\Theta_{\mathfrak{P}}$ above is a moment map for the action of $\mathbb{R}^{3g-3}$ on its fibers; see Goldman [23] for more details.

There is a similar description of the deformation space $\mathcal{P}(S)$ of convex $\mathbb{R}\mathbf{P}^0 2$-structures over S. Once again we consider a pair-of-pants decomposition

$$S = \bigcup_{i=1}^{2g-2} P_i$$

where $C_1, \ldots, C_{3g-3}$ are the decomposition curves. If M is a convex $\mathbb{R}\mathbf{P}^0 2$-manifold diffeomorphic to S, then the family $\{C_1, \ldots, C_{3g-3}\}$ is isotopic to a disjoint family of simple closed geodesic curves cutting M into a disjoint union of convex $\mathbb{R}\mathbf{P}^0 2$-manifolds, each diffeomorphic to a pair-of-pants. Furthermore if γ is the holonomy around such a curve C, the development of $\tilde{C} \subset \tilde{M}$ is a segment joining the attracting fixed point of γ on $\mathbb{R}\mathbf{P}^0 2$ to the repelling fixed point of γ on $\mathbb{R}\mathbf{P}^0 2$; such a curve is said to be *principal* in [26]. Let $\mathcal{P}(P)$ denote the deformation space of convex $\mathbb{R}\mathbf{P}^0 2$-structures on P with principal geodesic boundary. Recording the conjugacy class of the three boundary components defines a map $\Theta_{\partial P} : \mathcal{P}(P) \longrightarrow \mathcal{R}^3$. The following fact is proved in [26,§5].

Proposition 5.2. *The map*

$$\Theta_{\partial P} : \mathcal{P}(P) \longrightarrow \mathcal{R}^3$$

is a fibration with fiber $\mathbb{R}^2$.

If M_1 and M_2 are two $\mathbb{R}\mathbf{P}^0 2$-manifolds with principal geodesic boundary components C_1 and C_2 respectively and $f : N(C_1) \longrightarrow N(C_2)$ is a projective isomorphism between collar neighborhoods, then there is a unique $\mathbb{R}\mathbf{P}^0 2$-structure on the identification space of $M = M_1 \cup M_2$ (where C_1 and C_2 are to be identified to give a curve $C \subset M$) such that there is a reflection in C defined in a tubular neighborhood of C realizing f. One can show ([26, 3. 7]) that if M_1 and M_2 are convex, then so is M. Thus to build convex $\mathbb{R}\mathbf{P}^0 2$-manifolds out of convex structures on pants, one needs only to know the structures on the pants and the projective maps needed to identify collar neighborhoods of their boundary components.

If $C \subset M$ is a principal simple closed geodesic, then the germ of the $\mathbb{R}\mathbf{P}^0 2$ structure on a tubular neighborhood $N(C)$ of C (or equivalently on each component of $N(C) - C$) is completely determined by the conjugacy class of the holonomy $\rho(C)$. As noted above, the possible conjugacy classes are parametrized by an open 2-cell $\mathcal{R}$ and the possible identifications are described by the centralizer of a hyperbolic element of $\mathrm{PGL}(3;\mathbb{R})$, which is isomorphic to $\mathbb{R}^2$. Thus the restriction map

$$R_{\mathfrak{P}} : \mathcal{P}(S) \longrightarrow \prod_{i=1}^{2g-2} \mathcal{P}(P_i)$$

is a principal $(\mathbb{R}^2)^{3g-3}$-fibration onto its image, which can be determined as follows. The $\mathbb{R}\mathbf{P}^0 2$-structures near the decomposition curves C_j are classified by the map

$$R_{\mathfrak{P}}(\mathcal{P}(S)) \longrightarrow \mathcal{R}^{3g-3} \tag{5.3}$$

$$[\rho] \mapsto ([\rho(C_1)], \dots, [\rho(C_{3g-3})]) \tag{5.4}$$

whose fiber is a product of $2g-2$ cells of dimension 2, one for each P_i. Thus the image $R(\mathcal{P}(S))$ is a cell of dimension $2(3g-3)+2(2g-2) = 10g-10$ and $\mathcal{P}(S)$ is a cell of dimension $2(3g-3)+(10g-10) = 16g-16$.

Once again the map $\mathcal{P}(S) \longrightarrow \mathcal{R}^{3g-3}$ is the moment map for an action of $(\mathbb{R}^2)^{3g-3}$. It seems likely that a refinement of the coordinates obtained in [26] for $\mathcal{P}(S)$ will be canonical coordinates, just as the Fenchel-Nielsen coordinates are canonical coordinates for the Weil-Petersson Kähler form on Teichmüller space as shown by Wolpert [73]. Furthermore there are various constructions of Riemannian metrics and almost complex structures on $\mathcal{P}(S)$ which restrict to the Weil-Petersson structure on $\mathfrak{T}(S) \subset \mathcal{P}(S)$, suggesting that there is at least one natural Kähler structure on $\mathcal{P}(S)$ which extends the Weil-Petersson Kähler geometry of Teichmüller space.

There are many interesting problems concerning the geometry of the space $\mathcal{P}(S)$. The holonomy map

$$\mathrm{hol} : \mathcal{P}(S) \longrightarrow \mathrm{Hom}(\pi, \mathrm{PGL}(3;\mathbb{R}))/\,\mathrm{PGL}(3;\mathbb{R})$$

is an embedding onto an open subset; if the image is also closed, then hol embeds $\mathcal{P}(S)$ as a connected component of

$$\mathrm{Hom}(\pi, \mathrm{PGL}(3;\mathbb{R}))/\,\mathrm{PGL}(3;\mathbb{R}).$$

One might try to compactify the image by projective classes of families of valuations à la Morgan-Shalen [58]. A closely related problem is to find alternate an alternate proof that $\mathcal{P}(S)$ is a cell, along the lines of the analytic proof that $\mathfrak{T}(S)$ is a cell: points of Teichmüller space are identified as quasiconformal deformations of a Fuchsian group and such quasiconformal deformations are parametrized by an open ball in a finite-dimensional complex vector space of Beltrami differentials. Alternatively one might try to parametrize points of $\mathcal{P}(S)$ in an analogous way to Thurston's description of $\mathbb{C}\mathbf{P}^01(S)$ in terms of the "bending parameters" in $\mathfrak{T}(S) \times \mathfrak{ML}(S)$.

6. Triangle Groups

Finally we discuss the representations of Schwarz triangle groups in $G = \mathrm{PGL}(3;\mathbb{C})$ and their relation with geometric structures. The representations of these groups are easily computable and therefore one has an explicit description of the character varieties in this case.

Let $p, q, r \geq 3$ be integers satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1;$$

then there exists a triangle $\triangle$ (unique up to isometry) in the hyperbolic plane $\mathbf{H}^2$ with angles $\pi/p, \pi/q, \pi/r$. Let R_1, R_2, R_3 be the reflections in the sides of $\triangle$ and let Γ denote the group they generate. Then Γ is a discrete group of isometries of $\mathbf{H}^2$ with fundamental domain $\triangle$ and admits the presentation

$$\begin{aligned} \langle R_1, R_2, R_3 \mid (R_1)^2 = (R_2)^2 = (R_3)^2 &= (R_1R_2)^p \\ &= (R_2R_3)^q = (R_2R_3)^r = I \rangle \end{aligned}$$

(Although Γ is not the holonomy group of a hyperbolic structure on a manifold, any torsionfree subgroup of finite index is the holonomy group of a hyperbolic structure on a closed surface.)

The projective model for hyperbolic geometry identifies $\mathbf{H}^2$ with the region $\{[x, y, z] \in \mathbb{R}\mathbf{P}^02 | \, x^2 + y^2 - z^2 < 0\}$ and the group of isometries with $\mathrm{PSO}(2,1) \subset \mathrm{PGL}(3;\mathbb{R})$. We obtain a representation $\rho_1 : \Gamma \longrightarrow \mathrm{PSO}(2,1) \subset \mathrm{PGL}(3;\mathbb{C})$ and we are interested in the component X_1 of $X(\Gamma, \mathrm{PGL}(3;\mathbb{C}))$ containing the equivalence class $[\rho_1]$. In particular, since SO(2,1) is the intersection of the two real forms SU(2,1) and $\mathrm{SL}(3;\mathbb{R})$ of $\mathrm{SL}(3;\mathbb{C})$, we are particularly interested in the deformations of ρ_1 in the real representation varieties $\mathrm{Hom}(\Gamma, \mathrm{SL}(3;\mathbb{R}))$ and $\mathrm{Hom}(\Gamma, \mathrm{SU}(2,1))$.

Theorem 6.1. *The component X_1 of $X(\Gamma, \mathrm{PGL}(3;\mathbb{C}))$ containing the equivalence class $[\rho_1]$ is isomorphic to the curve $\mathbb{C}^*$. Indeed there is a* *map*

$$\begin{aligned} \mathbb{C}^* &\longrightarrow \operatorname{Hom}(\Gamma, \operatorname{SL}(3;\mathbb{C})) \\ s &\mapsto \rho_s \end{aligned}$$

such that for each $\chi \in X_1$ there is a unique s such that $\chi = [\rho_s]$. Furthermore the component of $X(\Gamma, \operatorname{PGL}(3;\mathbb{R}))$ containing ρ_1 maps isomorphically onto $\mathbb{R}^* \subset \mathbb{C}^*$; the representations corresponding to the positive reals define convex $\mathbb{R}\mathbf{P}^0 2$-structures on the Schwarz triangle orbifold. The component of $X(\Gamma, \operatorname{PU}(2,1))$ containing $[\rho_1]$ maps isomorphically onto the unit circle $\mathrm{S}^1 \subset \mathbb{C}^*$ and thus the component of $\operatorname{Hom}(\Gamma, \operatorname{PU}(2,1))$ containing ρ_1

Let $e_1, e_2, e_3 \in \mathbb{C}^3$ be the standard basis and let $s \in \mathbb{C}^*$. Consider the matrix

$$\mathbf{B}_s = \begin{bmatrix} 1 & -s^{-1}\cos\frac{2\pi}{p} & -\cos\frac{2\pi}{q} \\ -s\cos\frac{2\pi}{p} & 1 & -\cos\frac{2\pi}{r} \\ -\cos\frac{2\pi}{q} & -\cos\frac{2\pi}{r} & 1 \end{bmatrix}$$

When $s = 1$ this matrix is real, symmetric, and defines a nondegenerate quadratic form of signature (2,1). Define for $i = 1, 2, 3$

$$\rho_s(R_i) = -I + 2\mathbf{B}_s e_i e_i^{\dagger}$$

(where $A^{\dagger}$ denotes the transpose of A). A simple calculation shows that ρ_s defines a representation of Γ in $\operatorname{SL}(3;\mathbb{C})$ and that for $s = 1$ the representation ρ_s is conjugate to the standard representation $\rho_1 : \Gamma \to \operatorname{SO}(2,1) \subset \operatorname{SL}(3;\mathbb{C})$ defined above. When s is real, $\mathbf{B}_s$ is real and $\rho_s : \Gamma \longrightarrow \operatorname{SL}(3;\mathbb{R})$ is a real representation. When $|s| = 1$, the matrix $\mathbf{B}_s$ is Hermitian and the corresponding Hermitian form on $\mathbb{C}^3$ is nondegenerate and has signature (2,1); in that case representation $\rho_s : \Gamma \longrightarrow \operatorname{SL}(3;\mathbb{C})$ leaves invariant this Hermitian form and thus is conjugate to a representation $\Gamma \longrightarrow \operatorname{SU}(2,1)$.

For $s > 0$ the groups $\rho_s(\Gamma)$ preserve a convex domain $\Omega_s \subset \mathbb{R}\mathbf{P}^0 2$ and for torsionfree finite-index subgroups $\Gamma' \subset \Gamma$, the quotient $\Omega_s/\rho_s(\Gamma')$ is a compact convex $\mathbb{R}\mathbf{P}^0 2$-manifold. Pictured below are the tesselated domains Ω_s for $p = q = r = 4$ and various choices of s. These convex domains are bounded by C^1 curves which are never C^2 (unless they are conics); the first examples of such domains which cover closed surfaces were discovered by Kac-Vinberg [39]. In particular the family of representations $\rho_s : \Gamma \longrightarrow \operatorname{PGL}(3;\mathbb{R})$ for $s \in \mathbb{R}$ defines a noncompact family of properly discontinous actions on domains in $\mathbb{R}\mathbf{P}^0 2$; one can check easily that the family $[\rho_s]$ does not converge in the quotient $\operatorname{Hom}(\Gamma, \operatorname{PGL}(3;\mathbb{R}))/\operatorname{PGL}(3;\mathbb{R})$.

For $|s| = 1$ we obtain a family of representations of Γ in the projective unitary group $\mathrm{PU}(2,1)$; for s sufficiently close to 1 the representations ρ_s will be discrete embeddings. These representations (restricted to torsionfree finite-index subgroups) correspond to noncompact complex hyperbolic surfaces on nontrivial 2-disc bundles over a closed Riemann surfaces F; alternatively they correspond to locally spherical CR-structures on a nontrivial oriented S^1-bundle over F. It would be extremely interesting to understand for what values of s these representations correspond to geometric structures and for which values of s are these representations discrete embeddings.

The predictions of the asymptotic theory of character varieties developed by Culler-Shalen [9], Morgan-Shalen [58,59] and Morgan [57] and Brumfiel [4] can be explictly verified for these examples. Since Γ is generated by elements of finite order, every action of Γ on an $\mathbb{R}$-tree must have a fixed point. When G has $\mathbb{R}$-rank 1, then the character variety $X(\Gamma, G)$ must be compact — as in the case $G = \mathrm{PU}(2,1)$ above. On the other hand, these examples of convex $\mathbb{R}\mathbf{P}^0 2$-structures show that if G has rank 2, there is no such compactness and indeed there exist unbounded families of representations, which in this case are discrete embeddings.

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