Problem Set #6

1. doCarmo, Chapter 2, #1,2,3,8
2. let $S \subset \mathbb{R}^3$ be an embedded surface. For any vector fields X and Y on S, define $\nabla_X Y$ pointwise by
$$(\nabla_X Y)(p) = \pi \circ \left.\frac{d}{dt}\right|_{t=0} (Y \circ \alpha)$$
where $\alpha : (-\epsilon, \epsilon)$ is an integral curve of X with $\alpha(0) = p$ and $\pi : \mathbb{R}^3 \to T_p S$ is orthogonal projection. Prove that ∇ is an affine connection.
3. The following spaces are diffeomorphic: $SO(3)$, $\mathbb{R}P^3$, $T^1 S^2 = \{(p, v) \in TS^2 \mid \|v\|^2 = 1\}$, where S^2 is given it's canonical Riemannian metric ($T^1 S^2$ is called the *unit tangent bundle* or the *tangent sphere bundle*.)
   (a) Find explicit diffeomorphisms between these spaces.
   (b) $SO(3)$ and $\mathbb{R}P^3$ are both Lie groups. (In class, we showed that S^3 is a Lie group. Since $H = \{\pm 1\}$ is a normal subgroup of S^3, $\mathbb{R}P^3 = S^3/H$ is also a Lie group.) Describe a group structure on $T^1 S^2$ such that the diffeomorphisms from part (a) are Lie group isomorphisms.
4. Let G be a Lie group equipped with a bi-invariant metric. Prove that for any left–invariant vector fields, X and Y, on G, the Levi-Civita connection satisfies
$$\nabla_X Y = \frac{1}{2}[X, Y]$$