Problem Set #5

1. doCarmo, Chapter 1, #4, 5
2. The pseudo-sphere is the surface of revolution obtained by revolving the curve $\alpha(t) = (t - \tanh t, 0, \frac{1}{\cosh t})$ about the x–axis. Prove that the pseudo-sphere, with the metric induced from the embedding in $\mathbb{R}^3$, is locally isometric to the upper half plane with the Poincare metric.
3. (a) Prove that the tangent bundle of a Lie group is trivial.
   (b) Prove that S^2 is not a Lie group.
4. Consider the Lie Algebra $\mathfrak{sl}(n, \mathbb{R}) = \mathcal{L}(SL(n, R))$.
   (a) Prove that $\mathfrak{sl}(n, \mathbb{R})$ is isomorphic to $\{A \in M(n, \mathbb{R}) \mid \mathrm{tr}(A) = 0\}$.
   (b) Prove that $SL(2, \mathbb{R})$ does not admit a bi-invariant Riemannian metric. (Hint: choose a basis of $\mathfrak{sl}(n, \mathbb{R})$ and compute the bracket explicitly.)
5. Let M be a smooth manifold with a semi-Riemannian metric, g. Let $i : M \to \mathbb{Z}$ be the index of g.
   (a) Prove that i is constant on each connected component of M.
   (b) Prove that S^2 does not admit a semi-Riemannian metric of index 1.