# Problem Set #3

1. Prove that the Gauss and Codazzi equations are equivalent to the equations
$$(\mathbf{x}_i)_{jk} = (\mathbf{x}_i)_{kj} \qquad \mathbf{n}_{ij} = \mathbf{n}_{ji}$$
2. Use the result of problem 1 and the existence theorem for solutions of linear partial differential equations to prove the fundamental theorem of surfaces.
3. State and prove the strongest version of the pseudo-theorem "There are lots of geodesics" that you can.
4. Let $r, z : (a, b) \to \mathbb{R}$ be smooth functions with $r(t) > 0$. The surface of revolution, $S$, obtained by revolving the curve $(r(t), 0, z(t))$ about the $z$–axis is parameterized by $\mathbf{x}(u, v) = (r(u)\cos(v), r(u)\sin(v), z(u))$.
   (a) A *meridian* of $S$ is a curve $\alpha(t) = \mathbf{x}(t, v_0)$, for some fixed $v_0$. Prove that every meridian is a geodesic.
   (b) A *circle of latitute* of $S$ is a curve $C(\theta) = \mathbf{x}(u_0, \theta)$, for some fixed $u_0$. For any curve, $\gamma(s)$, let $r(s)$ be the distance from $\gamma(s)$ to the $z$–axis and let $\phi(s)$ be the angle between $\gamma$ and the circle of latitude passing through $\gamma(s)$. Prove that if $\gamma$ is a geodesic, then $r\cos\phi$ is constant.
   (c) Prove that the converse to part (b) is false. Not every curve for which $r\cos\phi$ is constant is a geodesic.
5. Let $\gamma : [a, b] \to S$ be a smooth curve in $S$, and let $V$ be a tangent vector field along $\gamma$. We say that $V$ is *parallel* if $\frac{DV}{dt} \equiv 0$. (Note: A geodesic is, by definition, a curve whose tangent vector field, $\gamma'(t)$, is parallel.)
   (a) Let $S$ be the $xy$–plane and let $\gamma$ be a curve connecting points $p$ and $q$. Prove that, for parallel vector fields, $V(q)$ depends only on $V(p)$ and is, in particular, independent of the curve $\gamma$. That is, prove that if $\gamma_1, \gamma_2 : [a, b] \to S$ are two curves with $\gamma_1(a) = \gamma_2(a) = p$ and $\gamma_1(b) = \gamma_2(b) = q$ and $V_i$ are parallel vector fields along $\gamma_i$ with $V_1(a) = V_2(a)$, then $V_1(b) = V_2(b)$.
   (b) Prove that the part (a) is false if $S$ is the unit sphere.