Problem Set #2

1. Prove that the Gauss and Codazzi equations are equivalent to the commutivity of all mixed third derivatives of the parameterization.

2. Let γ be a simple, closed curve whose curvature is everywhere positive. Consider the ϵ–neighborhood

$$N_{\gamma,\epsilon} = \{x \in \mathbb{R}^3 \mid \|x - \gamma(t)\| \leq \epsilon \text{ for some } t\}$$

   and let $S = \partial N_{\gamma,\epsilon}$.

   (a) Prove that if ϵ is sufficiently small, then M is a regular surface.

   (b) Find an explicit parameterization of M and compute the first fundamental form, second fundamental form, and Gaussian curvature of S in terms of the curvature and torsion of γ.

3. (a) Let G be the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$. Compute the Gaussian curvature of G in terms of f. When is the curvature positive? When is it negative? Where have you seen this calculation before?

   (b) Let S be a closed (ie. compact without boundary) surface. Let $p \in S$ be a point in the surface such that $K(p) > 0$. Prove that there is an open neighborhood, U, containing p, such that U lies entirely on one side of T_pS. (Hint: prove that the Gaussian curvature of a surface is unchanged by rotations of $\mathbb{R}^3$.)

   (c) Prove that if the curvature of a closed, orientable surface, S, is everywhere positive, then the Gauss map, $N : S \to S^2$, is a diffeomorphism. Use this to prove that all of S lies on one side of T_pS and that S is, therefore, the boundary of a convex region in $\mathbb{R}^3$.

4. The *third fundamental form*, III, is defined by

$$\mathrm{III}(v, w) = \langle (dn)^2(v), w\rangle$$

   Prove that for any pair of vectors, $v, w \in T_pS$,

$$\mathrm{III}(v, w) = 2H(p)\mathrm{II}(v, w) - K(p)\mathrm{I}(v, w)$$

   (Hint: Cayley–Hamilton theorem)

5. A surface is called *minimal* if its mean curvature is identically zero. Prove that the only surfaces of revolution that are minimal are (pieces of) the plane and the catenoid (ie. the surface obtained by revolving the curve $\alpha(t) = (a\cosh(t), 0, at)$ about the z–axis).