Problem Set #1

1. Show that for any regular curve, $\gamma(t)$, not necessarily parameterized by arclength, the Frenet apparatus is given by

$$T = \frac{\gamma'}{|\gamma'|} \qquad N = B \times T \qquad B = \frac{\gamma' \times \gamma''}{|\gamma' \times \gamma''|}$$
$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} \qquad \tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{|\gamma' \times \gamma''|^2}$$

2. Show that the Fundamental Theorem of Curves is false if we only assume that $\kappa(s) \geq 0$. Construct 2 planar curves, of class at least C^2, whose curvatures are equal, but which are not related by a rigid motion.

3. Let $\gamma(s)$ be a regular curve whose curvature and torsion are both non-zero. Prove that γ lies on the surface of a sphere if and only if

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau \kappa^2} \right)'$$

4. Let $\gamma(s)$ be a regular curve, and let C_{s_1,s_2,s_3} be the circle passing through the points $\gamma(s_1)$, $\gamma(s_2)$, and $\gamma(s_3)$.
   (a) Prove that if $\kappa(s_1) > 0$, then the circles C_{s_1,s_2,s_3} converge to a unique circle, C_{s_1}, as s_2 and s_3 approach s_1. This circle is called the *osculating circle*.
   (b) Prove that $\kappa(s_1)$ is equal to the curvature of the osculating circle.