SERRE SYMMETRY PERSISTS THROUGH THE FRÖLICHER SPECTRAL SEQUENCE

Abstract. Serre's duality theorem implies a symmetry $h^{p,q} = h^{n-p,n-q}$ between the Hodge numbers on a compact complex manifold. Equivalently, the first page of the associated Frölicher spectral sequence satisfies $\dim E_1^{p,q} = \dim E_1^{n-p,n-q}$ for all p, q. Adapting an argument of Chern, Hirzebruch, and Serre in an obvious way, we show that this "Serre symmetry" $\dim E_k^{p,q} = \dim E_k^{n-p,n-q}$ holds on all subsequent pages of the spectral sequence as well.

Definition 1. (cf. [1]) An n–dimensional *bigraded Poincaré ring* is a $\mathbb{Z}^2$–graded ring A with the following properties:

(1) Each $A^{p,q}$ is a finite-dimensional complex vector space, and there exists an n such that $A^{p,q} = \{0\}$ if $p > n$ or $q > n$ and $A^{n,n}$ is one-dimensional.
(2) Denote $A^k = \bigoplus_{p+q=k} A^{p,q}$. If $x \in A^k$ and $y \in A^l$ then $xy = (-1)^{kl} yx$.
(3) Fix a nonzero element $\xi \in A^{n,n}$. Define a bilinear pairing $A^{p,q} \otimes A^{n-p,n-q} \xrightarrow{\langle -,- \rangle} \mathbb{C}$ by $\langle x, y\rangle \xi = xy$. We require the linear map $i_{n-p,n-q}$ from $A^{n-p,n-q}$ to $A^{p,q*}$ (the complex-linear dual of $A^{p,q}$) which sends $y \in A^{n-p,n-q}$ to the functional $\langle -, y\rangle$ to be an isomorphism for all p, q. (Note that this property does not depend on the choice of nonzero ξ.)

Example 2. For X a compact complex manifold of complex dimension n, the Dolbeault cohomology $H_{\bar\partial}^{*,*}$ is a bigraded Poincaré ring. Indeed, the space of (p, q) forms for p, q not between 0 and n is trivial; each $H_{\bar\partial}^{p,q}$ is finite-dimensional by elliptic theory; $H_{\bar\partial}^{n,n}$ is one-dimensional, spanned by a volume form for the manifold. Therefore property (1) of the definition is satisfied. Property (2) follows from the same property holding on the algebra of all complex-valued differential forms on X and the fact that $\bar\partial$ is a derivation with respect to the product of forms. As for property (3), let $\xi = [\Omega]_{\bar\partial}$ where Ω is some fixed (n, n) volume form on X. Since $\bar\partial\Omega = 0$ and it is not $\bar\partial$-exact (since otherwise it would be d-exact for degree reasons, and this cannot be by Stokes' theorem), ξ spans $H_{\bar\partial}^{n,n}$. Now, we will show that $i_{n-p,n-q}$ is an isomorphism; since by Serre duality $\dim H_{\bar\partial}^{p,q} = \dim H_{\bar\partial}^{n-p,n-q}$, it will suffice

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to show injectivity. Take a non-zero $\alpha \in H^{n-p,n-q}_{\bar{\partial}}$ and take the $\bar{\partial}$-harmonic representative y of this class. Denote by $\overline{*y}$ the conjugate of its Hodge dual, i.e. the (p,q)-form such that $y \wedge \overline{*y} = \langle y, y \rangle_{L^2} \Omega$. Since $*\partial * y = -\bar{\partial}^* y = 0$, we have $\partial * y = 0$ and hence $\bar{\partial}\overline{*y} = 0$. Now $y \wedge \overline{*y} = ||y||_{L^2}\Omega$, and $||y||_{L^2} \neq 0$, and thus passing to cohomology we obtain that $[y]_{\bar{\partial}} \wedge [\overline{*y}]_{\bar{\partial}}$ is a non-zero multiple of ξ. We conclude $\langle [\overline{*y}]_{\bar{\partial}}, [y]_{\bar{\partial}} \rangle \neq 0$, i.e. $i_{n-p,n-q}(\alpha) \neq 0$.

Definition 3. (cf. [1]) A *differential* on an n–dimensional bigraded Poincaré ring A is a linear map $A \xrightarrow{d} A$ satisfying:

(α) $d(A^{p,q}) \subset A^{p+p',q+q'}$, i.e. d is of bidegree (p',q') for some $p', q' \in \mathbb{Z}$. We further require $p' + q' = 1$ (but not necessarily $p', q' \geq 0$).
(β) $d^2 = 0$.
(γ) $d(xy) = (dx)y + (-1)^r x(dy)$ for $x \in A^r$, $r \in \mathbb{Z}$.
(δ) $d(A^{2n-1}) = \{0\}$.

Example 4. For X a compact complex manifold, the induced differential ∂ on $A = H_{\bar{\partial}}(X)$ (given by $\partial[\alpha]_{\bar{\partial}} = [\partial\alpha]_{\bar{\partial}}$) is a differential of bidegree $(1,0)$ on the bigraded Poincaré ring $H_{\bar{\partial}}(X)$ in the above sense. Indeed, property (α) is satisfied, and properties (β) and (γ) are satisfied by ∂ on the level of forms (note that the derivation ∂ on forms induces a derivation on $H_{\bar{\partial}}(X)$). As for property (δ), we note that $A^{2n-1} = A^{n-1,n} \oplus A^{n,n-1}$. The differential vanishes on $A^{n,n-1}$ for degree reasons. Now take $[\alpha]_{\bar{\partial}} \in A^{n-1,n}$ and suppose $\partial[\alpha]_{\bar{\partial}} = [\partial\alpha]_{\bar{\partial}}$ is non-zero. Since $A^{n,n}$ is one-dimensional, this means there is a non-zero constant c such that $\partial\alpha - c\Omega = \bar{\partial}\beta$ for some $(n, n-1)$-form β. Rearranging, we get $\Omega = \partial(\frac{\alpha}{c}) - \bar{\partial}(\frac{\beta}{c})$. For degree reasons, this is the same as $\Omega = d(\frac{\alpha-\beta}{c})$, which by Stokes' theorem would imply $X = \emptyset$. (Alternatively, the vanishing of this differential follows from the convergence of the Frölicher spectral sequence to the complexified de Rham cohomology, and $H^{2n}_{dR}(X;\mathbb{C}) \cong \mathbb{C}$.)

Proposition 5. (cf. [1]) The cohomology ring $HA = \ker(d)/\operatorname{im}(d)$ of an n-dimensional bigraded Poincaré ring A with differential d is an n-dimensional Poincaré ring.

Proof. Since d has a well-defined bidegree we obtain a decomposition $A' = \bigoplus_{p,q} A'^{p,q}$, where $A'^{p,q} = (\ker d \cap A^{p,q})/\operatorname{im} d$. Note that property ($\delta$) implies that A' inherits properties (1) and (2). If ξ is the (n,n)-bidegree element used to define $\langle -,- \rangle$ on A, then we can define a bilinear pairing $HA^{p,q} \otimes HA^{n-p,n-q} \xrightarrow{\langle -,- \rangle} \mathbb{C}$ on the cohomology ring by $\langle [x],[y] \rangle [\xi] = [x][y]$. Note that $[\xi]$ is non-zero due to property (δ). Now define a linear map $HA^{n-p,n-q} \xrightarrow{i'_{n-p,n-q}} (HA^{p,q})^*$ by sending $[y]$ to the functional $\langle -, [y] \rangle$.

Now we show that HA has property (3), i.e. that $i'_{n-p,n-q}$ is an isomorphism for all p, q. Denote the bidegree of d by (p', q'); recall $p' + q' = 1$. For $x \in A^{p,q}$ and $y \in A^{n-p-p',n-q-q'}$, by properties (γ) and (δ) of the differential, we have $0 = d(xy) = (dx)y + (-1)^{p+q}x(dy)$. From the definition of $\langle -, - \rangle$, this implies $\langle dx, y \rangle = (-1)^{p+q-1}\langle x, dy \rangle$. Consider the following diagram (extending to left and right):

$$\begin{tikzcd}
{} \arrow[r] & A^{n-p-p',n-q-q'} \arrow[r, "d"] \arrow[d, "i_{n-p-p',n-q-q'}"] & A^{n-p,n-q} \arrow[r, "d"] \arrow[d, "i_{n-p,n-q}"] & A^{n-p+p',n-q+q'} \arrow[r] \arrow[d, "i_{n-p+p',n-q+q'}"] & {} \\
{} \arrow[r] & \left(A^{p+p',q+q'}\right)^* \arrow[r, "(-1)^{p+q-1}d^*"] & \left(A^{p,q}\right)^* \arrow[r, "(-1)^{p+q}d^*"] & \left(A^{p-p',q-q'}\right)^* \arrow[r] & {}
\end{tikzcd}$$

Here d^* denotes the dual morphism to d; we have $d^*(\langle -, y \rangle) = \langle d-, y \rangle$. Note that the calculation preceding the diagram implies that $i_{n-p,n-q}$ is a chain map for all p, q. Indeed, let us check commutativity for the left square: for $y \in A^{n-p-p',n-q-q'}$ we have $i_{n-p,n-q}(dy) = \langle -, dy \rangle$, while

$$(-1)^{p+q-1}d^*(i_{n-p-p',n-q-q'}(y)) = (-1)^{p+q-1}d^*(\langle -, y \rangle) = (-1)^{p+q-1}\langle d-, y \rangle.$$

Since $i_{n-p,n-q}$ is an isomorphism for all p, q by property (3), it follows that the chain map $i_{n-*,n-*}$ induces an isomorphism on cohomology. We would now like to relate this induced map to the map $i'_{n-p,n-q}$ we wish to establish bijectivity for.

First of all, for $y \in A^{n-p,n-q}$, we have $i_{n-p,n-q}(y) = \langle -, y \rangle$, and so the induced map $i^*_{n-p,n-q}$ sends $[y]$ to the d^*-cohomology class $[\langle -, y \rangle]$. Now we note that the target space $\ker d^* \cap (A^{p,q})^* / \operatorname{im} d^*$ is not quite $\left(A'^{p,q}\right)^* = (\ker d \cap A^{p,q} / \operatorname{im} d)^*$. Regardless, by the universal coefficient theorem, the map Ψ from the latter to the former given by $\Psi([\alpha])([V]) = \alpha(V)$ is an isomorphism (since we are over a field). Given $[x] \in A^{p,q}$, note that $\Psi \circ i^*_{n-p,n-q}([y])([x])$ equals $\Psi([\langle -, y \rangle])([x]) = \langle x, y \rangle$. However, as $\langle x, y \rangle \xi = xy$, passing to cohomology we see that $\langle x, y \rangle = \langle [x], [y] \rangle$. (For this equality we are using that the pairing on the cohomology ring is defined using $[\xi]$.) Therefore $\Psi \circ i^*_{n-p,n-q} = i'_{n-p,n-q}$. Since both maps on the left hand side are isomorphisms, we have the desired statement. □

Corollary 6. The E_2 page of the Frölicher spectral sequence for a compact complex manifold satisfies Serre duality. (The E_2 page here denotes the ∂-cohomology of the $\bar{\partial}$-cohomology.)

Recall that for a complex manifold X, denoting its algebra of complex-valued smooth forms by $A^{\bullet,\bullet}(X)$, the Frölicher spectral sequence is associated to the filtration $F^pA^{\bullet,\bullet}(X) = \bigoplus_{i \geq p, j} A^{i,j}(X)$. The complexified de Rham differential d preserves this filtration, $d(F^p) \subset F^p$, and if $\alpha \in F^p$,

$\beta \in F^q$, then $\alpha \wedge \beta \in F^{p+q}$. Now by [2, Theorem 2.14], it follows that the differential on the E_k page of the Frölicher spectral sequence, for any $k \geq 1$, satisfies property (γ). We know that it also satisfies properties (α) with bidegree $(k, 1-k)$ and (β). Property (δ) is trivially satisfied for degree reasons (or by the alternative argument as given in Example 4). In particular, (E_2, d_2) is a bigraded Poincaré ring with differential, and so inductively we have the following:

Corollary 7. For every $k \geq 1$, the kth page of the Frölicher spectral sequence of a compact complex n–manifold satisfies Serre symmetry, i.e. $\dim E_k^{p,q} = \dim E_k^{n-p,n-q}$ for all p, q.

Remarks. The above result also follows, by different methods, from forthcoming work of Stelzig [3] using combinatorial arguments and Popovici–Stelzig–Ugarte via harmonic theory. We note that this result shortens some of the calculations done in [5] and implies a Serre symmetry for all pages of the spectral sequence for left-invariant almost complex structures on nilpotent Lie groups as considered in [4, Section 5]. It is as of yet unknown whether Serre symmetry holds for the Dolbeault cohomology of an almost complex manifold as studied in [4].

References

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