THE MINIMAL MODELS OF $\mathbb{CP}^2 \# \mathbb{CP}^2$ AND $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

ABSTRACT. We calculate the minimal models of $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ in detail.

Recall that $\mathbb{CP}^2$ is formal, and that the connect sum of formal manifolds is formal. So, $\mathbb{CP}^2 \# \mathbb{CP}^2$ is formal, meaning we can form its minimal model directly from its rational cohomology algebra $H^*(\mathbb{CP}^2 \# \mathbb{CP}^2, \mathbb{Q})$. To obtain the cohomology algebra of a connect sum, tensor the cohomology algebras of the spaces you are connecting, and impose the relations that all products of elements coming from different connect-summands are zero, and identify the volume forms. In the example of $\mathbb{CP}^2$, the cohomology algebra is $H^*(\mathbb{CP}^2) = \Lambda(a;\ \deg(a) = 2, a^3 = 0)$. So, the cohomology algebra of the connect sum of two CP^2's is

$$H^*(\mathbb{CP}^2 \# \mathbb{CP}^2) = \Lambda(a, b;\ \deg(a) = \deg(b) = 2, a^3 = 0, b^3 = 0, ab = 0, a^2 - b^2 = 0).$$

Think of this algebra as a dga with zero differential, and let's find its minimal model. First, introduce an η in degree 3 to kill $a^2 - b^2$ in cohomology. Also, introduce a degree 3 element ε to kill ab. So, for now our minimal model candidate is (subscripts denote degrees) $M = \Lambda(a_2, b_2, \eta_3, \varepsilon_3;\ da = 0, db = 0, d\eta = a^2 - b^2, d\varepsilon = ab)$, with a map $M \xrightarrow{f} H^*(\mathbb{CP}^2 \# \mathbb{CP}^2)$ given by $f(a) = a, f(b) = b, f(\eta) = 0, f(\varepsilon) = 0$. Now, observe that the relations $a^3 = 0$ and $b^3 = 0$ in the cohomology algebra are already accounted for in M (meaning, these expressions are exact), since $a^3 = d(\eta a + \varepsilon b)$ and $b^3 = d(-\eta b + \varepsilon a)$. Similarly, $a^2 b$ and ab^2 are exact (in case we're worried these elements might give unwanted cohomology in M).

Now observe that we are done: M (with the given map f) is a minimal model for $\mathbb{CP}^2 \# \mathbb{CP}^2$. We just have to check that there are no closed but non-exact elements in degree 7 or higher in M. Indeed, we've already checked that all cohomology in M up to degree 6 coincides with cohomology in $\mathbb{CP}^2 \# \mathbb{CP}^2$ (we didn't check $\eta\varepsilon$ in degree 6, but that isn't even closed). A generic element in M looks like

$$g(a, b) + h(a, b)\eta + l(a, b)\varepsilon + k(a, b)\eta\varepsilon,$$

where g, h, k, l are polynomials in a and b. We want to show that any closed such element of degree 7 or greater is exact, so we can reduce to looking at homogeneous elements. Note that $g(a, b)$ and $k(a, b)\eta\varepsilon$ are of even degree, and $h(a, b)\eta$ and $l(a, b)\varepsilon$ are of odd degree. So we can look at elements of the form $g(a, b) + k(a, b)\eta\varepsilon$ independently from elements of the form $h(a, b)\eta + l(a, b)\varepsilon$:

• Suppose $d(g(a, b) + k(a, b)\eta\varepsilon) = 0$; we want to show it is exact. Observe that $d(g(a, b)) = 0$, but it is also exact, since it is a homogeneous polynomial in a and b of degree at least 7, so each term contains an a^3, b^3, $a^2 b$, or ab^2, which are exact (and a and b themselves are closed). As for $k(a, b)\eta\varepsilon$, observe

$$d(k(a, b)\eta\varepsilon) = k(a, b) \cdot (a^2 - b^2) \cdot \epsilon - k(a, b) \cdot \eta \cdot ab.$$

The assumption that the element we are considering is closed means that this expression is zero, which by freeness (considering the coefficients along η and ε) implies that $k(a,b) = 0$. So, if $g(a,b) + k(a,b)\eta\varepsilon$ is closed, it is in fact just $g(a,b)$, which is exact as previously argued.

- Suppose $d(h(a,b)\eta + l(a,b)\varepsilon) = 0$. So, we have $h(a,b)\cdot(a^2-b^2) + l(a,b)\cdot ab = 0$, that is,

$$(a^2-b^2)h(a,b) = -l(a,b)ab.$$

Since the polynomial ring in two variables is a unique factorization domain, we can conclude $h(a,b) = ab\cdot h'(a,b)$ and $l(a,b) = (a^2-b^2)l'(a,b)$. From the above relation we have $l'(a,b) = -h'(a,b)$. So, the element we started with is of the form $h(a,b)\eta + l(a,b)\varepsilon = h'(a,b)\cdot(ab\eta - (a^2-b^2)\eta)$. Observe that this is exact; indeed, it is $d(h'(a,b)\cdot\eta\varepsilon)$.

So, the minimal model of $\mathbb{CP}^2\#\mathbb{CP}^2$ is the one given above.

To obtain the minimal model of $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, everything is the same as above, except $d\eta = a^2+b^2$. Other than this change of sign, the model looks the same.

As an immediate consequence, note that $\pi_2(\mathbb{CP}^2\#\mathbb{CP}^2)\otimes\mathbb{Q} = \pi_3(\mathbb{CP}^2\#\mathbb{CP}^2)\otimes\mathbb{Q} = \mathbb{Q}^2$; compare this with $\pi_3(\mathbb{CP}^2) = 0$.