MORE COMPUTATIONS WITH HOMOGENEOUS SPACES

Abstract. We follow an argument presented in [Am13] to conclude the non-formality of $SO(2n)/SU(n)$ for $n \geq 8$.

1. Homogeneous spaces are not necessarily formal

First let us observe that the quotient G/H of two closed Lie groups is not necessarily a formal space (in the sense of rational homotopy theory). We consider the 19–manifold $SU(6)/(SU(3) \times SU(3))$. We compute its minimal model and will see that it contains a non-trivial Massey product (as in [Am13]).

We approach the computation of the model in the standard way; consider the map $BSU(3) \times BSU(3) \to BSU(6)$ on classifying spaces induced by the inclusion $SU(3) \times SU(3) \hookrightarrow SU(6)$, and obtain a model for $SU(6)/SU(3) \times SU(3)$ as a model for the fiber of this map. On the level of minimal models, the map $\mathrm{Model}(BSU(6)) \to \mathrm{Model}(BSU(3)) \otimes \mathrm{Model}(BSU(3))$ is given by the universal relations among the Chern classes of an $SU(6)$ bundle and the two $SU(3)$ bundles it might split into a sum of. Let us denote the universal Chern classes in $H^*(BSU(6);\mathbb{Q})$ by c_2, c_3, c_4, c_5, c_6, and those in the two copies of $BSU(3)$ by a_2, a_3 and b_2, b_3. So, $\mathrm{Model}(BSU(6)) = \Lambda(c_2, c_3, c_4, c_5, c_6)$ and $\mathrm{Model}(BSU(3) \times BSU(3)) = \Lambda(a_2, a_3, b_2, b_3)$, where both graded algebras have trivial differential. The map we are considering is given by sending c_i to the polynomial in a_i and b_i obtained from the relation

$$1 + c_2 + c_3 + c_4 + c_5 + c_6 = (1 + a_2 + a_3) \cdot (1 + b_2 + b_3).$$

We have

$$\begin{aligned}
c_2 &= a_2 + b_2 \\
c_3 &= a_3 + b_3 \\
c_4 &= a_2 b_2 \\
c_5 &= a_2 b_3 + a_3 b_2 \\
c_6 &= a_3 b_3,
\end{aligned}$$

and this gives the map $\mathrm{Model}(BSU(6)) \to \mathrm{Model}(BSU(3)) \otimes \mathrm{Model}(BSU(3))$. A model of $SU(6)/SU(3) \times SU(3)$ is now obtained by forming the mapping cone. Factor our map of dga's into an inclusion (of dga's) $\Lambda(c_i) \hookrightarrow E$ followed by a quasi-isomorphism $E \xrightarrow{\sim} \Lambda(a_i, b_i)$. We see that we can build E and a quasi-isomorphism to $\Lambda(a_i, b_i)$ by including variables $\overline{a_2}$ and $\overline{a_3}$ mapping to a_2 and a_3 alongside the variables c_i mapping to the corresponding polynomials in the a_i and b_i. Since this map now sends $c_2 - \overline{a_2}$ to b_2 and sends $c_3 - \overline{a_3}$ to b_3, we see that it is surjective. To make it injective, we introduce variables $\eta_7, \eta_9, \eta_{11}$ into E to kill all relations, i.e. we set

$$\begin{aligned}
d\eta_7 &= c_4 - \overline{a_2}\overline{b_2}, \\
d\eta_9 &= c_5 - \overline{a_2}c_3 + 2\overline{a_2}\overline{a_3} - \overline{a_3}c_2, \\
d\eta_{11} &= c_6 - \overline{a_3}c_3 + \overline{a_3}^2.
\end{aligned}$$

Now we form the mapping cone by quotienting this newly formed dga E by the ideal generated by the c_i (along with taking the induced differential). Renaming $\overline{a_2}$ and $\overline{a_3}$ into a and b (of degrees 4 and 6) respectively, we obtain

$$\text{Model}(SU(6)/SU(3)\times SU(3)) = \Lambda(a, b, \eta_7, \eta_9, \eta_{11};\ da = db = 0, d\eta_7 = a^2, d\eta_9 = 2ab, d\eta_{11} = b^2).$$

Now observe that we have the non-trivial Massey product $\text{Massey}(a, a, b)$ given by the non-trivial cohomology class $[\eta_7 b + \frac{1}{2}a\eta_9] \in H^{13}(SU(6)/SU(3) \times SU(3); \mathbb{Q})$ (we can see directly that the ambiguity in the Massey product cannot make it zero).

2. A criterion for non-formality

We will show that the manifolds $SO(2n)/SU(n)$ are non-formal for $n \geq 8$. For $n \leq 7$ we can carry out a (tedious) computation of the model as in the previous section and determine formality directly.

From the short exact sequence of groups $0 \to SU(n) \hookrightarrow U(n) \overset{\det}{\to} S^1 \to 0$ we obtain the induced fibration

$$S^1 \to SO(2n)/SU(n) \to SO(2n)/U(n).$$

We will use this fibration and our better understanding of $SO(2n)/U(n)$ to say something about $SO(2n)/SU(n)$.

First let us say something about the cohomology of $SO(2n)/U(n)$. Interpreting $SO(2n)/U(n)$ as the space of almost complex structures on R^{2n} (compatible with some fixed orientation), we see that it fibers over S^{2n-2} with fiber $SO(2n-2)/U(n-1)$. (In fact, sections of this fibration correspond to almost complex structures on the base sphere; i.e. $SO(2n)/U(n)$ is the twistor space of the base sphere S^{2n-2} and is thus a Kähler complex manifold.) Namely, fix a unit vector $e \in R^{2n}$. To begin determining an almost complex structure J on R^{2n}, we have to choose a unit vector Je in the sphere S^{2n-2} in the hyperplane orthogonal to e. Once Je is chosen, it remains to choose an almost complex structure on the R^{2n-2} orthogonal to $\{e, Je\}$.

From here we can see that the manifolds $SO(2n)/U(n)$ have cohomology concentrated in even degrees. Indeed, we can proceed inductively: $SO(2)/U(1)$ is a point, and $SO(2n+2)/U(n+1)$ fibers over an even sphere with fiber $SO(2n)/U(n)$. The claim now follows by observing that all non-trivial entries in the fibration spectral sequence have even coordinates.

Now let us return to the fibration $S^1 \to SO(2n)/SU(n) \to SO(2n)/U(n)$. Denote by $e \in H^2(SO(2n)/U(n), \mathbb{R})$ the real Euler class of the fibration

$$S^1 \to SO(2n)/SU(n) \overset{p}{\to} SO(2n)/U(n),$$

and consider the Gysin sequence

$$\cdots \longrightarrow H^*(SO(2n)/U(n),\mathbb{R}) \xrightarrow{\cdot e} H^{*+2}(SO(2n)/U(n),\mathbb{R}) \xrightarrow{p^*} H^{*+2}(SO(2n)/SU(n),\mathbb{R}) \xrightarrow{\int_{S^1}} H^{*+1}(SO(2n)/U(n),\mathbb{R}) \xrightarrow{\cdot e} \cdots$$

Since the natural map $SO(2n)/U(n) \to SO(2n+2)/U(n+1)$ is $2n-2$-connected, and $SO(4)/U(2) = S^2$, we have that $H^2(SO(2n)/U(n), \mathbb{R}) = \mathbb{R}$ for $n \geq 2$. Now if we can conclude that Euler class is non-zero, then we will have that it is a non-zero multiple of the Kähler class $\omega \in H^2(SO(2n)/U(n), \mathbb{R})$, whence we can consider a modified Gysin sequence with the Kähler class replacing the Euler class. To see that the Euler class is non-trivial in $H^2(SO(2n)/U(n), \mathbb{R})$, consider what would happen if it was purely torsion. Then we would have that the fibration $S^1 \to SO(2n)/SU(n) \to SO(2n)/U(n)$ is rationally trivial, and so $\pi_1(SO(2n)/SU(n))\otimes\mathbb{Q} = \pi_1(S^1)\otimes\mathbb{Q}\oplus\pi_1(SO(2n)/U(n))\otimes\mathbb{Q}$. The manifolds $SO(2n)/U(n)$ are simply-connected – again, consider the 2-connected inclusion $S^2 = SO(4)/U(2) \to SO(2n)/U(n)$ – and $\pi_1(SO(2n)/SU(n)) = \mathbb{Z}_2$ as visible from the fibration $SU(n) \to SO(2n) \to SO(2n)/SU(n)$. Therefore the Euler class cannot be purely torsion, and so it is a non-zero multiple of the (by definition non-trivial) Kähler class ω. Consider again the Gysin sequence, but now with the cupping-with-e map replaced by cupping-with-ω,

$$\cdots \longrightarrow H^*(SO(2n)/U(n),\mathbb{R}) \xrightarrow{\cdot\omega} H^{*+2}(SO(2n)/U(n),\mathbb{R}) \xrightarrow{p^*} H^{*+2}(SO(2n)/SU(n),\mathbb{R}) \xrightarrow{\int_{S^1}} H^{*+1}(SO(2n)/U(n),\mathbb{R}) \xrightarrow{\cdot\omega} \cdots$$

By Hard Lefschetz, cupping with the appropriate power of the Kähler class gives an isomorphism between complementary cohomology groups. In particular, cupping only once with the Kähler class gives an injection $H^k(-,\mathbb{R}) \hookrightarrow H^{k+2}(-,\mathbb{R})$ for k less than half the dimension of the manifold. From here, our long exact Gysin sequence splits into short exact sequences

$$0 \longrightarrow H^k(SO(2n)/U(n),\mathbb{R}) \xrightarrow{\cdot e} H^{k+2}(SO(2n)/U(n),\mathbb{R}) \longrightarrow H^{k+2}(SO(2n)/SU(n),\mathbb{R}) \longrightarrow 0$$

for $k+2 \leq \frac{1}{2}\dim(SO(2n)/U(n)) = \frac{1}{2}(\binom{2n}{n} - n^2) = \frac{n^2-n}{2}$, giving us

$$H^{k+2}(SO(2n)/U(n);\mathbb{R}) = H^k(SO(2n)/U(n);\mathbb{R}) \oplus H^{k+2}(SO(2n)/SU(n);\mathbb{R}).$$

We observed that $SO(2n)/U(n)$ has cohomology concentrated in even degrees, so we conclude that $H^k(SO(2n)/SU(n);\mathbb{R}) = 0$ for odd $k \leq \frac{n^2-n}{2}$.

Now consider a real minimal model Λ for $SO(2n)/SU(n)$. The above observation that odd cohomology in degree less than half the dimension vanishes tells us that there are no closed generators up to half the dimension in Λ. Indeed, any closed generator in degree less than $\frac{n^2-n}{2}$ would have to be exact, but minimality prevents generators from being exact.

Quotients of connected Lie groups are sufficiently nice (they are simple) as to guarantee that the generators in a minimal model correspond to the real homotopy groups of the space. Now let us observe that the real homotopy of $SO(2n)/SU(n)$ vanishes above half the dimension and so no generators in Λ lie in that range. We will see this from the fibration $SU(n) \to SO(2n) \to SO(2n)/SU(n)$. Indeed, a model for $SU(n)$ can be obtained from the fibration $SU(n) \to ESU(n) \to BSU(n)$, and we conclude that it is a free algebra on odd generators (obtained as the universal higher Chern classes in $BSU(n)$ shifted down in degree by one), $\mathrm{Model}(SU(n)) = \Lambda(\overline{c_2}, \ldots, \overline{c_n})$, $\deg(\overline{c_i}) = 2i-1$. A model for $SO(2n)$ is obtained analogously from the fibration $SO(2n) \to ESO(2n) \to BSO(2n)$, and we have $\mathrm{Model}(SO(2n)) = \Lambda(\overline{p_1}, \ldots, \overline{p_{n-1}}, \overline{e})$, where the generators are shifted Pontryagin classes in degrees $4i-1$ and a shifted Euler class in degree $2n-1$. From these models we see that $\pi_{\geq 2n}(SU(n)) \otimes \mathbb{R} = 0$ and $\pi_{\geq 4n-4}(SO(2n)) \otimes \mathbb{R} = 0$, and so $\pi_{\geq 4n-4}(SO(2n)/U(n)) \otimes \mathbb{R} = 0$.

Now, if $4n-4 \leq \frac{n^2-n}{2}$, that is $n \geq 8$, we will have concluded that the real model Λ for $SO(2n)/SU(n)$ has no closed odd generators below half its dimension, and no generators at all above half its dimension. Therefore, the volume form of $SO(2n)/U(n)$, which is a closed non-exact odd element in Λ, will have to lie in the ideal generated by non-closed generators, and will therefore represent a non-trivial Massey product. Therefore $SO(2n)/SU(n)$ is not formal over $\mathbb{R}$ for $n \geq 8$, and so it is not formal over $\mathbb{Q}$.

The above argument is presented as Theorem B in [Am13]:

Let E be a simple circle fibration with finite dimensional rational cohomology fibering over a simply connected space B such that B satisfies the hard Lefschetz property.

If the hard Lefschetz class of B is a non-trivial multiple of the Euler class of the circle fibration, if the rational cohomology of B is concentrated in even degrees, and if the rational homotopy of B vanishes above half the dimension of B, then E is not formal.

References

[Am13] Amann, M., 2013. Non-formal homogeneous spaces. Mathematische Zeitschrift, 274(3-4), pp.1299-1325.