The minimal model of the free loop space of a simply connected space

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Following [1], we construct the minimal model of the free loop space LX of a simply connected space X from the minimal model of X. Recall that the free loop space of X is the space of all maps of the circle into X,

$$LX = \mathrm{Map}(S^1, X).$$

In order to describe the minimal model of LX, we will express its universal property via some diagram, then dualize to obtain a diagram of differential algebras. The universal object in this dual diagram will thus be a model for LX.

The object LX is such that for any test space K there is a bijective correspondence between maps from K to LX and maps from $K \times S^1$ to X (by "currying"). Indeed, we have the following diagram:

$$\begin{tikzcd}
LX \times S^1 \arrow[rr, "\mathrm{ev}"] & & X \\
& K \times S^1 \arrow[ul, "f \times \mathrm{id}"] \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}$$

Here $\mathrm{ev}(\gamma, z) = \gamma(z)$ for a loop γ and $z \in S^1$, and $\tilde{f}(k, z) = f(k)(z)$. Turning this into a diagram of minimal models, we have

$$\begin{tikzcd}
M(LX)(\xi) \arrow[dr, "f^* \otimes \mathrm{Id}_\xi"'] & & M(X) \arrow[ll, "u"'] \arrow[dl, "\tilde{f}^*"] \\
& M(K)(\xi) &
\end{tikzcd}$$

Here by $M(-)$ we denote the minimal model of a given space. The minimal model of S^1 is the free differential algebra in a single generator in degree one, denoted ξ, with trivial differential. For the product $K \times S^1$, we have $M(K \times S^1) = M(K) \otimes M(S^1)$, where the differential in the latter is the tensor product of the differentials in the components. So, $M(K \times S^1) = M(K) \otimes \Lambda(\xi)$, which we denote for simplicity by $M(K)(\xi)$ (and likewise for $M(LX)(\xi)$). On the level of differential algebras we denote by f^* the map induced by f.

Now let us make a guess at what the minimal model $M(LX)$ of the free loop space should look like. Consider the fibration

$$\begin{tikzcd}
\Omega X \arrow[r] & LX \arrow[d] \\
& X
\end{tikzcd}$$

where ΩX denotes the based loop space. The vertical map sends a loop to its starting (equivalently, ending) point. Note that this fibration admits a section. Namely, send a point in X to the constant loop at that point. Due to this section, the long exact sequence in homotopy

$$\cdots \to \pi_*(\Omega X) \to \pi_*(LX) \to \pi_*(X) \to \pi_{*-1}(\Omega X) \to \cdots$$

splits at the LX terms, giving us

$$\pi_*(LX) = \pi_*(\Omega X) \oplus \pi_*(X).$$

Since $\pi_k(\Omega X) = [S^k, \Omega X] = [\Sigma S^k, X] = [S^{k+1}, X] = \pi_{k+1}(X)$, we have

$$\pi_*(LX) = \pi_*(X) \oplus \pi_{*+1}(X).$$

The generators in the minimal model are dual to (a prescribed set of) generators of the rational homotopy groups. What the above formula tells us, rationally, is that for every generator x of the minimal model of X, in the minimal of LX we have both x (since X naturally sits inside LX) and a generator $\bar{x}$ somehow related to x, in one degree lower than x. So, the minimal model $M(LX)$ of the free loop space of X, is the exterior algebra on these x and $\bar{x}$, which we will denote by $\Lambda(x, \bar{x})$. Now we have to figure out what the differential in this algebra should be.

A natural candidate for the universal map $u : M(X) \to M(LX)(\xi)$ in the "dual" diagram above, since it should be degree-preserving (and $\bar{x}$ is related to x), is the map $u(x) = x + \xi\bar{x}$. We have not yet checked that u is multiplicative, nor have we considered what $\bar{xy}$ should be, for generators x and y in $M(X)$. So consider

$$\begin{aligned} u(x)u(y) &= (x + \xi\bar{x})(y + \xi\bar{y}) \\ &= xy + \xi(\bar{x}y + x\bar{y}) \pm \xi^2\bar{x}\bar{y} \\ &= xy \pm \xi(\bar{x}y + x\bar{y}), \end{aligned}$$

where we write $\pm$ since graded commutativity took effect when we moved ξ around, and we used $\xi^2 = 0$ since ξ is of odd degree. So, if we define a function i on $M(LX) = \Lambda(x, \bar{x})$ by $i(x) = \bar{x}$ and $i(\bar{x}) = 0$, we have that i is both a derivation and a differential on $M(LX)$. Thus defining $u(a) = a + \xi \cdot i(a)$ for an arbitrary $a \in M(X)$ (not necessarily a generator) gives us a multiplicative map.

Now let us verify that this universal map u indeed gives us a bijective correspondence, for an arbitrary test space K, between maps $f^* : M(LX) \to M(K)$ and $\tilde{f}^* : M(X) \to M(K)(\xi)$. Given f^*, define $\tilde{f}^*$ by

$$\tilde{f}^*(a) = f^*(a) + f^*(i(a))\xi.$$

This makes the dual diagram commute. On the other hand, given $\tilde{f}^*$, define

$$\begin{aligned} f^*(x) &= \tilde{f}^*(x)|_{\xi=0} \\ f^*(\bar{x}) &= \tilde{f}^*(x)|_{\xi=1}, \end{aligned}$$

for generators x and $\bar{x}$ of $\Lambda(x, \bar{x}) = M(LX)$.

Finally, let us see what the differential in $\Lambda(x, \bar{x})$ should be. The differential in $M(LX)$ should be an extension of the one in $M(X)$, so let us call both of them d. For the universal map u to be a map of differential algebras, we need u to respect the differential, i.e. $d(u(x)) = d(x = \xi i(x) = dx + d\xi \cdot i(x) - \xi \cdot d(i(x))$ has to equal $u(dx) = dx + \xi \cdot i(dx)$. From here we read that $di + id = 0$. This uniquely defines d on $\Lambda(x, \bar{x})$. Indeed, on the generators x it is already defined, and for $\bar{x}$ we have

$$d(i(x)) + i(dx) = 0,$$

so therefore

$$d\bar{x} = -i(dx).$$

To summarize, the minimal model of the free loop space LX of X simply connected is given by taking the minimal model $\Lambda(x)$ of X, adding a new generator $\bar{x}$ for every generator x in $\Lambda(x)$, so that $\deg(\bar{x}) = \deg(x) - 1$, and extending the differential d from $\Lambda(x)$ to $\Lambda(x, \bar{x})$ by setting $d\bar{x} = -i(dx)$, where i is a differential and derivation of degree -1 given by $i(x) = \bar{x}$.

Example. Using the above we can immediately express the minimal model of the free loop space of, say, the six-sphere. First of all, $M(S^6) = \Lambda(x, y)$, where $\deg(x) = 6, \deg(y) = 11, dx = 0, dy = x^2$. To create $M(LS^6)$, we introduce $\bar{x}$ and $\bar{y}$ in degrees 5 and 10, respectively, and set $d\bar{x} = -i(dx) = 0$ and $d\bar{y} = -i(dy) = i(x^2) = -2xi(x) = -2x\bar{x}$. Therefore,

$$M(LS^6) = \Lambda(x_6, \bar{x}_5, y_{11}, \bar{y}_{10}\ ;\ dx = 0, d\bar{x} = 0, dy = x^2, d\bar{y} = -2x\bar{x}).$$

Example. In another note (on Lie groups being products of odd spheres, rationally) we showed that rationally $SO(6) = S^3 \times S^5 \times S^7$. Therefore on the level of minimal models,

$$M(LSO(6)) = M(LS^3 \times LS^5 \times LS^7) = M(LS^3) \otimes M(LS^5) \otimes M(LS^7) = \Lambda(a_3, \bar{a}_2, b_5, \bar{b}_4, c_7, \bar{c}_6)$$

with trivial differential (triviality of the differential follows easily from $di + id = 0$). In [1] it is shown that the Betti numbers of the free loop space of a given space can be arbitrarily large if and only if the cohomology ring of the given space requires at least two generators. The cohomology ring of $S^3 \times S^5 \times S^7$ requires three generators, so we have the Betti numbers of $LSO(6)$ are unbounded.

References.

[1] Vigué-Poirrier, M. and Sullivan, D., 1976. The homology theory of the closed geodesic problem. Journal of Differential Geometry, 11(4), pp.633-644.