

# ON THE TOPOLOGY OF THE SPACE OF ALMOST COMPLEX STRUCTURES ON THE SIX SPHERE

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ABSTRACT. The space of orientation-compatible almost complex structures on  $S^6$  naturally contains a copy of  $\mathbb{R}\mathbb{P}^7$ . We show that the inclusion induces an isomorphism on fundamental groups and rational homotopy groups. We also compute the homotopy fiber of the inclusion and the homotopy groups of the space of almost complex structures in terms of the homotopy groups of  $S^7$ . Our approach lends itself to generalization to components of almost complex structures with  $c_1 = 0$  on six-manifolds.

## 1. INTRODUCTION

In this note we consider the topology of the space of almost complex structures  $\mathcal{J}(S^6)$  on the six-dimensional sphere  $S^6$ . By an almost complex structure we will mean one which induces a fixed orientation and which is orthogonal with respect to some fixed metric on  $S^6$ ; swapping the orientation, varying the metric, or removing the metric compatibility condition altogether results in a homotopy equivalent space via the appropriate natural map. For this reason we from now on take  $S^6$  to be the unit sphere in the imaginary octonions endowed with the round metric and the standard orientation.

We will employ the useful description of  $\mathcal{J}(S^6)$  as the space of sections of the natural  $SO(6)/U(3) \cong \mathbb{C}\mathbb{P}^3$  bundle over  $S^6$  whose fibers are the complex structures on the given tangent vector space. Likely one would be interested primarily in the smooth sections of this bundle, corresponding to smooth almost complex structures; the inclusion of the space of smooth sections into the space of continuous sections is a homotopy equivalence, so we will study the latter without loss of generality.

Let  $J^{\text{cn}}$  denote the canonical almost complex structure given on  $T_p S^6$  by octonion multiplication

$$J_p^{\text{cn}}(x) = px.$$

The group  $SO(7)$  acts naturally on the space of sections by

$$(A \cdot J)_p(v) = AJ_{A^{-1}p}(A^{-1}v).$$

For our canonical almost complex structure  $J^{\text{cn}}$  we have that  $A \cdot J^{\text{cn}} = J^{\text{cn}}$  means

$$A(pv) = (Ap)(Av),$$

and so the isotropy of this action is the group  $G_2$  of real algebra automorphisms of the octonions. We thus obtain an inclusion

$$SO(7)/G_2 \cong \mathbb{R}\mathbb{P}^7 \xrightarrow{j} \mathcal{J}(S^6).$$

This subspace was previously studied by Battaglia [Ba90]. Her description of the space of complex structures on  $\mathbb{R}^6$  in terms of octonion multiplication is crucial to the present work (see Proposition 2.1).

The aim of this note is to show that the homotopy fiber of the inclusion  $\mathbb{R}\mathbb{P}^7 \xrightarrow{j} \mathcal{J}(S^6)$  is a path component of the sevenfold based loop space  $\Omega^7 S^7$  of the seven-sphere. As the homotopy groups of the latter space are torsion, we recover the result of the first named author [F18, §4] that  $\mathcal{J}(S^6)$  has

the rational homotopy groups of  $S^7$  (or  $\mathbb{RP}^7$ ), and see that the natural inclusion of  $\mathbb{RP}^7$  induces an isomorphism on rational homotopy groups and on fundamental groups. This also answers a question communicated by Dennis Sullivan and stated in *loc. cit.*, on whether this inclusion is a genuine (i.e. integral) homotopy equivalence.

We then touch upon the general problem of the topology of the space of almost complex structures on a closed six-manifold in the component of an almost complex structure with  $c_1 = 0$ . This, along with the case of  $c_1 \neq 0$ , will be studied more systematically in upcoming work.

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## 2. COMPLEX STRUCTURES ON $\mathbb{R}^6$

We use [CS] as a reference for the normed division algebra  $\mathbb{O}$  of the octonions. This algebra is obtained from the quaternions  $\mathbb{H}$  by the Cayley–Dickson construction: given quaternions  $a, b$ , an octonion is a pair of quaternions  $(a, b)$ , written  $a + Ib$ , with multiplication rule given by

$$(1) \quad (a + Ib)(c + Id) = (ac - d\bar{b}) + I(cb + \bar{a}d)$$

We identify  $\mathbb{R}^6$  with the orthogonal complement to  $\mathbb{C} = \langle 1, i \rangle$ ,

$$\mathbb{R}^6 = \langle 1, i \rangle^\perp \subset \mathbb{O}.$$

A complex structure  $J$  on  $\mathbb{R}^6$  is said to be compatible with the standard orientation if a complex basis  $(u, v, w)$  for  $\mathbb{R}^6$  gives rise to a real basis  $(u, Ju, v, Jv, w, Jw)$  with the standard orientation. Given a unit octonion  $x$ , let  $R_x \in SO(8)$  denote right multiplication by the octonion  $x$ .

**Proposition 2.1.** Let  $J$  be an orthogonal complex structure on  $\mathbb{R}^6$  compatible with the standard orientation. The following equivalent statements hold:

- (1)  $J$  is obtained by conjugating the standard complex structure (given by left multiplication by  $i$ ) by the map  $R_{\bar{x}}$  for some unit octonion  $x$ , which is unique up to the action of  $S^1 \subset \langle 1, i \rangle = \mathbb{C}$ . Namely,  $J$  is of the form

$$J_x(v) = (i(vx))\bar{x} =: i^{R_{\bar{x}}}(v).$$

- (2)  $J$  has a complex line in common with  $i$ , i.e. there is a complex line  $L$  on  $\mathbb{R}^6$  (with complex structure given by  $i$ ) such that  $J$  restricted to  $L$  is equal to (left multiplication by)  $i$ .

*Proof.* The formula above for  $J_x$  can be understood as follows using quaternion coordinates: let  $A$  denote a quaternion subalgebra of  $\mathbb{O}$  containing  $1, i$  and  $x$  (which is unique unless  $x \in \mathbb{C}$ ). Fix a standard basis  $\langle 1, i, j, ij \rangle$  for this algebra and write

$$x = \cos \theta + w \sin \theta$$

with  $w$  a unit vector in the  $\langle i, j, ij \rangle$  plane. Let  $I \in \mathbb{O}$  be a unit vector orthogonal to  $A$  and write an element  $v \in \mathbb{R}^6$  as

$$v = a + Ib$$

with<sup>1</sup>  $a, b \in A$ . Using (1) we see that

$$\begin{aligned} (i(vx))\bar{x} &= (i((a + Ib)(\cos \theta + w \sin \theta)))(\cos \theta - w \sin \theta) \\ &= (ia(\cos \theta + w \sin \theta) + I((-i)((\cos \theta + w \sin \theta)b)))(\cos \theta - w \sin \theta) \\ &= ia + I((\cos \theta - w \sin \theta)(-i)(\cos \theta + w \sin \theta)b) \\ &= ia + ((\cos \theta - w \sin \theta)i(\cos \theta + w \sin \theta))(Ib) \end{aligned}$$

<sup>1</sup>We are interested in the case  $a = zj$  with  $z \in \mathbb{C}$  but the computation holds for an arbitrary  $a \in A$ .

This means that  $J_x$  acts as

- left multiplication by  $i$  on the quaternion subalgebra  $A$ ,
- left multiplication by  $l$  on the orthogonal plane  $A^\perp$  where  $l$  is the unit quaternion obtained from  $i$  by rotating the unit imaginary quaternions in  $A$  by an angle of  $2\theta$  around the axis  $w$ .

If  $x$  and  $y$  are unit octonions such that  $J_x = J_y$  then  $x$  and  $y$  must both be contained in the same quaternion subalgebra  $A = \langle 1, i, j, ij \rangle \subset \mathbb{O}$ . If we write  $x = z_1 + z_2j$  and  $y = w_1 + w_2j$  with  $z_i, w_i \in \mathbb{C}$ , the equality of  $J_x$  and  $J_y$  on  $A^\perp$  translates to

$$(\overline{z_1} - z_2j)i(z_1 + z_2j) = (\overline{w_1} - w_2j)i(w_1 + w_2j),$$

and one easily checks this amounts to the existence of  $e^{i\theta}$  such that  $y = e^{i\theta}x$ .

We now prove the equivalence of statements (1) and (2). That (1) implies (2) is immediate from the description of  $J_x$  in quaternion coordinates. Conversely, assume that  $J$  is an almost complex structure agreeing with  $i$  on the complex line  $L \subset \mathbb{R}^6$ . Then  $A = \mathbb{C} \oplus L$  is a quaternion subalgebra of  $\mathbb{O}$ . Since  $J$  is orthogonal and preserves orientation, its restriction to  $A^\perp$  must be given by left multiplication by a unit imaginary element  $l \in A$ . Clearly there exists  $x = \cos\theta + w \sin\theta \in A$  such that  $\overline{w}ix = l$  and, for such an  $x$ , we have  $J = J_x$ . We have already seen that  $x$  is unique up to left multiplication by a unit complex number.

Statement (1) follows immediately from [Ba90, Proposition 3.5], but for completeness we sketch a proof of (2). A complex structure  $J$  on  $\mathbb{R}^6$  determines a subspace

$$\mathbb{C}\mathbb{P}_J^2 \subset \text{Gr}_2^+(\mathbb{R}^6)$$

consisting of  $J$ -complex lines inside  $\text{Gr}_2^+(\mathbb{R}^6)$  (the Grassmannian of oriented planes). Since the space of complex structures compatible with the orientation is path connected, the fundamental class  $[\mathbb{C}\mathbb{P}_J^2] \in H_4(\text{Gr}_2^+(\mathbb{R}^6))$  is independent of  $J$ . The self intersection of  $\mathbb{C}\mathbb{P}^2$  in  $\text{Gr}_2^+(\mathbb{R}^6)$  is given by the Euler class of the normal bundle, which one can check to be 1. It follows that, for any two complex structures  $J, J'$ , the subspaces  $\mathbb{C}\mathbb{P}_J^2$  and  $\mathbb{C}\mathbb{P}_{J'}^2$  intersect, i.e.  $J$  and  $J'$  have a complex line in common. If  $J$  and  $J'$  are orthogonal, this line is necessarily unique, and  $J$  and  $J'$  must agree on that line.  $\square$

**Remark 2.2.** The previous Proposition gives rise to the following geometric description of the space  $J(\mathbb{R}^6)$  of orthogonal complex structures on the vector space  $\mathbb{R}^6$  compatible with the standard orientation: Let  $\mathbb{C}\mathbb{P}^2$  be the space of complex lines in  $\mathbb{R}^6$  for the complex structure  $i$ . The tautological line bundle  $T \rightarrow \mathbb{C}\mathbb{P}^2$  gives rise to a bundle of quaternion algebras  $T \oplus \langle 1, i \rangle \rightarrow \mathbb{C}\mathbb{P}^2$ . There is a map from the unit sphere bundle  $S(T \oplus \mathbb{C}) \rightarrow J(\mathbb{R}^6)$  sending  $x \mapsto i^{R_x}$ , which descends to the quotient by the natural action of  $S^1 \subset \mathbb{C}$ . This yields a surjective map  $\phi: P(T \oplus \mathbb{C}) \rightarrow J(\mathbb{R}^6)$  which collapses the 0-section to the point  $i$  and is bijective away from the 0-section. The map  $\phi$  expresses  $J(\mathbb{R}^6)$  as a blow-down of  $\mathbb{C}\mathbb{P}^3 \# \mathbb{C}\mathbb{P}^3$ .

Alternatively, a complex structure other than  $i$  determines an element of  $\mathbb{C}\mathbb{P}^2$  (the line on which it coincides with  $i$ ). Identifying  $J(\mathbb{R}^6)$  with  $\mathbb{C}\mathbb{P}^3$ , we obtain a map

$$\mathbb{C}\mathbb{P}^3 \setminus \{i\} \rightarrow \mathbb{C}\mathbb{P}^2$$

with fiber the affine line  $\mathbb{C}\mathbb{P}^1 \setminus *$  of all possible orthogonal complex structures on the orthogonal  $\mathbb{R}^4$  other than  $i$  itself. Considering the section given by assigning to  $L \in \mathbb{C}\mathbb{P}^2$  the complex structure given by  $i$  on  $L$  and  $-i$  on  $L^\perp$ , we may regard this fibration as the standard fibration of  $\mathbb{C}\mathbb{P}^3 \setminus *$  by linear projection onto  $\mathbb{C}\mathbb{P}^2$  from a point at  $\infty$  on a complementary line.

### 3. ALMOST COMPLEX STRUCTURES ON $S^6$

Recall that  $S^6$  denotes the unit sphere in the imaginary octonions equipped with the round metric and the standard orientation. Each  $p \in S^6$  determines (by left octonion multiplication) an orthogonal

complex structure on  $\mathbb{R}^8$  and on  $T_p S^6 = \langle 1, p \rangle^\perp \subset \mathbb{O}$ . Since there is an automorphism of the octonions taking  $i$  to  $p$ , it follows from Proposition 2.1 that an arbitrary orthogonal complex structure on  $T_p S^6$  can be written as  $p^{R_{\bar{x}}}$  for some unit octonion  $x$ .

We have the following description of the twistor space  $Z(S^6) \cong SO(7)/U(3)$ :

**Proposition 3.1.** Consider the  $S^1$ -action on  $S^6 \times S^7$  defined by

$$e^{i\theta} \cdot (p, x) = (p, (\cos \theta + p \sin \theta)x)$$

The map

$$(S^6 \times S^7)/S^1 \xrightarrow{\psi} Z(S^6)$$

given by

$$[(p, x)] \mapsto p^{R_{\bar{x}}}$$

is a diffeomorphism.

*Proof.* The quaternion subalgebras containing  $p$  and  $x$  are the same as the quaternion subalgebras containing  $p$  and  $(\cos \theta + p \sin \theta)x$ . The computation in the proof of Proposition 2.1 shows that

$$(p(v((\cos \theta + p \sin \theta)x)))\overline{((\cos \theta + p \sin \theta)x)} = (p(vx))\bar{x}$$

so that  $\psi$  is well defined. Again by Proposition 2.1, the map  $\psi$  is a fiberwise bijection and hence a fiberwise homeomorphism. Clearly  $\psi$  is a smooth map. We leave it to the reader to check that  $\psi^{-1}$  is also smooth.  $\square$

**Remark 3.2.** The twistor space of  $S^6$  is the space of orthogonal complex structures on the vector space  $\mathbb{R}^8 \simeq \mathbb{O}$  compatible with the standard orientation. Proposition 2.1 gives the following picture for an element  $J$  of this space: there is a unique unit octonion  $p \in S^6$  with which  $J$  agrees on the plane  $\langle 1, p \rangle$  (and then necessarily on some four-plane containing this) and  $J$  is then determined by a six-dimensional complex structure on  $\langle 1, p \rangle^\perp$ .

**Remark 3.3.** Recall that it follows from the Cayley–Dickson construction that  $G_2$  acts transitively on  $S^6$  and the isotropy group of  $i$  can be identified with  $SU(3)$ . The fibration

$$SO(6)/U(3) \rightarrow SO(7)/U(3) \rightarrow S^6$$

is associated to the principal fibration

$$(2) \quad SU(3) \rightarrow G_2 \rightarrow S^6$$

where  $SU(3)$  acts on  $\mathbb{C}\mathbb{P}^3 = SO(6)/U(3)$  fixing a distinguished point (in the fiber over  $p$  this is the tautological complex structure  $p$ ). Indeed, we have a map

$$G_2 \times S^7 \xrightarrow{\Psi} S^6 \times S^7$$

given by

$$(\phi, x) \mapsto (\phi(i), \phi(x))$$

If  $\lambda \in SU(3)$ , we have  $\Psi(\phi\lambda, x) = \Psi(\phi, \lambda(x))$  so  $\Psi$  descends to a diffeomorphism  $\bar{\Psi}: G_2 \times_{SU(3)} S^7 \rightarrow S^6 \times S^7$ . As  $\Psi$  conjugates the standard  $S^1$  action on the second coordinate of  $G_2 \times S^7$  to the action of Proposition 3.1, we see that  $\bar{\Psi}$  induces a diffeomorphism

$$G_2 \times_{SU(3)} \mathbb{C}\mathbb{P}^3 \rightarrow (S^6 \times S^7)/S^1.$$

Note that the clutching map for (2) is a generator of  $\pi_5 SU(3)$  (see for instance [G19]) which becomes trivial in  $\pi_5 G_2$ , so the bundle of normed division algebras associated to (2) is trivial. The map  $\bar{\Psi}$  is an explicit trivialization.

Consider the map of fibrations over  $S^6$

$$\begin{array}{ccc} S^6 \times S^7 & \longrightarrow & SO(7)/U(3) \\ \downarrow & & \downarrow \\ S^6 & \longrightarrow & S^6 \end{array}$$

Taking sections and evaluating at  $i$  gives rise to a square diagram of fibrations

$$\begin{array}{ccc} \text{Map}(S^6, S^7) & \xrightarrow{\varpi} & \mathcal{J}(S^6) \\ \downarrow \text{ev}_i & & \downarrow \text{ev}_i \\ S^7 & \longrightarrow & \mathbb{C}\mathbb{P}^3 \end{array}$$

This can be completed to a  $3 \times 3$  diagram of fiber sequences

$$(3) \quad \begin{array}{ccccc} \text{Map}_*(S^6, S^1) & \longrightarrow & \Omega^6 S^7 & \xrightarrow{\varpi'} & \mathcal{J}_*(S^6) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Map}(S^6, S^1) & \xrightarrow{\iota} & \text{Map}(S^6, S^7) & \xrightarrow{\varpi} & \mathcal{J}(S^6) \\ \downarrow \text{ev}_i & & \downarrow \text{ev}_i & & \downarrow \text{ev}_i \\ S^1 & \longrightarrow & S^7 & \longrightarrow & \mathbb{C}\mathbb{P}^3 \end{array}$$

where  $\mathcal{J}_*(S^6) = \{J \in \mathcal{J}(S^6) : J(i) = i\}$ ; notice that  $\mathcal{J}_*(S^6) \simeq \Omega^6 \mathbb{C}\mathbb{P}^3 \simeq \Omega^6 S^7$ . The fiber of  $\varpi$  is the space of sections of a trivial  $S^1$ -bundle over  $S^6$ , and we have used the canonical trivialization

$$(p, e^{i\theta}) \mapsto (p, \cos \theta + p \sin \theta) \in S^6 \times S^7$$

to identify this space of sections with  $\text{Map}(S^6, S^1)$ . By  $\text{Map}_*(S^6, S^1)$  we denote the (contractible) space of maps from  $S^6$  to  $S^1$  which send  $i$  to 1.

**Lemma 3.4.** The map  $\iota$  in (3) induces multiplication by 2 on  $\pi_1$ .

*Proof.* A generator of  $\pi_1 \text{Map}(S^6, S^1)$  is given by

$$e^{i\theta} \mapsto (p \mapsto e^{i\theta})$$

and its image in  $\pi_1 \text{Map}(S^6, S^7)$  is given by

$$(4) \quad e^{i\theta} \mapsto (p \mapsto (\cos \theta + p \sin \theta)).$$

Note that  $\pm p$  both get sent to the same circle in  $S^7$ . Therefore, upon the natural identification  $\pi_1 \Omega^6 S^7 = \pi_0 \Omega^7 S^7$ , the element in  $\pi_1 \Omega^6 S^7$  corresponding to the loop (4) via the natural inclusion  $\Omega^6 S^7 \hookrightarrow \text{Map}(S^6, S^7)$  becomes the component of a map of degree 2 from  $S^7$  to  $S^7$ .  $\square$

**Remark 3.5.** We can also compute  $\pi_1 \mathcal{J}(S^6)$ , and hence obtain an alternative proof of the above lemma, using the following result of Crabb and Sutherland [CrSu84, Proposition 2.7 and Theorem 2.12]:

*Let  $X$  be an oriented closed connected  $2n$ -manifold and  $\xi$  a complex rank  $n+1$  bundle over  $X$ . Denote by  $N\xi$  the space of sections of the projective bundle  $P\xi$  which lift to sections of  $\xi$ ; this is a non-empty connected space. Then  $\pi_1(N\xi)$  is a central extension*

$$0 \rightarrow \mathbb{Z}/c_n(\xi)[X] \rightarrow \pi_1(N\xi) \rightarrow H^1(X; \mathbb{Z}) \rightarrow 0.$$

We note that our  $\mathbb{C}\mathbb{P}^3$  bundle  $\mathbb{C}\mathbb{P}^3 \rightarrow SO(7)/U(3) \rightarrow S^6$  is the projectivization of the rank 4 complex vector bundle of positive pure spinors  $\mathcal{S}^+$ , see [LM, Proposition IV.9.8 and Remark IV.9.12]. Taking  $\xi = \mathcal{S}^+$ , since  $H^2(S^6; \mathbb{Z}) = 0$ , every section of  $P\xi$  lifts to a section of  $\xi$ , i.e.  $N\xi = \mathcal{J}(S^6)$ , and so by the above result and knowledge of the cohomology of  $SO(7)/U(3)$  (see e.g. [P91, Proposition 3.2]) we

conclude from the projective bundle formula that  $\pi_1 \mathcal{J}(S^6) = \mathbb{Z}/c_3(\mathcal{S}^+)[S^6] = \mathbb{Z}/2$ . Lemma 3.4 follows from here, i.e. the map  $\text{Map}(S^6, S^1) \xrightarrow{\sim} \text{Map}(S^6, S^7)$  in diagram 3 must induce  $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$  on fundamental groups.

**Corollary 3.6.** The homotopy groups of the space  $\mathcal{J}(S^6)$  are given by

$$\pi_k \mathcal{J}(S^6) = \begin{cases} \mathbb{Z}/2 & \text{if } k = 1, \\ \pi_k(S^7) \oplus \pi_{k+6}(S^7) & \text{otherwise.} \end{cases}$$

*Proof.* Since  $S^7$  is an  $H$ -space we have  $\text{Map}(S^6, S^7) \simeq S^7 \times \text{Map}_*(S^6, S^7) = S^7 \times \Omega^6 S^7$ . The result follows from the long exact sequence in the middle row of (3) and Lemma 3.4.  $\square$

Since the map  $\text{Map}(S^6, S^7) \xrightarrow{\pi} \mathcal{J}(S^6)$  in diagram (3) factors through  $\text{Map}(S^6, \mathbb{R}P^7)$  (as  $-1$  is in the center of  $\mathbb{O}$ ), the constant sections of  $S^6 \times S^7 \xrightarrow{\pi_1} S^6$  give rise to an inclusion

$$(5) \quad \mathbb{R}P^7 \xrightarrow{j} \mathcal{J}(S^6).$$

The image of this map is what Battaglia [Ba90] calls the space of linear almost complex structures on  $S^6$  and, as we explain in Proposition 4.1 below, equals the subspace  $SO(7)/G_2$  mentioned in the introduction.

**Theorem 3.7.** The inclusion  $j$  in (5) sits in a homotopy fiber sequence

$$\Omega_0^7 S^7 \rightarrow \mathbb{R}P^7 \xrightarrow{j} \mathcal{J}(S^6),$$

where  $\Omega_0^7 S^7$  denotes a path component of the sevenfold based loop space of  $S^7$ .

*Proof.* Proposition 2.1 identifies the evaluation map  $\mathbb{R}P^7 \xrightarrow{\text{ev}_i} \mathbb{C}P^3$  with the Hopf fibration  $\mathbb{R}P^1 \rightarrow \mathbb{R}P^7 \rightarrow \mathbb{C}P^3$ .

The map of fibrations

$$\begin{array}{ccc} \mathbb{R}P^7 & \longrightarrow & \mathcal{J}(S^6) \\ \downarrow \text{ev}_i & & \downarrow \text{ev}_i \\ \mathbb{C}P^3 & \xrightarrow{=} & \mathbb{C}P^3 \end{array}$$

can be extended (see [N10, Section 3.2]) to a  $3 \times 3$  diagram of homotopy fiber sequences

$$(6) \quad \begin{array}{ccccc} G & \longrightarrow & \mathbb{R}P^1 & \longrightarrow & \mathcal{J}_*(S^6) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & \mathbb{R}P^7 & \longrightarrow & \mathcal{J}(S^6) \\ \downarrow & & \downarrow \text{ev}_i & & \downarrow \text{ev}_i \\ * & \longrightarrow & \mathbb{C}P^3 & \xrightarrow{=} & \mathbb{C}P^3 \end{array}$$

We will first prove that it suffices to show that  $\mathbb{R}P^1 \rightarrow \mathcal{J}_*(S^6)$  sends a generator of  $\pi_1 \mathbb{R}P^1$  to a generator of  $\pi_1 \mathcal{J}_*(S^6) \cong \pi_1 \Omega^6 S^7 \cong \mathbb{Z}$ . Indeed, if  $g: S^1 \rightarrow \Omega^6 S^7$  generates  $\pi_1$  we may extend the homotopy commutative square

$$\begin{array}{ccc} S^1 & \xrightarrow{g} & \Omega^6 S^7 \\ \downarrow = & & \downarrow \\ S^1 & \xrightarrow{=} & S^1 \end{array}$$

(where the right vertical arrow classifies the generator of  $H^1(\Omega^6 S^7) = \mathbb{Z}$  to a  $3 \times 3$  square of homotopy fiber sequences. It is then clear that the space  $G$  in (6) is homotopy equivalent to  $\Omega_0^7 S^7$ , from which it follows that the same is true of  $F$ .

The image of a generator of  $\pi_1(\mathbb{R}\mathbb{P}^1)$  under the map  $\mathbb{R}\mathbb{P}^1 \rightarrow \mathcal{J}_*(S^6)$  is represented by the loop  $\gamma: [0, 1] \rightarrow \mathcal{J}_*(S^6)$  given by

$$t \mapsto (p \mapsto p^{R_{\cos \pi t - i \sin \pi t}}).$$

A lift  $\tilde{\gamma}$  of this loop to  $\Omega^6 S^7$  along the map  $\varpi'$  in (3) is given by the formula

$$t \mapsto (p \mapsto (\cos \pi t - p \sin \pi t)(\cos \pi t + i \sin \pi t)).$$

The image of  $\tilde{\gamma}$  under the canonical identifications  $\pi_1 \Omega^6 S^7 = \pi_0 \Omega^7 S^7 = [S^7, S^7]$  is the product of the homotopy classes of the maps

$$\cos \pi t + p \sin \pi t \mapsto \cos \pi t - p \sin \pi t \quad \text{and} \quad \cos \pi t + p \sin \pi t \mapsto \cos \pi t + i \sin \pi t$$

using the  $H$ -space structure on  $S^7$ . The first map has degree 1, while the second one has degree 0. This completes the proof.  $\square$

**Corollary 3.8.** The inclusion  $\mathbb{R}\mathbb{P}^7 \xrightarrow{j} \mathcal{J}(S^6)$  is an isomorphism on fundamental groups and rational homotopy groups.

#### 4. TWO $\mathbb{R}\mathbb{P}^7$ S MAPPING TO $\mathcal{J}(S^6)$

We will now explain why the subspace  $j(\mathbb{R}\mathbb{P}^7) \subset \mathcal{J}(S^6)$  in Theorem 3.7, coming from the constant sections of the fibration  $S^6 \times S^7 \rightarrow S^6$  agrees with the orbit  $SO(7)/G_2$  of the natural  $SO(7)$ -action on the canonical element of  $\mathcal{J}(S^6)$ . This is also explained in [Ba90] but we include an argument for completeness.

Recall from [CS, Section 8.2] that  $\text{Spin}(8)$  is the group of orthogonal isotopies of the octonions

$$\text{Spin}(8) = \{(\phi, \psi, \lambda) \in SO(8)^3: \phi(x)\psi(y) = \lambda(xy) \text{ for all } x, y \in \mathbb{O}\}.$$

Each coordinate in a triple  $(\phi, \psi, \lambda)$  determines the other two up to a global sign [CS, Theorem 8.3]. The projections  $\text{Spin}(8) \rightarrow SO(8)$  onto the first two coordinates are the spin representations of  $\text{Spin}(8)$  while the projection onto the third coordinate is the natural projection to  $SO(8)$ .

Given  $\lambda \in SO(8)$ , it is proved in [CS, Theorem 8.11] that there are unit octonions  $a_\lambda, b_\lambda \in S^7$  that are unique up to sign, called *companions* of  $\lambda$ , such that the elements  $(\phi, \psi, \lambda) \in \text{Spin}(8)$  satisfy

$$\phi(x) = \lambda(x)a_\lambda, \quad \psi(y) = b_\lambda \lambda(y) \quad \text{for all } x, y \in \mathbb{O}.$$

Note that the classes of the companions in  $\mathbb{R}\mathbb{P}^7$  can be regarded as the ‘‘difference’’ between the  $SO(8)$  and the spin representations of  $\text{Spin}(8)$  and therefore vary smoothly with  $\lambda$ .

Let  $SO(7) = \{\lambda \in SO(8): \lambda(1) = 1\}$ . Then for  $(\phi, \psi, \lambda) \in \text{Spin}(8)$  with  $\lambda \in SO(7)$  we have  $\phi(x)\psi(\bar{x}) = 1$  for all  $x \in \mathbb{O}$ , and hence

$$\psi(x) = \overline{\phi(\bar{x})}, \quad b_\lambda = \bar{a}_\lambda.$$

**Proposition 4.1.** Let  $J^{\text{cn}}$  denote the canonical almost complex structure from the introduction, and let  $j$  denote the inclusion in (5). Then  $SO(7) \cdot J^{\text{cn}} = j(\mathbb{R}\mathbb{P}^7)$ .

*Proof.* Given  $\lambda \in SO(7)$ ,  $p \in S^6$ ,  $v \in \mathbb{O}$  and writing  $a_\lambda$  for a companion of  $\lambda$ , we have

$$\begin{aligned} \lambda(\lambda^{-1}(p)\lambda^{-1}(v)) &= (\lambda(\lambda^{-1}(p))a_\lambda)(\bar{a}_\lambda \lambda(\lambda^{-1}(v))) \\ &= (pa_\lambda)(\bar{a}_\lambda v) \\ &= (pa_\lambda)(\bar{a}_\lambda(va_\lambda)\bar{a}_\lambda) \\ &= (p(va_\lambda))\bar{a}_\lambda \end{aligned}$$

where the last two equalities follow from the Moufang laws [CS, Section 7.4].

Hence  $\lambda \cdot J^{\text{cn}} = j([a_\lambda])$  and we have  $SO(7) \cdot J^{\text{cn}} \subset j(\mathbb{RP}^7)$ . The subgroup  $\{(\phi, \psi, \lambda) \in \text{Spin}(8) : \lambda \in SO(7)\}$  is  $\text{Spin}(7)$  and the projection

$$(\phi, \psi, \lambda) \mapsto \phi$$

is the restriction of a spin representation of  $\text{Spin}(8)$  to  $\text{Spin}(7)$ . This is the spin representation of  $\text{Spin}(7)$ . Evaluating at 1 we see that

$$\phi(1) = \pm a_\lambda.$$

Since the spin representation of  $\text{Spin}(7)$  acts transitively on  $S^7$  the proof is complete.  $\square$

We have been considering the inclusion  $SO(7)/G_2 \cong \mathbb{RP}^7 \hookrightarrow \mathcal{J}(S^6)$ . We can also consider the map  $\mathbb{RP}^7 \rightarrow \mathcal{J}(S^6)$  given by the composite

$$(7) \quad S^7/\mathbb{Z}_2 = \mathbb{RP}^7 \xrightarrow{c} SO(7) \rightarrow SO(7)/G_2 \hookrightarrow \mathcal{J}(S^6)$$

where the map  $c$  takes (the equivalence class of) a unit octonion to its action on the imaginary octonions by conjugation.

**Proposition 4.2.** The composite  $\mathbb{RP}^7 \xrightarrow{c} SO(7) \rightarrow SO(7)/G_2$  is a degree three self-map of  $\mathbb{RP}^7$ . Hence the composite  $\mathbb{RP}^7 \hookrightarrow \mathcal{J}(S^6)$  in (7) also induces an isomorphism on fundamental groups and on rational homotopy groups.

*Proof.* Given a unit octonion  $x$  and  $p \in S^6$ , let  $A$  be a quaternion algebra containing  $p, x$  with standard basis  $\langle 1, p, j, pj \rangle$  and decompose  $\mathbb{R}^8 \cong \mathbb{O}$  as  $A \oplus IA$ . We will compute the action of  $c(x)$  on the canonical complex structure  $J^{\text{cn}}$  in these coordinates.

$$(c(x) \cdot J^{\text{cn}})_p(v) = \bar{x}((xp\bar{x})(xv\bar{x}))x$$

If  $v \in A$ , then associativity implies that  $(c(x) \cdot J^{\text{cn}})_p(v) = pv$ . Suppose now that  $v = Iw$  with  $w \in A$ . Then, using (1) we obtain

$$\begin{aligned} (c(x) \cdot J^{\text{cn}})_p(v) &= x((\bar{x}px)(\bar{x}(Iw)x))\bar{x} \\ &= x((\bar{x}px)((I(xw))x))\bar{x} \\ &= x((\bar{x}px)(I(x^2w)))\bar{x} \\ &= x(I((\bar{x}\bar{p}x)(x^2w)))\bar{x} \\ &= x(I(\bar{x}\bar{p}x^3w))\bar{x} \\ &= I((\bar{x}^3\bar{p}x^3)w). \end{aligned}$$

We conclude that, with the notation of Proposition 2.1, we have

$$(c(x) \cdot J^{\text{cn}})_p = p^{R_{x^3}}.$$

Thus the image of the map (7) equals  $j(\mathbb{RP}^7)$  for the inclusion  $j$  in (5), but (7) is generically a 3-to-1 map. Given unit octonions  $x, y$ , the elements  $[x], [y] \in \mathbb{RP}^7$  have the same image under (7) if and only if  $x^6 = y^6$ .  $\square$

## 5. THE SPACE OF ALMOST COMPLEX STRUCTURES WITH $c_1 = 0$ ON A SIX MANIFOLD

We now consider the more general setting of the space of almost complex structures on a six-manifold homotopic to a given  $J$  with  $c_1(J) = 0$ . As before, all almost complex structures will be assumed to be orthogonal with respect to some fixed background metric and compatible with a fixed orientation.



**Proposition 5.1.** Let  $M$  be a six-manifold with an almost complex structure  $J$  so that  $c_1(J) = 0$ . Let  $f: M \rightarrow BSU(3)$  be a lift<sup>2</sup> of the classifying map of  $TM$  determined by  $J$ . Let  $\mathcal{J}(M)$  denote the path component of  $J$  in the space of almost complex structures on  $M$  and

$$S^7 \rightarrow E \xrightarrow{q} M$$

be the  $S^7$ -bundle on  $M$  classified by the composite

$$M \xrightarrow{f} BSU(3) \rightarrow BG_2$$

(where we consider the standard action of  $G_2$  on the unit octonions). Then there is a fiber sequence

$$\text{Map}(M, S^1) \rightarrow \Gamma(q) \rightarrow \mathcal{J}(M).$$

*Proof.* The map  $M \rightarrow BG_2$  gives  $TM \oplus \mathbb{R}^2$  the structure of a bundle of normed division algebras on  $M$ . Writing  $1, i$  for the sections corresponding to the last two coordinates, we have that  $1$  is the unit on each fiber while  $i$  acts as multiplication by  $J$  on  $TM$ .

Proposition 2.1 then expresses the bundle of orthogonal almost complex structures on  $M$  (compatible with the orientation) as the quotient of  $E$  by the  $S^1$  action given by fiberwise multiplication by  $\cos \theta + i \sin \theta$ . The statement follows.  $\square$

**Remark 5.2.** Note that  $\Gamma(q)$  is an  $H$ -space. If  $M$  is simply connected then  $\text{Map}(M, S^1) \simeq S^1$  and so Proposition 5.1 shows that each component of the space of almost complex structures on  $M$  with  $c_1 = 0$  is homotopy equivalent to the quotient of an  $H$ -space by an  $S^1$ -action. The space  $\Gamma(q)$  is the space of sections of the sphere bundle of  $TM \oplus \mathbb{R}^2$ , and is therefore independent of the choice of almost complex structure. On the other hand, its  $H$ -space structure (and the corresponding  $S^1$ -action) will depend on the path component of the almost complex structure.

If in addition  $c_2(J) = 0$ , then the bundle  $q$  in Proposition 5.1 is trivial as the next lemma shows. In that case we have in the previous statement  $\Gamma(q) = \text{Map}(M, S^7)$ .

**Lemma 5.3.** Let  $X$  be a six-dimensional cell complex. The composite

$$X \xrightarrow{f} BSU(3) \rightarrow BG_2$$

is null if and only if  $c_2(f) = 0$ .

*Proof.* Recall that we have a fiber sequence  $S^6 \rightarrow BSU(3) \rightarrow BG_2$  (cf. (2)). Consider the diagram

$$\begin{array}{ccccc} S^6 & \longrightarrow & K(\mathbb{Z}, 6) & \longrightarrow & P_6BSU(3) \\ \downarrow & & \nearrow p_6 & & \downarrow \\ BSU(3) & \xrightarrow{c_2} & & \longrightarrow & K(\mathbb{Z}, 4) \end{array}$$

where the map  $P_6BSU(3) \rightarrow K(\mathbb{Z}, 4)$  is the bottom of the Postnikov tower of  $BSU(3)$ , with homotopy fiber  $K(\mathbb{Z}, 6)$ . If  $c_2 = 0$  then the map  $p_6 \circ f$  factors through  $K(\mathbb{Z}, 6)$  and hence through  $S^6$ . Since a map from a six complex into  $P_6BSU(3)$  which lifts to  $BSU(3)$  does so uniquely, we see that  $f$  lifts to  $S^6$ . Conversely, if  $f$  factors through the homotopy fiber  $S^6$  of  $BSU(3) \rightarrow BG_2$ , then clearly  $c_2 = 0$ .  $\square$

**Example 5.4.** As an illustration we can compute the homotopy groups of  $\mathcal{J}(X_g)$ , where  $X_g$  denotes the connected sum of  $g$  copies of  $S^3 \times S^3$ , in terms of the homotopy groups of  $S^7$ . Note that the space of orientation-compatible almost complex structures on  $X_g$  is non-empty and connected, and

<sup>2</sup>Note that this lift is unique up to homotopy through sections, as the homotopy fiber  $S^1 \rightarrow BSU(3)$  of the map  $BSU(3) \rightarrow BU(3)$  is null.

since  $c_1 = c_2 = 0$ , by Lemma 5.3 the tangent bundle map  $X_g \rightarrow BU(3)$  factors through  $S^6$ . Since  $\chi(X_g) = 2 - 2g$ , this implies the map  $X_g \rightarrow S^6$  is of degree  $1 - g$ . We thus have the pullback diagram

$$(8) \quad \begin{array}{ccc} Z(X_g) & \longrightarrow & SO(8)/U(4) \\ \downarrow & & \downarrow \\ X_g & \xrightarrow{1-g} & S^6 \end{array}$$

where  $Z(X_g) \rightarrow X_g$  denotes the twistor space of  $X_g$ , and a corresponding map of fiber sequences

$$(9) \quad \begin{array}{ccccc} \text{Map}(S^6, S^1) & \longrightarrow & \text{Map}(S^6, S^7) & \longrightarrow & \mathcal{J}(S^6) \\ \downarrow (1-g)^* & & \downarrow (1-g)^* & & \downarrow (1-g)^* \\ \text{Map}(X_g, S^1) & \longrightarrow & \text{Map}(X_g, S^7) & \longrightarrow & \mathcal{J}(X_g) \end{array}$$

For a closed oriented 6-manifold  $W$ , we have  $\pi_1 \text{Map}(W, S^7) \cong \pi_1 \text{Map}_*(W, S^7) \cong \pi_0 \text{Map}_*(\Sigma W, S^7) \cong \mathbb{Z}$ , with the last isomorphism sending  $f \in [\Sigma W, S^7]_*$  to its degree  $n$ , i.e. the unique integer  $n$  such that  $f_*(\Sigma[W]) = n[S^7] \in H_7(S^7)$ . It follows that the map  $(1-g)^*$  in the middle column of (9) induces multiplication by  $(1-g)$  on fundamental groups. Lemma 3.4 together with the long exact sequence of the bottom row of (9) then implies that

$$\pi_1 \mathcal{J}(X_g) = \mathbb{Z}/(2 - 2g).$$

There is a cofiber sequence

$$S^5 \xrightarrow{w} \bigvee_{i=1}^{2g} S^3 \rightarrow X_g$$

where the attaching map  $w$  of the top cell is a sum of Whitehead products of inclusions of  $S^3$  in the wedge

$$w = [\iota_1, \iota_2] + \dots + [\iota_{2g-1}, \iota_{2g}]$$

As  $\Sigma w$  is null (see for instance [Wh, Theorem X.8.20]), we have

$$\Sigma X_g \simeq \left( \bigvee_{j=1}^{2g} S^4 \right) \vee S^7.$$

As  $S^7$  is an  $H$ -space,  $\text{Map}(X_g, S^7) \simeq S^7 \times \text{Map}_*(X_g, S^7)$  and hence for all  $i \geq 1$  we have

$$\pi_i \text{Map}(X_g, S^7) \cong \pi_i(S^7) \oplus \pi_{i-1} \text{Map}_*(\Sigma X_g, S^7) \cong \pi_i(S^7) \oplus \bigoplus_{j=1}^{2g} \pi_{i+3} S^7 \oplus \pi_{i+6} S^7.$$

The homotopy long exact sequence of the bottom row of (9) then gives us

$$\pi_2 \mathcal{J}(X_g) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } g = 1, \\ \mathbb{Z}/2 & \text{otherwise,} \end{cases}$$

while the remaining higher homotopy groups agree with those of  $\text{Map}(X_g, S^7)$ .

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