EXAMPLES OF ALMOST COMPLEX FOUR MANIFOLDS WITH NO COMPLEX STRUCTURE

ABSTRACT. Using Kodaira's classification of complex surfaces and the BMY inequality along with some rational homotopy theory and obstruction theory, we provide three examples of four-manifolds that admit almost-complex structures but no integrable complex structures.

1. Example using rational homotopy theory

Consider the Lie algebra spanned by $\{X, Y, Z, W\}$, with bracket given by $[X, Y] = -Z$, $[X, Z] = W$, and all other brackets zero. Take a four-dimensional simply-connected Lie group G corresponding to this Lie algebra, and find a cocompact subgroup Γ (whose existence is guaranteed by Malcev, since the structure constants are rational). Now consider the quotient manifold $M^4 = G/\Gamma$. This is a (closed) nilmanifold of dimension 4 to which the vector fields X, Y, Z, W descend to provide a parallelization of the tangent bundle. On the quotient, we can define an almost complex structure J by setting $JX = Y, JY = -X, JZ = W, JW = -Z$.

The minimal model of M is obtained by dualizing the Lie bracket, i.e.

$$\text{Model}(M) = \Lambda(x, y, z, w), \text{ with } dx = dy = 0, dz = xy, dw = -xz.$$

Computing cohomology, we see that

$$\begin{aligned} H^1(M, \mathbb{R}) &= \text{span}([x], [y]), \\ H^2(M, \mathbb{R}) &= \text{span}([xw], [yz]), \\ H^3(M, \mathbb{R}) &= \text{span}([xzw], [yzw]), \\ H^4(M, \mathbb{R}) &= \text{span}([xyzw]). \end{aligned}$$

So, the first Betti number of this nilmanifold is 2. If M admitted an integrable complex structure, by Kodaira's classification of surfaces we would conclude that it in fact (possibly after a deformation through diffeomorphisms) admitted a Kähler structure. However, a Kähler manifold is formal, and a formal nilmanifold is a torus. So, if M were complex, it would be a four-torus, and so we would have $b_1(M) = 4$.

Remark. Observe that, by the nilpotence condition, in the minimal model of a nilmanifold there is always at least one closed element (namely, the first one). So, b_1 of a nilmanifold cannot be 0. On the other hand, a nilmanifold with maximal possible first Betti number (that of a torus), is itself a torus, since all the generators have to be closed. So, in the case of four dimensional nilmanifolds, $b_1 = 2$ is the only choice of Betti number that could make the above argument work. Any such nilmanifold can be given an almost complex structure (just define it as we did in the construction). So, any four dimensional nilmanifold with $b_1 = 2$ admits an almost-complex structure but no complex structure. Let us consider how many such nilmanifolds exist. Name and order the basis elements of the minimal model of such a nilmanifold by x, y, z, w. Since we require that $b_1 = 2$, this tells us $dx = dy = 0$. By nilpotence and $b_1 = 2$, we conclude $dz = xy$. As for w, we already considered the case of $dw = -xz$. The only other possibility (after a change of variables) given the conditions seems to be

$dw = yz$. An isomorphism between this minimal model and the one we originally considered is given (up to sign) by swapping x and y. So, there is a unique example in our class of nilmanifolds.

2. Example using obstruction theory

Consider $X = (S^2 \times S^2)\#(S^1 \times S^3)\#(S^1 \times S^3)$. (This example was pointed out to me by Michael Albanese.) We show that this smooth manifold admits an almost complex structure, but does not admit an integrable complex structure. If a closed four-manifold were to admit an almost complex structure J, then the first Chern class c_1 would satisfy two properties:

- its mod 2 reduction would be the second Stiefel-Whitney class w_2,
- and it would satisfy $\int c_1^2 = 2\chi + 3\sigma$, where χ is the Euler characteristic and σ is the signature.

A theorem of Wu states that these necessary conditions on the class c_1 are in fact sufficient for the existence of an almost complex structure. That is, given a closed four-manifold and a class c in second integral cohomology satisfying properties (i) and (ii), there is an almost-complex structure on the manifold with c as its first Chern class. In our example, we can choose $c = 0$ and thus obtain an almost complex structure. Indeed, the intersection form on X is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and so the signature is 0, and X admits a cell decomposition with one 0–cell, two 1–cells, two 2–cells, two 3-cells, and three 4–cells, so $\chi = 0$ as well. Since $w_2(X) = w_2(S^2 \times S^2) + w_2(S^1 \times S^3) + w_2(S^1 \times S^3)$, and all the summands are stably parallelizable, so $w_2(X) = 0$ and $c = 0$ satisfies the two conditions stated above.

Now suppose X admitted an integrable complex structure. Then, since the first Betti number is 2 (importantly, even), by Kodaira's classification of surfaces, X is deformable to a complex surface admitting a Kähler metric. So, just take X to be that surface admitting a Kähler metric (since the underlying smooth type is the same). Now, note that $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$. Rotation by 180° gives a 2-to-1 covering of a wedge of three circles over a wedge of two circle, and so $\mathbb{Z} * \mathbb{Z}$ admits an index two subgroup that is isomorphic to $\mathbb{Z}*\mathbb{Z}*\mathbb{Z}$. Therefore X admits a double cover $\widetilde{X}$ whose fundamental group is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. Pulling back the Kähler structure from X to $\widetilde{X}$, we have that $\widetilde{X}$ is a compact Kahler manifold with first Betti number equal to 3, which cannot be.

3. Example using the BMY inequality

Consider $X = \mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$. We show that X admits an almost complex structure using the criteria of Wu listed above, and show that it does not admit an integrable complex structure.

First, since $w_2(\mathbb{CP}^2) = 1 \in \mathbb{Z}_2 = H^2(\mathbb{CP}^2, \mathbb{Z}_2)$, we have that $w_2(X) = (1, 1, 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = H^2(X, \mathbb{Z}_2)$. So, any class $c \in H^2(X, \mathbb{Z})$ which would be the first Chern class of an almost complex structure on X would be of the form $c = \alpha + \beta + \gamma$, where α, β, γ are (identified with) odd integers. We compute the Euler characteristic $\chi(X) = 5$ and signature $\sigma(X) = 3$. So, this tentative class c must satisfy $\int c^2 = 19$. And indeed, choosing $\alpha = 3, \beta = 3, \gamma = 1$, we have $c^2 = (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 = 9 + 9 + 1 = 19$. Therefore X admits an almost complex structure J with $c_1(X, J) = (3, 3, 1)$ (and $c_2(X, J) = 5$).

Now, suppose X admitted an integrable complex structure. Then since its first Betti number is even, X would be deformation equivalent to a Kähler surface. In fact, a complex surface with even b_1 is deformation equivalent to an *algebraic surface* by Kodaira. So, take X to be this hypothetical algebraic surface with smooth type $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$. We can consider the Kodaira dimension $\kappa \in \{-\infty, 0, 1, 2\}$ of this surface, and we will rule out the possibilities $\kappa = -\infty, 0, 1$.

Suppose $\kappa = -\infty$. For complex surfaces, the Frölicher spectral sequence degenerates on the first page (by Kodaira), and so $h^{0,1} = h^{1,0} = b_1$. Since $b_1(X) = 0$, we have $h^{0,1} = 0$. Castelnuovo's theorem applied to complex surfaces with $\kappa = -\infty$ now implies that X is a rational surface, and so its Hodge diamond would have the form

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & n & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(with $n \geq 1$). However, Noether's formula tells us that the analytic Euler characteristic $h^{0,0} - h^{0,1} + h^{0,2}$ is given by $\int \frac{1}{12}(c_1^2 + c_2)$. Above we saw $c_1^2 = 19$ and $c_2 = 5$, so we have $h^{0,0} - h^{0,1} + h^{0,2} = 2$, which cannot be with such a Hodge diamond. Therefore $\kappa \neq -\infty$.

Complex surfaces of Kodaira dimension 0 or 1 satisfy $\int c_1^2 = 0$, so we can rule out these cases. We conclude that $\kappa = 2$, i.e. X would be a surface of general type. However, here we can apply the Bogomolov-Miyaoka-Yau inequality $\int c_1^2 \leq 3 \int c_2$ to obtain the contradiction $19 \leq 15$. Therefore X does not admit a complex structure.

Remark 3.1. We can ask which manifolds with the homotopy type of a connect sum $k\mathbb{CP}^2$ of k projective planes admit an almost complex structure. By Hirzebruch, a $4m$–dimensional almost complex manifold satisfies $\chi = (-1)^m \sigma \bmod 4$. In the case of 4–manifolds, we obtain the simple requirement that $\chi + \sigma$ is divisible by 4. We calculate $\chi(k\mathbb{CP}^2) + \sigma(k\mathbb{CP}^2) = 2k + 2$, and so if (a manifold with the homotopy type of) $k\mathbb{CP}^2$ were to admit an almost complex structure, k would have to be odd. Conversely, for $k = 2l + 1$ we see that the class

$$c = (3, 1, 3, 1, \ldots, 1, 3) \in H^2(k\mathbb{CP}^2, \mathbb{Z})$$

satisfies the Wu criteria stated above. That is, since $c \bmod 2$ equals $(1, 1, 1, \ldots, 1) = w_2(k\mathbb{CP}^2)$ and $\int c^2 = 9(l + 1) + l = 10l + 9 = 3\chi(k\mathbb{CP}^2) + 2\sigma(k\mathbb{CP}^2)$, there is an almost complex structure on $(2l + 1)\mathbb{CP}^2$ with first Chern class c.

Now consider an odd connect sum $(2l + 1)\mathbb{CP}^2$. If it were to admit an integrable complex structure, we would conclude that this complex surface would be of general type in the same manner as in the case of $3\mathbb{CP}^2$ considered above. Then the BMY inequality would tell us $\int c_1^2 \leq 3 \int c_2$, i.e. $10l + 9 \leq 3(2l + 3)$. We conclude $l = 0$. Therefore any smooth manifold with the homotopy type of $(2l + 1)\mathbb{CP}^2$ with $l \geq 1$ admits an almost complex structure but no complex structure.