COMPUTATIONS WITH THE FRÖLICHER SPECTRAL SEQUENCE

ALEKSANDAR MILIVOJEVIC

Abstract. We go over the basic tools necessary to work with the Frölicher spectral sequence, and do some example computations with complex nilmanifolds and a hypothetical complex structure on the six-sphere. This is for the most part a survey of some of the results in [1], [2], [3], [4], [5].

1. Complex Nilmanifolds

Suppose we have a real $2n$–dimensional nilpotent Lie group G with an integrable complex structure. We can choose a basis of the space of real one-forms that has the form $\{\omega_1, \omega_2, \ldots, \omega_n, \bar{\omega_1}, \bar{\omega_2}, \ldots, \bar{\omega_n}\}$. To be complex means we have

$$d\omega_i = \sum_{j,k} \alpha^i_{j,k} \omega_j \omega_k + \sum_{j,k} \beta^i_{j,k} \omega_j \bar{\omega_k}$$

for some real coefficients α and β and to be nilpotent means the above basis can in fact be chosen so that

$$d\omega_i = \sum_{j<i,k<i} \alpha^i_{j,k} \omega_j \omega_k + \sum_{j<i,k<i} \beta^i_{j,k} \omega_j \bar{\omega_k}.$$

The differential on the conjugate basis elements is determined by

$$d\bar{\omega} = \partial\bar{\omega} + \bar{\partial}\bar{\omega} = \overline{\bar{\partial}\omega} + \overline{\partial\omega},$$

where $\partial\omega$ is the first term in the above formula, and $\bar{\partial}\omega$ is the second.

If we can choose a basis of one-forms so that the coefficients α and β are rational, then a result of Malcev tells us we can find a cocompact discrete subgroup Γ of our Lie group G such that G/Γ is a closed $2n$–manifold, called a *nilmanifold.* If the complex structure we have on G is left-invariant, then it descends to this nilmanifold, and so we obtain a complex n-fold which we call a *complex nilmanifold.*

The real minimal model (in the sense of rational homotopy theory) of a nilmanifold is particularly simple to compute. Namely, the model of a nilmanifold obtained as a quotient of a Lie group with the above form is the free graded commutative algebra generated in degree one by elements corresponding to the basis one-forms chosen above, with the same differential. The differential is, in turn, obtained by dualizing the Lie bracket on vector fields on the Lie group.

Example 1.1. The Lie algebra to $SU(2)$ has basis vectors X, Y, Z with $[X, Y] = Z, [X, Z] = -Y, [Y, Z] = X$, and so the differential on the dual one-forms is given by $dx = yz, dy = -xz, dz = xy$. The real minimal model of $SU(2)$ is *not* the exterior algebra on x, y, z with this differential, though, since such an algebra is not minimal. In fact, since $SU(2)$ is diffeomorphic to S^3, the minimal model of $SU(2)$ is given by the exterior algebra on a single generator of degree 3, with trivial differential.

Example 1.2. A nilmanifold M of dimension three can be obtained by taking the real Heisenberg group of matrices of the form

$$A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

and quotienting by the cocompact discrete subgroup of such matrices with integer coefficients. The Lie algebra of the Heisenberg group can be read off as the components of

$$A^{-1}dA = \begin{pmatrix} 1 & -x & xy - z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & dx & dz \\ 0 & 1 & dy \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & dx & dz - xdy \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}.$$

So, a basis of left-invariant one-forms is given by $dx, dy, dz - xdy$. Denoting them by α, β, γ, we have $d\alpha = 0, d\beta = 0, d\gamma = -\alpha\beta$. These forms descend to the quotient nilmanifold, which thus has minimal model given by

$$\Lambda(\alpha, \beta, \gamma,\ d\alpha = 0, d\beta = 0, d\gamma = -\alpha\beta).$$

Example 1.3. We can think of the 3–manifold in the previous example as obtained by a circle bundle over a torus, where this base torus has minimal model $\Lambda(\alpha, \beta)$ with trivial differential. This circle bundle is the one obtained by the classifying map $T^2 \to BS^1 = \mathbb{CP}^\infty$ corresponding to $-1 \in \mathbb{Z} = H^2(T^2, \mathbb{Z}) = [T^2, \mathbb{CP}^\infty]$, so we have $d\gamma = -\alpha\beta$, where γ is a volume form for the circle we are fibering in. Crossing this 3–manifold with a circle, i.e. trivially fibering in another circle, we obtain a 4–manifold with minimal model $\Lambda(\alpha, \beta, \gamma, \xi,\ d\alpha = 0, d\beta = 0, d\gamma = -\alpha\beta, d\xi = 0)$ called the *Kodaira-Thurston manifold*, famous for being one of the first examples of a symplectic non-Kähler manifold.

Example 1.4. We have the following complex nilmanifold analogue of the nilmanifold considered in Example 2. Consider the complex matrices of the form

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The same computation as in Example 2 gives us that $dz_1, dz_2, dz_3 - z_1 dz_2$ are a complex basis for the left-invariant one-forms. Denoting again $x = dz_1, y = dz_2, z = dz_3 - z_1 dz_2$, we have that the underlying six-dimensional Lie group has $x, y, z, \bar{x}, \bar{y}, \bar{z}$ as a real basis for the left-invariant one-forms. Quotienting by any cocompact discrete subgroup, for example the subgroup of matrices with integer entries, yields a six-dimensional nilmanifold with minimal model

$$\Lambda(x, \bar{x}, y, \bar{y}, z, \bar{z},\ dx = 0, dy = 0, dz = -xy).$$

This manifold is known as the *Iwasawa manifold*.

2. The Frölicher spectral sequence

The zeroth page of the Frölicher spectral sequence is simply a 2–dimensional grid in the first quadrant, with the (p, q)-forms in the (p, q)-th slot. The differential on this page is $\bar{\partial}$, and points one slot vertically.

We go to the E_1 page by taking cohomology on the E_0 page. That is, the (p, q)-slot on the E_1 page, denoted $E_1^{p,q}$, consists of $\bar{\partial}$-cohomology classes of degree (p, q). The differential on the E_1 page is the map induced by ∂ on $\bar{\partial}$-cohomology, and points one

$$\begin{tikzcd}
3 & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] \\
1 & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] \\
0 & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] & \cdot \arrow[u, "\bar{\partial}"] \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

FIGURE 1. The E_0 page with its differential $d_0 = \bar{\partial}$.

$$\begin{tikzcd}
3 & \cdot \arrow[r, "\partial"] & \cdot \arrow[r, "\partial"] & \cdot \arrow[r, "\partial"] & \cdot \\
2 & \cdot \arrow[r, "\partial"] & \cdot \arrow[r, "\partial"] & \cdot \arrow[r, "\partial"] & \cdot \\
1 & \cdot \arrow[r, "\partial"] & \cdot \arrow[r, "\partial"] & \cdot \arrow[r, "\partial"] & \cdot \\
0 & \cdot \arrow[r, "\partial"] & \cdot \arrow[r, "\partial"] & \cdot \arrow[r, "\partial"] & \cdot \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

FIGURE 2. The E_1 page with its differential $d_1 = \partial$.

slot to the right. The map ∂ indeed descends to $\bar{\partial}$-cohomology, since $\bar{\partial}$-exact forms get mapped to $\bar{\partial}$-exact forms. Namely, $\partial(\bar{\partial}\alpha) = \bar{\partial}(-\partial\alpha)$, since $d^2 = (\partial + \bar{\partial})^2 = 0$ and so $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

By taking cohomology on the E_1 page with respect to this ∂ differential, we obtain the E_2 page. On the E_2 page, the differential d_2 points two slots right and one slot down. To get to the E_3 page, we take cohomology with respect to the d_2 differential. On E_3, the differential d_3 points three slots right, and two slots down, and so on. The differentials on the second page and later do not have as nice descriptions as on the zeroth and first pages, but we can still describe them. First let us record when a given (p, q)-form even defines an element on the E_r page, for varying r. Note that a (p, q) form on the E_0 page defines an element on the E_1 page if it defines a $\bar{\partial}$-cohomology class, i.e. if it is $\bar{\partial}$-closed. This element on the first page then defines an element on E_2 if it is ∂-closed. Further, it defines an element on E_3 if it is d_2-closed, and so on. We say a (p, q) form *lives to page* r if it gives a well-defined element on E_r (i.e. if it is $\bar{\partial}$-closed, and then that induced class is ∂-closed, and *that* induced class is d_2-closed, $\ldots$, and *that* induced class is d_{r-1}-closed).

Note that all the slots outside of this first quadrant on each page is 0. So, if an arrow points to or from a slot where no (p, q)-forms could live to, it is a zero arrow.

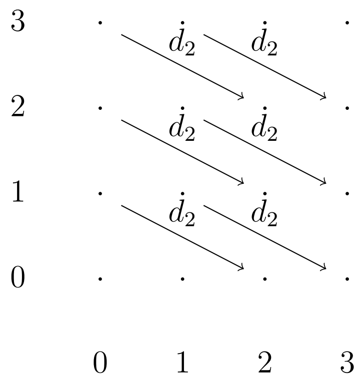

FIGURE 3. The E_2 page with its differential d_2.

Proposition 2.1. *A $\bar{\partial}$-closed (p,q)-form α_1 defines an element on E_r if there exists a chain of forms $\alpha_2, \dots, \alpha_r$ such that $\partial\alpha_1 = -\bar{\partial}\alpha_2$, $\partial\alpha_2 = -\bar{\partial}\alpha_3$, $\dots$, $\partial\alpha_{r-1} = \bar{\partial}\alpha_r$. (Here the forms α_i are required to be homogeneous of the appropriate degrees).*

If α_1 defines an element on E_r, then this element is zero if we can find a chain of elements $\alpha_0, \alpha_2, \dots, \alpha_r$ (again, homogeneous of the appropriate degrees) such that $\bar{\partial}\alpha_0 + \partial\alpha_2 = \alpha_1$, $\bar{\partial}\alpha_2 = -\partial\alpha_3$, $\bar{\partial}\alpha_3 = -\partial\alpha_4$, $\dots$, $\bar{\partial}\alpha_{r-1} = -\partial\alpha_r$, and $\bar{\partial}\alpha_r = 0$.

The Frölicher spectral sequence for any complex n-fold eventually stabilizes. Namely, when the arrows start to point $n+1$ slots or more to the right, then they are all zero arrows, and taking cohomology does nothing to the slots. We denote this stable page by E_∞. This spectral sequence converges to the (complexified) de Rham cohomology, here meaning

$$\Omega^k_{\mathrm{dR}}(X) \otimes \mathbb{C} = \bigoplus_{p+q=k} E_\infty^{p,q}.$$

We say the spectral sequence *degenerates at page r* if E_r is the earliest page on which all the arrows are zero.

Remark 2.2. On a closed Kähler manifold, we have the $\partial\bar{\partial}$-lemma, which says that a form α which is ∂-closed, $\bar{\partial}$-closed, and ∂- or $\bar{\partial}$-exact, is of the form $\alpha = \partial\bar{\partial}\beta$ for some β. This implies that the differential d_1 on E_1 of the Frölicher spectral sequence associated to a Kähler manifold is trivial. Namely, for a $\bar{\partial}$-closed form α, we have $d_1([\alpha]_{\bar{\partial}}) = [\partial\alpha]_{\bar{\partial}}$. (Here $[-]_{\bar{\partial}}$ denotes a form's $\bar{\partial}$-cohomology class.) Now, $\partial\alpha$ is closed under both ∂ and $\bar{\partial}$, and is $\bar{\partial}$-exact. Therefore, $\partial\alpha = \partial\bar{\partial}\beta = -\bar{\partial}\partial\beta$, and so $[\partial\alpha]_{\bar{\partial}} = [\bar{\partial}(-\partial\beta)]_{\bar{\partial}} = 0$. So, the spectral sequence degenerates on the first page.

3. Nilmanifold computations

We compute some terms of the Frölicher spectral sequence for some complex nilmanifolds X. Let us adopt the notation $h_r^{p,q} = \dim_{\mathbb{C}} E_r^{p,q}$. Note that $h_1^{p,q}$ are the usual Hodge numbers $h^{p,q} = \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X)$.

We note that at each slot (p,q), the vector space $E_{r+1}^{p,q}$ can be considered as a subspace of $E_r^{p,q}$. All forms that live to $E_{r+1}^{p,q}$ must also live to $E_r^{p,q}$, and if they are to be non-zero on E_{r+1}, they must be non-zero on E_r. In particular, as r increases, the numbers $h_r^{p,q}$ decrease.

Example 3.1. Let us show that the Iwasawa manifold from Example 1.4 is not Kähler by showing that its spectral sequence does not degenerate at the first page. For this, it suffices to find a slot (p,q) such that $E_1^{p,q} \not\cong E_2^{p,q}$. Recall the minimal model

$$\Lambda(x, \bar{x}, y, \bar{y}, z, \bar{z},\ dx = 0, dy = 0, dz = -xy).$$

Consider the $(2,0)$-form xy. It is $\bar{\partial}$-closed, and for degree reasons it cannot be $\bar{\partial}$-exact, so it defines a non-zero element in $E_1^{2,0}$. Since $\partial(xy) = 0$, this form lives to E_2. It defines the zero class in $E_2^{2,0}$, though, since $\partial z = xy$, and $\bar{\partial} z = 0$. Keeping in mind the above remark, we conclude that $E_2^{2,0}$ is a proper subspace of $E_1^{2,0}$, and so the spectral sequence does not degenerate on the first page.

In general, the Frölicher spectral sequence of a complex-parallelizable nilmanifold degenerates at latest on the E_2 page. (A complex manifold X is complex-parallelizable if the holomorphic tangent bundle $T^{1,0}X$ is trivial. In the case of nilmanifolds, this is equivalent to there existing a choice of basis one-forms $\{\omega_1, \ldots, \omega_n, \bar{\omega_1}, \ldots, \bar{\omega_n}\}$ such that $\{\omega_1, \ldots, \omega_n\}$ is closed under d, i.e. $\bar{\partial}$ is trivial on the $(1,0)$-forms.)

Example 3.2. Let us consider now an example of a complex threefold whose spectral sequence degenerates on E_3. Take the simply connected nilpotent complex Lie group of complex dimension three with a basis $\{x, y, z\}$ of left-invariant degree $(1,0)$-forms satisfying $dx = 0, dy = x\bar{y}, dz = xy + x\bar{y} + \bar{x}y$. By Malcev, we can find a cocompact discrete subgroup, and taking the corresponding quotient we obtain a complex nilmanifold of dimension three. Its real minimal model is given by $\Lambda(x, y, z, \bar{x}, \bar{y}, \bar{z}, dx = 0, dy = x\bar{y}, dz = xy + x\bar{y} + \bar{x}y)$.

Consider $xy \in E_0^{2,0}$. Note that $\partial(xy) = 0$, so this form lives to $E_1^{2,0}$. Further, since $\bar{\partial}(xy) = 0$, in particular the induced $\bar{\partial}$ on E_1 is zero when applied to xy. So, xy defines a class in $E_2^{2,0}$. Note that for degree reasons, by Proposition 2.1., the form xy will live to every page (altough we do not a priori know if it will live to become a zero class on some page). For this class to be zero on E_2, there would have to be a $\bar{\partial}$-closed $(1,0)$ form β such that $\partial\beta = xy$. The only such forms are $z + cy$, where c is any constant. Then $\partial(z + cy) = xy$, but $\bar{\partial}(z + cy) = x\bar{y} + \bar{x}y + cx\bar{y}$, which cannot be zero. However, by setting $c = -1$ and taking the $(0,1)$ form $-\bar{y}$, we see tthat we have $\partial(z - y) = xy$, $\bar{\partial}(z - y) = \bar{x}y$, and $\partial{-\bar{y}} = -\bar{x}y$, $\bar{\partial}(-\bar{y}) = 0$, so by Proposition 2.1 we have that xy gives the zero class in $E_3^{2,0}$. Therefore $E_2^{2,0} \not\cong E_3^{2,0}$.

We can check directly that degeneration for the spectral sequence happens precisely on the third page (by considering every slot), but we will revisit this example and check this with less work after we make some observations later.

Example 3.3. We consider a complex sixfold whose Frölicher spectral sequence degenerates at least on the fourth page (meaning, E_4). Take the sixfold (again going through the procedure of taking a Lie group and quotienting) whose real minimal model is given by

$$\Lambda(x_1, x_2, x_3, y_1, y_2, y_3, \bar{x_1}, \bar{x_2}, \bar{x_3}, \bar{y_1}, \bar{y_2}, \bar{y_3}),$$
$$dx_1 = dx_2 = dx_3 = 0, dy_1 = \bar{x_1}x_2, dy_2 = \bar{x_1}x_2, dy_3 = x_1y_1 + x_1\bar{x_3}.$$

Consider the $(3,0)$-form $x_1x_2y_1$. It is closed with respect to both $\bar{\partial}$ and ∂, so it defines a class on the E_1 page and also defines a class (albeit possibly zero) on every subsequent page. To determine if this class is 0 in $E_2^{3,0}$, we look for $(2,0)$ forms whose ∂ part is equal to $x_1x_2y_1$. A direct inspection yields that all such forms are of the form $x_2y_3 + c_1x_2y_3 + c_2x_3y_2$, for arbitrary constants c_1, c_2. Then, note that

$\bar{\partial}(x_2y_3 + c_1x_2y_3 + c_2x_3y_2) = x_1x_2\bar{x_3} + c_1x_1x_2\bar{x_2} - c_2\bar{x_1}x_2x_3$. Another inspection shows that $y_1\bar{x_3}$ is the simple $(1,1)$ form with a summand of $x_1x_2\bar{x_3}$ in $\partial(y_1\bar{x_3})$. (Here by *simple* we mean that it is not a non-trivial combination of two or more $(1,1)$-forms in the basis for $(1,1)$-forms obtained by multiplying an element of $\{x_i, y_i\}$ with an element of $\{\bar{x_i}, \bar{y_i}\}$.) However, $\bar{\partial}(y_1\bar{x_3}) = x_1\bar{x_2}\bar{x_3}$. In particular, it is non-zero, so we cannot have that $x_1x_2y_1$ is the zero class on E_3. But, in the notation of Proposition 2.1, if we set $\alpha_0 = 0$, $\alpha_2 = x_2y_3$, $\alpha_3 = -y_1\bar{x_3}$, $\alpha_4 = \bar{x_3}\bar{y_2}$, we see that, since $\bar{\partial}(\bar{x_3}\bar{y_2}) = 0$, we have that $x_1x_2y_1$ represents the zero class on the fourth page. Therefore $E_3^{3,0} \not\cong E_4^{3,0}$, so degeneration happens no sooner than on E_4.

4. Geography of spectral sequence degeneration

We can ask the following question: For given n and r, is there a closed n-fold for which the spectral sequence degenerates on E_r? First, as previously observed, the spectral sequence associated to a complex n-fold degenerates at the $n+1$-st page at the latest, simply because at that page all the differentials become 0 by virtue of being too long.

A particular case of our question is then: For given n, is there a complex n-fold whose spectral sequence degenerates no sooner than on E_{n+1}? Some negative results are given by the following two results.

Proposition 4.1. *The Frölicher spectral sequence of a closed complex manifold of dimension one or two degenerates on the first page.*

The result for $n = 1$ follows from the observation that every complex curve is Kähler. The $n = 2$ result is due to Kodaira (and note that not every complex surface is Kähler). The question is unanswered for $n \geq 3$.

Question 4.2. For $n \geq 3$, is there a closed complex n-fold whose Frölicher spectral sequence degenerates on page $n + 1$?

If an n-fold were to have its spectral sequence degenerate on E_{n+1}, then there would have to be non-zero arrows on E_n. Due to the length of the arrows, we see that there can only be at most two non-trivial arrows on E_n, namely the arrow from $(0, n)$ to $(n, 1)$ and the one from $(0, n-1)$ to $(n, 0)$. However, the following observation shows that the latter arrow is in fact always zero.

Lemma 4.3. *If an $(n, 0)$ form on a closed complex manifold X defines a non-zero class on E_1, then it defines a non-zero class on E_{n+1} as well. In particular, the spectral sequence in the $(n, 0)$ stabilizes already on the first page.*

Proof. For an $(n, 0)$ form α to define a non-zero class on E_1, it just has to be $\bar{\partial}$-closed. (It cannot be $\bar{\partial}$-exact for degree reasons.) Observe that, again for degree reasons, this form is ∂-closed, and so defines a class on every page $E_r^{n,0}$. Now, from Proposition 2.1, we can see that in order for this class to be zero on some page E_r, there would have to be forms $\alpha_2, \ldots, \alpha_r$ in the appropriate degrees such that $\alpha = d(\alpha_2 + \cdots + \alpha_n)$ (here $\alpha_0 = 0$). So, let us show that α is not d-exact, and it will follow that α defines a non-zero class on every page (and so in particular on E_{n+1}).

Suppose $\alpha = d\beta$. Then, using the Hermitian inner product on the holomorphic part of the exterior algebra on the complexified cotangent bundle, we have

$$||\alpha||^2 = \int_X \alpha\bar{\alpha} = \int_X d\beta d\bar{\beta} = \int_X d(\beta\bar{\beta}) = \int_{\partial X} \beta\bar{\beta} = 0.$$

So, $\alpha = d\beta$ would have to be the zero form, which is not the case. □

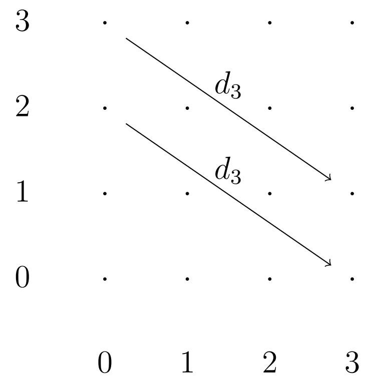

FIGURE 4. The only two possibly non-trivial differentials on E_3 for a complex threefold.

Degeneration on page $n+1$ can therefore only be detected at the $(0, n)$ slot. It would have to be the case that there is a $(0, n)$-form which gives a well-defined class in $E_1^{0,n}, \ldots, E_n^{0,n}$, but does not give a well-defined element in $E_{n+1}^{0,n}$. Observe that this *cannot happen in the case of complex nilmanifolds*. Namely, the $E_1^{0,n}$ slot for a nilmanifold would be spanned by a form like $\bar{x_1} \cdots \bar{x_n}$, where the differential applied to an x_k or $\bar{x_k}$ is spanned by the x_i and $\bar{x_i}$ such that $i < k$. (So, here the real minimal model would have underlying algebra $\Lambda(x_1, \bar{x_1}, \ldots, x_n, \bar{x_n})$. From there we see that $\partial(\bar{x_1} \cdots \bar{x_n}) = 0$ since applying the differential results in a sum of products of x_i and $\bar{x_i}$, and in each of these summands some $\bar{x_i}$ will repeat and hence make the summand zero. Therefore $\bar{x_1} \cdots \bar{x_n}$ lives to E_{n+1}. Namely, we can take $\alpha_2 = \ldots = \alpha_r = 0$ in the notation of Proposition 2.1 to observe that $\bar{x_1} \cdots \bar{x_n}$ lives to any E_r. The key point here is that $\partial(\bar{x_1} \cdots \bar{x_n})$ is *equal to* zero, and not just $\bar{\partial}$-closed. We record our conclusion in the following remark.

Remark 4.4. The Frölicher spectral sequence of a complex nilmanifold of complex dimension n degenerates on the E_n page or earlier.

In the positive direction, it has been shown (see [2]) that for any n there is a closed complex manifold whose spectral sequence degenerates no sooner than on E_n. The examples provided are complex nilmanifolds of complex dimension $4n - 2$.

5. HYPOTHETICAL COMPLEX S^6

A famous open problem is the following: For $n \geq 3$, is there an almost-complex manifold of (real) dimension $2n$ which does not admit an integrable complex structure?

A particular case of this is the question of whether S^6 admits an integrable complex structure. The six-sphere is known to admit an almost-complex structure (explicitly, by interpreting the sphere as the unit imaginary octonions), and every other almost-complex structure is homotopic to this one. It can also be directly checked that this almost-complex structure induced by the octonions is not integrable. The question is if there is perhaps some other integrable (almost) complex structure on S^6.

Even though this question is unresolved, we can still say something about the Hodge numbers $h^{p,q}$ of any complex structure on S^6, if it did exist. Recall the notation $h_r^{p,q} = \dim_{\mathbb{C}} E_r^{p,q}$, and that by definition $h_1^{p,q} = h^{p,q}$.

7	·	·	·	·	·	?
6	·	·	·	·	?	?
5	·	·	·	?	?	?
4	·	·	?	?	?	✓
3	·	×	✓	✓	?	?
2	×	×	✓	✓	✓	✓
1	✓	✓	✓	✓	✓	✓
	1	2	3	4	5	6

FIGURE 5. The geography of degeneration. The horizontal axis counts the dimension, and the vertical axis the page of degeneration. A ✓ means an example of a manifold with the given dimension and degeneration page is known. An × means such degeneration cannot occur. A ? indicates that no examples of such manifolds are known or explicitly recorded. (An entry in the $(6,3)$ slot, and possibly in the $(5,3)$ and $(5,4)$ slots, should not be difficult to find.)

The following tools are useful in obtaining relations on the Hodge numbers of a complex n-fold X:

- *Serre duality*, which can be succinctly stated as $h^{p,q} = h^{n-p,n-q}$, or visually as a central symmetry of the Hodge diamond (about $h^{n,n}$). Serre duality *does not hold* in general for the numbers $h_r^{p,q}$, $r \geq 2$.
- The "Euler characteristic" of every page E_r in the Frölicher spectral sequence is equal to the Euler characteristic of the manifold. Namely,
$$\chi(X) = \sum_k (-1)^k b_k = \sum_k \sum_{p+q=k} (-1)^k h_r^{p,q}.$$
- The *arithmetic genus* χ_0 of the complex manifold X is defined to be the holomorphic Euler characteristic of the (trivial) vector bundle of holomorphic functions on X,
$$\chi_0(X) = \sum_k (-1)^k \dim_{\mathbb{C}} H^k(X, \Omega^0).$$
By the Dolbeault theorem, $H_{\bar\partial}^{p,q} \cong H^q(X, \Omega^p)$, and so $H^k(X, \Omega^0) \cong H_{\bar\partial}^{0,k}(X)$, so we can rewrite the arithmetic genus as
$$\chi_0(X) = (-1)^k \sum_k h^{0,k}.$$

By Hirzebruch-Riemann-Roch, we have

$$\chi(X) = \int_X \mathrm{ch}(\Omega^0)\,\mathrm{td}(X),$$

where $\mathrm{ch}(\Omega^0)$ is the Chern character of the bundle and $\mathrm{td}(X)$ the Todd class of X. Here, $\mathrm{ch}(\Omega^0) = 1$ since Ω^0 is a trivial bundle. The first few terms of $\mathrm{td}(X)$ are given by

$$\mathrm{td}(X) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \frac{3c_2^2 + c_1 c_3 + 4c_1^2 c_2 - c_1^4 - c_4}{720} + \cdots .$$

Since the Chern character is trivial in our case, when evaluating $\chi_0(X)$ only the degree n term in $\mathrm{td}(X)$ matters in the integral $\int_X \mathrm{ch}(\Omega^0)\,\mathrm{td}(X)$. So, for example, for a complex threefold we have $\chi_0(X) = \int_X \frac{c_1 c_2}{24}$.

- The Euler characteristic along any line of differentials on E_r is equal to the Euler characteristic of the corresponding cohomology on E_{r+1}. In particular, for the transition from page one to page two, for all k we have

$$\sum_p (-1)^p h_1^{p,k} = \sum_p (-1)^p h_2^{p,k}.$$

Using the above toolkit, we obtain some conclusions on the Hodge numbers of a hypothetical complex structure on S^6. Many more inequalities can be obtained by similar arguments, and are recorded in the references, and all Hodge numbers from now on will be those corresponding to this complex manifold.

Proposition 5.1. $h^{0,1} \geq 1$.

Proof. The arithmetic genus of S^6 is given by $\chi_0(S^6) = h^{0,0} - h^{0,1} + h^{0,2} - h^{0,3}$. The closed $(0,0)$-forms are just the constants, so $h^{0,0} = 1$. Furthermore, by Lemma 4.3, if $h^{0,3}$ was non-zero, then $h_4^{0,3}$ would be non-zero. Since for a complex threefold (as S^6 would be), $E_4 = E_\infty$, we have

$$b_3 = h_4^{0,3} + h_4^{1,2} + h_4^{2,1} + h_4^{3,0} \geq h_4^{3,0},$$

and so we would have $b_3(S^6) > 0$, which is not true. Therefore, $\chi_0(S^6) = 1 - h^{0,1} + h^{0,2}$. By Hirzebruch-Riemann-Roch, $\chi_0(S^6) = \int_{S^6} \frac{c_1 c_2}{24}$. Since $c_1 \in H^2(S^6, \mathbb{Z})$ and $c_2 \in H^4(S^6, \mathbb{Z})$, both of these classes are trivial, and so $\chi_0(S^6) = 0$, from which we conclude $1 - h^{0,1} + h^{0,2} = 0$, i.e.

$$h^{0,1} = 1 + h^{0,2} \geq 1.$$

□

The argument just used to show $h^{0,3} = 0$ gives the following Proposition.

Proposition 5.2. *The Frölicher spectral sequence for S^6 degenerates either on page two or on page three.*

Proof. As $h_1^{0,1} = h^{0,1} \geq 1$ as we just saw, and $0 = b_1 = h_4^{1,0} + h_4^{0,1}$, we must have that $E_1^{0,1} \not\cong E_4^{0,1}$, and so degeneration happens on page two, three, or four. The discussion following Lemma 4.3 shows that degeneration on page four can happen only if $h_1^{0,3} \neq 0$. However, by Serre duality, $h_1^{0,3} = h_1^{3,0}$, which we saw was zero. □

Proposition 5.3. *If $h^{1,1} = 0$, then degeneration happens (exactly) on page three.*

Proof. Computing the Euler characteristic along the lines $k = 0, 1, 2, 3$ with respect to the differential d_1 in the transition from E_1 to E_2, we obtain the following four equations:

$$h^{0,0} - h^{1,0} + h^{2,0} - h^{3,0} = h_2^{0,0} - h_2^{1,0} - h_2^{2,0} + h_2^{3,0}$$
$$h^{0,1} - h^{1,1} + h^{2,1} - h^{3,1} = h_2^{0,1} - h_2^{1,1} + h_2^{2,1} - h_2^{3,1}$$
$$h^{0,1} - h^{1,1} + h^{2,1} - h^{3,1} = h_2^{0,2} - h_2^{1,2} + h_2^{2,2} - h_2^{3,2}$$
$$h^{0,3} - h^{1,3} + h^{2,3} - h^{3,3} = h_2^{0,3} - h_2^{1,3} + h_2^{2,3} - h_2^{3,3}$$

Let us say what we can about some individual terms in the above equations. First of all, since $h^{0,0} = 1$, $b_0(S^6) = h_3^{0,0} = 1$, and $h^{0,0} \geq h_2^{0,0} \geq h_3^{0,0}$, we have $h_2^{0,0} = 1$. Similary, $h_2^{3,3} = 1$ since $h^{3,3} = h^{0,0} = 1$ by Serre duality, and $h_2^{3,0} = h_2^{0,3} = 0$ since $h^{3,0} = h^{0,3} = 0$. Notice that on the second page, both arrows pointing to and from $E_2^{1,1}$ and $E_2^{2,2}$ are zero since $h_2^{3,0} = h_2^{0,3} = 0$. Therefore $h_2^{1,1} = h_3^{1,1} \leq b_3(S^6) = 0$, and so $h_2^{1,1} = 0$ and analogously $h_2^{2,2} = 0$. Similarly we obtain $h_2^{1,0} = 0$ from $b_1(S^6) = 0$. Using these observations and the equation $h^{0,1} = 1 + h^{0,2}$ from Proposition 5.1, along with Serre duality on the $h^{p,q}$, we can rewrite our four large equations as

$$1 - h^{1,0} + h^{2,0} = 1 + h_2^{2,0}$$
$$h^{0,2} + 1 - h^{1,1} + h^{1,2} - h^{0,2} = h_2^{0,1} + h_2^{2,1} - h_2^{3,1}$$
$$h^{0,2} - h^{1,2} + h^{1,1} - h^{0,2} - 1 = h_2^{0,2} - h_2^{1,2} - h_2^{3,2}$$
$$-h^{1,3} + h^{1,0} - 1 = -h_2^{1,3} - 1$$

Summing the first and fourth equations gives us $h^{2,0} - h^{1,3} = h_2^{2,0} - h_2^{1,3}$, which becomes $h_2^{2,0} = h_2^{1,3}$ by Serre duality.

Summing the second and third equalities we obtain

$$h_2^{0,1} + h_2^{2,1} + h_2^{0,2} = h_2^{3,1} + h_2^{1,2} + h_2^{3,2}.$$

Now, observe that on page two, we have the following sequence of arrows,

$$0 \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow 0.$$

Because the spectral sequence must degenerate or have already degenerated at the next page, and due to $b_2(S^6) = b_3(S^6) = 0$, we conclude that the middle arrow must be injective and surjective. Therefore, $h_2^{0,1} = h_2^{2,0}$. Similarly we conclude $h_2^{1,3} = h_2^{3,2}$, $h_2^{2,1} = h_2^{0,2}$, and $h_2^{3,1} = h_2^{1,2}$. Combining this with the end of the previous paragraph, we have $h_2^{0,1} = h_2^{3,2}$. Now the equation $h_2^{0,1} + h_2^{2,1} + h_2^{0,2} = h_2^{3,1} + h_2^{1,2} + h_2^{3,2}$ becomes $h_2^{2,1} + h_2^{0,2} = h_2^{3,1} + h_2^{1,2}$, so we have $h_2^{0,2} = h_2^{1,2}$.

We can rewrite the third large equation, using $h_2^{3,2} = h_2^{0,1}$, as

$$h_2^{0,1} = h_2^{0,2} - h_2^{0,1} + h^{1,2} - h^{1,1} + 1.$$

Because of $h_2^{0,2} = h_2^{1,2}$, this becomes

$$h_2^{3,2} = h^{1,2} + 1 - h^{1,1}.$$

If $h^{1,1} = 0$, then $h_2 3, 2 = h^{1,2} + 1 \geq 1$, and since $h_3^{3,2} \leq b_5(S^6) = 0$, we conclude $E_2 \not\cong E_3$. $\square$

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