THE SIXTH k-INVARIANT OF $BSO(3)$

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Abstract. We give some concrete information on the sixth k-invariant of $BSO(3)$. In the first section we review Postnikov towers in order to fix notation. In the second section we give the k-invariants up to k^5, following [1]. We carry out some calculations done in that paper in more detail, and explicitly identify the mod 2 reduction of the fourth k-invariant k^4. In the third section we determine k^6 up to two possibilities.

1. Postnikov towers

Recall that the Postnikov tower of a (let us restrict to simply connected) cell complex X is a sequence of fibrations

$$\begin{tikzcd}
& \vdots \\
& X[n] \arrow[d] \\
& X[n-1] \arrow[d] \\
& \vdots \arrow[d] \\
& X[3] \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] \arrow[uuuur] & X[2],
\end{tikzcd}$$

where the arrow $X \to X[n]$ induces an isomorphism on $\pi_{\leq n}$ and $\pi_{>n}X[n] = 0$. The fibrations $X[n] \to X[n-1]$ are principal $K(\pi_n X, n)$ fibrations classified by maps

$$X[n-1] \xrightarrow{k_n} K(\pi_n X, n+1),$$

i.e. by cohomology classes in $H^{n+1}(X[n-1], \pi_n X)$ which we call the k-invariants of X. The nth k-invariant is the cohomology class that tells us how to fiber an Eilenberg–Maclane space over $X[n-1]$ in order to create $X[n]$.

Observe that since the fiber of $X[n+1] \to X[n]$ is n–connected, any map of a finite complex Y to X is homotopic to a map from Y to some $X[n]$. Given a map $Y \to X[n]$, this will lift through the fibration $X[n+1] \to X[n]$ if and only if the pullback of the k-invariant k^{n+1} vanishes in Y.

The k-invariants of the Postnikov tower of X can be identified in the following way: Suppose we have the map $X \overset{f_n}{\to} X[n]$ in the Postnikov tower. Since $\pi_{n+1}X = 0$ and the map induces an isomorphism on $\pi_{\leq n}$, the fiber F of the map is n–connected (meaning, $\pi_{\leq n}F = 0$). So, $H_{n+1}(F;\mathbb{Z}) = \pi_{n+1}F$, which in turn equals $\pi_{n+1}X$. Now, in $H^{n+1}(F;\pi_{n+1}F)$ there is a canonical class, observed through the identification

$$H^{n+1}(F;\pi_{n+1}F) = Hom_{\mathbb{Z}}(H_{n+1}F, \pi_{n+1}F).$$

The canonical class is the inverse of the Hurewicz homomorphism $\pi_{n+1}F \to H_{n+1}F$ (which is an isomorphism here since F is n–connected). Call this cohomology class the *fundamental class* of F. The k–invariant k^{n+2} is then the transgression of the fundamental class of F in the spectral sequence for the fibration $F \to X \to X[n]$. We can approach this class by considering the following segment of the Serre long exact sequence:

$$H^{n+1}(F;\pi_{n+1}F) \xrightarrow{\tau} H^{n+2}(X[n];\pi_{n+1}F) \xrightarrow{f_n^*} H^{n+2}(X;\pi_{n+1}F).$$

This transgression of the fundamental class, i.e. the k-invariant, will also be the transgression of the fundamental class (by construction) of the fiber in the fibration $K(\pi_{n+1}F, n+1) \to X[n+1] \to X[n]$ that we obtain.

2. The k-invariants for $BSO(3)[5]$

Let us now restrict to the space $BSO(3)$. Calculating its Postnikov tower is a somewhat doable task– we know the cohomology of $BSO(3)$, and the fundamental class of the fiber of $BSO(3) \to BSO(3)[n]$ is the generator of $\pi_{n+1}BSO(3) = \pi_n SO(3) = \pi_n S^3$ (for $n \geq 2$). So, as long as we know $\pi_n S^2$, there is hope for calculating k^{n+2}.

To begin, $BSO(3)[2]$ will be a $K(\pi_2 BSO(3), 2)$, i.e. a $K(\mathbb{Z}_2, 2)$. Since $H^2(BSO(3);\mathbb{Z}_2)$ $= \mathbb{Z}_2$, the map $BSO(3) \to BSO(3)[2] = K(\mathbb{Z}_2, 2)$ can be none other than w_2, the second Stiefel–Whitney class. Now, consider the fibration $F \to BSO(3) \overset{w_2}{\to} BSO(3)[2]$, where F is the homotopy fiber of w_2. Since $\pi_3 BSO(3) = 0$ (this is in fact the only higher homotopy group of $BSO(3)$ that vanishes), the fiber F is in fact 3–connected, not just 2–connected. We see that $\pi_4 F = \pi_4 BSO(3) = \pi_3 S^3 = \mathbb{Z}$. Therefore the k-invariant to obtain $BSO(3)[3] = BSO(3)[4]$ from $BSO(3)[2]$ will be a map

$$BSO(3)[2] \xrightarrow{k^4} K(\mathbb{Z}, 5),$$

i.e. a class in $H^5(K(\mathbb{Z}_2, 2);\mathbb{Z})$.

The bad news is that integral k–invariants are hard to identify, stemming from the complicated integral cohomology of Eilenberg–Maclane spaces. The good news is that this is the only integral k–invariant that will show up in the Postnikov tower of $BSO(3)$ (since $\pi_{\geq 5}BSO(3)$ is torsion), and this particular k–invariant was calculated classicaly (see [3]) for much of what is discussed in this section.) The k–invariant is the Pontryagin square

$$H^2(-;\mathbb{Z}_2) \xrightarrow{P} H^4(-;\mathbb{Z}_4)$$

followed by the coboundary map $H^4(-;\mathbb{Z}_4) \xrightarrow{\delta} H^5(-;\mathbb{Z})$ coming from the long exact sequence in cohomology associated to the short exact sequence of groups

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_4 \to 0.$$

Now we move on to k^5. The homotopy fiber F of $BSO(3) \to BSO(3)[4]$ is 4–connected, with $\pi_5 F = \pi_5 BSO(3) = \mathbb{Z}_2$, and so k^5 is a cohomology class in $H^6(BSO(3)[4];\mathbb{Z}_2)$. We calculate $H^6(BSO(3)[4];\mathbb{Z}_2)$ by using the spectral sequence in $\mathbb{Z}_2$ cohomology associated to the fibration $K(\mathbb{Z},4) \to BSO(3)[4] \to K(\mathbb{Z}_2,2)$.

Recall that $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is singly generated by the fundamental class as an algebra that is a module over the Steenrod algebra. So, a basis of $H^*(K(\mathbb{Z}_2,2);\mathbb{Z}_2)$ is given by products of $Sq^I i_2$, where Sq^I is any admissible composition of Steenrod squares, and i_2 is the fundamental class. Similarly, $H^*(K(\mathbb{Z},n);\mathbb{Z}_2)$ is singly generated by the mod 2 reduction of the fundamental class i_n as an algebra that is a module over the Steenrod algebra, with the caveat that $Sq^1 i_n = 0$ (since i_n admits an integral lift). Here is the relevant part of the E_2 page of the spectral sequence for $K(\mathbb{Z},4) \to BSO(3)[4] \to K(\mathbb{Z}_2,2)$ over $\mathbb{Z}_2$:

$$
\begin{array}{ccccccccc}
7 & Sq^3 i_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & Sq^2 i_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4 & i_4 & \cdot & i_2 i_4 & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & i_2 & Sq^1 i_2 & i_2^2 & Sq^2Sq^1 i_2, & (Sq^1 i_2)^2, & (Sq^1 i_2) i_2^2, \\
 & & & & & & i_2 Sq^1 i_2 & i_2^3 & (Sq^2Sq^1 i_2) i_2 \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

It will help to first determine the transgression of the fundamental class i_4, i.e. $d_5 i_4$ (where d_5 denotes the differential on E_5, which goes 5 units right and 4 units down). Recall that integrally, the transgression of i_4 is $\delta P i_2$. To figure out what it is modulo 2, we observe a few things: First of all, the transgression mod 2 is non-zero. Indeed, otherwise $K(\mathbb{Z}_2,2) \times K(\mathbb{Z},4)$ would be the homotopy 4–type of $BSO(3)$ at the prime 2. However, the relation $w_2^2 = p_1$ mod 2 in the cohomology of $BSO(3)$ tells us that this cannot be. Secondly, note that $Sq^1(d_5 i_4) = 0$ since $d_5 i_4$ has an integral lift, namely $\delta P i_2$. A basis for $H^5(K(\mathbb{Z}_2,2);\mathbb{Z}_2)$ is given by $Sq^2 Sq^1 i_2$ and $i_2 Sq^1 i_2$, and we see, using the Adem relations $Sq^1Sq^2 = Sq^3$ and $Sq^1Sq^1 = 0$, that

$$Sq^1(Sq^2Sq^1i_2) = Sq^3Sq^1i_2 = (Sq^1i_2)^2 \neq 0,$$
$$Sq^1(i_2Sq^1i_2) = Sq^1i_2Sq^1i_2 + i_2Sq^1Sq^1i_2 = (Sq^1i_2)^2 \neq 0.$$

We conclude that

$$d_5i_4 = \delta Pi_2 \bmod 2 = Sq^2Sq^1i_2 + i_2Sq^1i_2.$$

As we see in the above diagram for the spectral sequence of the fibration $K(\mathbb{Z},4) \to BSO(3)[4] \to K(\mathbb{Z}_2,2)$, the classes $(Sq^1i_2)^2$ and i_2^3 will survive to E_∞ and define classes in $H^6(BSO(3)[4];\mathbb{Z}_2)$, namely $p^*(Sq^1i_2)^2$ and $p^*i_2^3$, where p is the map $BSO(3)[4] \to K(\mathbb{Z}_2,2)$. Recall that p was in fact the map that picks out the second Stiefel–Whitney class w_2, and so these classes in $H^6(BSO(3)[4];\mathbb{Z}_2)$ are w_2^3 and $(Sq^1w_2)^2 = w_3^2$. (The map $BSO(3) \to BSO(3)[4]$ is a 4–equivalence, so we do indeed have w_2 and w_3 in the cohomology of $BSO(3)[4]$.)

The other classes that might contribute to $H^6(BSO(3)[4];\mathbb{Z}_2)$ are i_2i_4 and $\mathrm{Sq}^2\, i_4$. We see that

$$d_5(i_2i_4) = i_2d_5i_4 = i_2Sq^2Sq^1i_2 + i_2^2Sq^1i_2,$$

which is non-zero since $H^*(K(\mathbb{Z}_2,2);\mathbb{Z}_2)$ is free as a $\mathbb{Z}_2$ polynomial ring over the variables $\mathrm{Sq}^I\, i_2$. So, i_2i_4 does not contribute to cohomology. As for Sq^2i_4, we first remark that $Sq^kd = dSq^k$ in spectral sequence calculations, whenever both sides of the equation make sense. For example, we have

$$\begin{aligned} d_7Sq^2i_4 = Sq^2d_5i_4 &= Sq^2Sq^2Sq^1i_2 + Sq^2(i_2Sq^1i_2) \\ &= i_2^2Sq^1i_2 + Sq^1i_2Sq^1Sq^1i_2 + i_2Sq^2Sq^1i_2 = i_2(i_2Sq^1i_2 + Sq^2Sq^1i_2). \end{aligned}$$

We used the Adem relation $Sq^2Sq^2Sq^1 = 0$. Now, recall that $d_5i_4 = i_2Sq^1i_2 + Sq^2Sq^1i_2$, and so on E_7, $i_2Sq^1i_2 + Sq^2Sq^1i_2$ is the zero class. Therefore $d_7Sq^2i_4 = 0$ and Sq^2i_4 gives the third and last basis element in $H^6(BSO(3)[4];\mathbb{Z}_2)$.

$$H^6(BSO(3)[4];\mathbb{Z}_2) = \mathrm{span}(w_2^3, w_3^2, x),$$

where x is a class that pulls back to Sq^2i_4 by the inclusion $K(\mathbb{Z},4) \to BSO(3)[4]$ in the fibration $K(\mathbb{Z},4) \to BSO(3)[4] \to K(\mathbb{Z}_2,2)$.

Now, the k-invariant $BSO(3)[4] \xrightarrow{k^5} K(\mathbb{Z}_2,6)$ is some $\mathbb{Z}_2$–combination of w_2^3, w_3^2, x. In [1] it is argued that this class is non-zero (an alternative way of showing that a k-invariant is non-zero will be used in the next section), and the coefficient along x is non-zero.

3. The sixth k-invariant of $BSO(3)$

The map $BSO(3) \to BSO(3)[5]$ has 5–connected homotopy fiber F which satisfies $\pi_6F = \pi_6BSO(3) = \pi_5S^3 = \mathbb{Z}_2$, and so the sixth k–invariant for $BSO(3)$ is a class $k^6 \in H^7(BSO(3)[5];\mathbb{Z}_2)$. We calculate $H^7(BSO(3)[5];\mathbb{Z}_2)$ from the spectral sequence for the fibration $K(\mathbb{Z}_2,5) \to BSO(3)[5] \to BSO(3)[4]$. Recall our notation for the cohomology of $BSO(3)[4]$: We determined it from the fibration $K(\mathbb{Z},4) \xrightarrow{i} BSO(3)[4] \xrightarrow{p} K(\mathbb{Z}_2,2)$, and $H^6(BSO(3)[4];\mathbb{Z}_2)$ is spanned by $p^*i_2^3$, $p^*(Sq^1i_2)^2$, and x, where $i^*x = Sq^2i_4$. Here i_2 and i_4 denote the mod 2 fundamental classes of $K(\mathbb{Z}_2,2)$ and $K(\mathbb{Z},4)$ respectively.

Before considering the spectral sequence for $K(\mathbb{Z}_2,5) \to BSO(3)[5] \to BSO(3)[4]$, we will need to figure out $H^7(BSO(3)[4];\mathbb{Z}_2)$, so we revisit the spectral sequence $K(\mathbb{Z},4) \to BSO(3)[4] \to K(\mathbb{Z}_2,2)$.

$$
\begin{array}{c|cccccccc}
7 & Sq^3 i_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & Sq^2 i_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4 & i_4 & \cdot & \cdot & (Sq^1 i_2) i_4 & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & i_2 & Sq^1 i_2 & i_2^2 & Sq^2Sq^1 i_2, & (Sq^1 i_2)^2, & (Sq^1 i_2) i_2^2, \\
 & & & & & & i_2 Sq^1 i_2 & i_2^3 & (Sq^2Sq^1 i_2) i_2 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

Recall that $d_5 i_4 = Sq^2Sq^1 i_2 + i_2 Sq^1 i_2$. We also saw that $d_7 Sq^2 i_4 = (Sq^1 i_2) i_2^2 + (Sq^2Sq^1 i_2) i_2$, and so $K(\mathbb{Z}_2,2)$ itself will contribute a single class to $H^7(BSO(3)[4];\mathbb{Z}_2)$, namely $[(Sq^1 i_2) i_2^2] = [(Sq^2Sq^1 i_2) i_2)]$. There are two more potential contributors to $H^7(BSO(3)[4];\mathbb{Z}_2)$, namely $Sq^3 i_4$ and $(Sq^1 i_2) i_4$. First, we see that

$$
\begin{aligned}
d_8 Sq^3 i_4 &= Sq^3 d_5 i_4 = Sq^3(Sq^2Sq^1 i_2 + i_2 Sq^1 i_2) \\
&= Sq^3Sq^2Sq^1 i_2 + i_2 Sq^3 Sq^1 i_2 + Sq^1 i_2 Sq^2 Sq^1 i_2 \\
&= i_2 (Sq^1 i_2)^2 + Sq^1 i_2 Sq^2 Sq^1 i_2 \\
&= (Sq^1 i_2)(i_2 Sq^1 i_2 + Sq^2Sq^1 i_2.).
\end{aligned}
$$

This equation descends to one on the E_8 page, and since the right hand factor is the zero class (thanks to $d_7 Sq^2 i_4$), we have that $Sq^3 i_4$ gives a class in $H^7(BSO(3)[4];\mathbb{Z}_2)$.

As for $(Sq^1 i_2) i_4$, we have

$$
\begin{aligned}
d_5((Sq^1 i_2) i_4) &= Sq^1 i_2 d_5 i_4 \\
&= Sq^1 i_2 (Sq^2 Sq^1 i_2 + i_2 Sq^1 i_2) \\
&= Sq^1 i_2 Sq^2 Sq^1 i_2 + i_2 (Sq^1 i_2)^2,
\end{aligned}
$$

which is non-zero on E_5.

Let us denote the class that $Sq^3 i_4$ creates by y. This is a class that pulls back to $Sq^3 i_4$ under the inclusion $K(\mathbb{Z},4) \to BSO(3)[4]$. Now we go to the spectral sequence for $K(\mathbb{Z}_2,5) \to BSO(3)[5] \to BSO(3)[4]$.

$$
\begin{array}{c|cccccccc}
7 & Sq^2 i_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
6 & Sq^1 i_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
5 & i_5 & \cdot & i_2 i_5 & \cdot & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & i_2 & \cdot & \cdot & \cdot & x, p^* i_2^3, & p^*(Sq^1 i_2) i_2^2, \\
 & & & & & & & p^*(Sq^1 i_2)^2 & y \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

Since $d_6 i_5 = x$, and $i^* x = Sq^2 i_4$, we have that $i^* d_7 Sq^1 i_5 = Sq^1 i^* d_6 i_5 = Sq^1 Sq^2 i_4$, and so

$$d_7 Sq^1 i_5 = y + \epsilon \cdot p^*(Sq^1 i_2) i_2^2,$$

where $\epsilon \in \mathbb{Z}_2$.

Before we proceed, let us argue that k^6 must be non-zero. Indeed, otherwise $BSO(3) \to BSO(3)[4] \times K(\mathbb{Z}_2, 5)$ would be a 5–equivalence. However,

$$H^5(BSO(3)[4] \times K(\mathbb{Z}_2, 5); \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

(spanned by $Sq^2 Sq^1 i_2 \otimes 1$ and $1 \otimes i_5$), while $H^5(BSO(3); \mathbb{Z}_2) = \mathbb{Z}_2$ (spanned by $w_2 w_3$).

Since $d_7 Sq^1 i_5 = y + \epsilon \cdot p^*(Sq^1 i_2) i_2^2$, we conclude that there are only two possible classes in $H^7(BSO(3)[5]; \mathbb{Z}_2)$; namely the class $p^*(Sq^1 i_2) i_2^2$ (which might be cohomologous to y) and $Sq^2 i_5$. Now, $Sq^2 i_5$ must survive to define a class, since otherwise the k–invariant k^6 (which we saw must be non-zero) would have to be (cohomologous to) $p^*(Sq^1 i_2) i_2^2$. But $p^*(Sq^1 i_2) i_2^2$ must survive through all stages of the Postnikov tower of $BSO(3)$, since it will become $(Sq^1 w_2) w_2^2 = w_3 w_2^2$, a non-zero class in $H^7(BSO(3); \mathbb{Z}_2)$.

Denote by z the class that $Sq^2 i_5$ becomes in $H^7(BSO(3)[5]; \mathbb{Z}_2)$. We conclude that the k-invariant k^6 is equal to $z + \varepsilon \cdot y$, where $\varepsilon \in \mathbb{Z}_2$. If it is the case that $d_7 Sq^1 i_5 = y$, then $k^6 = z$.

References

[1] Antieau, B. and Williams, B., 2014. On the classification of oriented 3-plane bundles over a 6-complex. Topology and its Applications, 173, pp.91-93.

[2] Alex Kruckman's Adem relations calculator, `https://math.berkeley.edu/ kruckman/adem/`

[3] `https://mathoverflow.net/questions/250303/the-fifth-k-invariant-of-bso3`