# The Decomposition Theorem, perverse sheaves and the topology of algebraic maps

Mark Andrea A. de Cataldo* and Luca Migliorini†

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**Abstract**

We give a motivated introduction to the theory of perverse sheaves, culminating in the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber. A goal of this survey is to show how the theory develops naturally from classical constructions used in the study of topological properties of algebraic varieties. While most proofs are omitted, we discuss several approaches to the Decomposition Theorem, indicate some important applications and examples.

## Contents



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# 1 Overview

The subject of perverse sheaves and the Decomposition Theorem have been at the heart of a revolution which took place over the last thirty years in algebra, representation theory and algebraic geometry.

The Decomposition Theorem is a powerful tool to investigate the homological properties of proper maps between algebraic varieties. It is the deepest fact known concerning the homology, Hodge theory and arithmetic properties of algebraic varieties. Since its discovery in the early 1980's, it has been applied in a wide variety of contexts, ranging from algebraic geometry to representation theory to combinatorics. It was conjectured by Gelfand and MacPherson in [78] in connection with the development of Intersection Homology theory. The Decomposition Theorem was then proved in very short order by Beilinson, Bernstein, Deligne and Gabber in [8]. In the sequel of this introduction we try to motivate the statement as a natural outgrowth of the deep investigations on the topological properties of algebraic varieties, begun with Riemann, Picard, Poincaré and Lefschetz, and culminated in the spectacular results obtained with the development of Hodge Theory and étale cohomology. This forces us to avoid many crucial technical details, some of which are dealt with more completely in the following sections. We have no pretense of historical completeness. For an account of the relevant history, see the historical remarks in [82], and the survey by Kleiman [111].

As to the contents of this survey, we refer to the detailed table of contents.

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## 1.1 The topology of complex projective manifolds: Lefschetz and Hodge Theorems

Complex algebraic varieties provided an important motivation for the development of algebraic topology from its earliest days (Riemann, Picard Poincaré). However, algebraic varieties and algebraic maps enjoy many truly remarkable topological properties that are not shared by other classes of spaces. These special features were first exploited by Lefschetz ([115]) (who claimed to have "planted the harpoon of algebraic topology into the body of the whale of algebraic geometry" [116], pag.13.) and they are almost completely summed up in the statement of the Decomposition Theorem and its embellishments.

**Example 1.1** (**Formality**) The relation between the cohomology and homotopy groups of a topological manifold is extremely complicated. Deligne, Griffiths, Morgan and Sullivan [60] have discovered that for a nonsingular simply connected complex projective variety $X$, the real homotopy groups $\pi_i(X) \otimes \mathbb{R}$ can be computed formally in terms of the cohomology ring $H^*(X)$.

In the next few paragraphs we shall discuss the Lefschetz theorems and the Hodge decomposition and index theorems. Together with Deligne's degeneration theorem These are precursors of the Decomposition Theorem and they are essential tools in the proofs, described in §4.1,4.2,4.3, of the Decomposition Theorem.

Let $X$ be a complex smooth $n$-dimensional projective variety and let $D = H \cap X$ be the intersection of $X$ with a generic hyperplane $H$. We use cohomology with rational coefficients. A standard textbook reference for what follows is [88]; see also [162] and [40].

The *Lefschetz Hyperplane Theorem* states that the restriction map $H^i(X) \to H^i(D)$ is an isomorphism for $i < n-1$ and is injective for $i = n-1$.

The cup product with the first Chern class of the hyperplane bundle gives a mapping $\cap c_1(H) : H^i(X) \to H^{i+2}(X)$ which can be identified with the composition $H^i(X) \to H^i(D) \to H^{i+2}(X)$, the latter being a "Gysin" homomorphism. The *Hard Lefschetz Theorem* states that for all $0 \leq i \leq n$ the $i$-fold iteration is an isomorphism

$$(\cap c_1(H))^i : H^{n-i}(X) \xrightarrow{\simeq} H^{n+i}(X).$$

Lefschetz had "proofs" of both theorems (see [112] for an interesting discussion of Lefschetz's proofs). R. Thom outlined a Morse-theoretic approach to the hyperplane theorem which was worked out in detail by Andreotti and Frankel (see [132]) [1] and Bott [18]. Lefschetz's proof of the Hard Lefschetz Theorem is incorrect. Hodge [95] (see also [165] gave the first complete proof. Deligne [58] gave a proof and a vast generalization of the Hard Lefschetz Theorem using varieties defined over finite fields.

The *Hodge Decomposition* is the canonical decomposition $H^i(X, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}(X)$. The summand $H^{p,q}(X)$ consists of cohomology classes on $X$ which admit a $g$-harmonic $(p,q)$ differential form as a representative, where $g$ is any fixed Kähler metric on $X$.

For a fixed index $0 \leq i \leq n$, there is the bilinear form on $H^{n-i}(X)$ defined by

$$(a, b) \longmapsto \int_X (c_1(H))^i \wedge a \wedge b.$$

While the Hard Lefschetz Theorem is equivalent to the non degeneracy of these forms, the *Hodge-Riemann Bilinear Relations* (cf. §5.1, Theorem 5.1.(3) establish their signature properties.

## 1.2 Singular algebraic varieties.

The Lefschetz and Hodge theorems fail if $X$ is singular. Two (somewhat complementary) approaches to modifying them for singular varieties involve Mixed Hodge Theory [55, 56] and Intersection Cohomology [83, 84], [16].

In *Mixed Hodge Theory* the topological invariant studied is the same investigated for nonsingular varieties, namely, the cohomology groups of the variety, whereas the structure with which it is endowed changes. See [66] for an elementary and nice introduction. The $(p,q)$ decomposition of classical Hodge Theory is replaced by a more complicated structure which is, roughly speaking, a finite iterated extension of $(p,q)$-decompositions.

The cohomology groups $H^i(X)$ are endowed with an increasing filtration (the weight filtration) $W$, and the graded pieces $W_k/W_{k-1}$ have a $(p,q)$ decomposition of weight $k$, that is $p+q=k$. Such a structure, called a *mixed Hodge structure*, exists canonically on any algebraic variety and satisfies several fundamental weight restrictions. Here are few:

1. if the not necessarily compact $X$ is nonsingular, then the weight filtration on $H^k(X)$ starts at $W_k$, that is $W_r H^k(X) = 0$ for $r < k$;

2. if $X$ is compact and possibly singular, then the weight filtration on $H^k(X)$ ends at $W_k$, that is $W_r H^k(X) = W_k H^k(X) = H^k(X)$ for $r \geq k$.

**Example 1.2** Let $X = \mathbb{C}^*$. Then $H^1(X) \simeq \mathbb{Q}$ has weight 2. All the classes in $H^1(X)$ are of type $(1,1)$. Let $X$ be a rational irreducible curve with a node. Then $H^1(X) \simeq \mathbb{Q}$ has weight 0. All the classes in $H^1(X)$ are of type $(0,0)$.

By contrast, in *Intersection Cohomology Theory*, it is the topological invariant which is changed, whereas the $(p,q)$-structure turns out to be the same. The *intersection cohomology groups* $IH^i(X)$ can be described using geometric "cycles" and this gives a concrete way to compute simple examples. There is a natural homomorphism $H^i(X) \to IH^i(X)$ which is an isomorphism when $X$ is nonsingular. The groups $IH^i(X)$ are finite dimensional, satisfy the Mayer-Vietoris Theorem and the Künneth Formula. These groups are not homotopy invariant but, in compensation, they have the following additional features: they satisfy Poincaré Duality ([83, 84]) and the Lefschetz theorems. The Lefschetz Hyperplane Theorem is proved in [82]). The Hard Lefschetz Theorem for intersection cohomology is proved in [8]. M. Saito ([144, 145]) has proved that they admit a $(p,q)$-Hodge Decomposition, has re-proved that they satisfy the Hard Lefschetz Theorem and has established the Hodge-Riemann Bilinear Relations. We have also proved these results in [44, 46] by different methods, and our proofs are strongly intertwined with our proof of the Decomposition Theorem.

**Example 1.3** Let $X$ be the nodal curve of Example 1.2. Then $IH^1(X) = 0$.

**Example 1.4** Let $E \subseteq \mathbb{P}^N_{\mathbb{C}}$ be a nonsingular projective variety of dimension $n-1$, and let $Y \subseteq \mathbb{C}^{N+1}$ be its affine cone with vertex $o$. The Intersection Cohomology groups can be easily computed (see [16] and also Example 3.1):

$$IH^i(Y) = 0 \text{ for } i \geq n \qquad IH^i(Y) = H^i(Y \setminus \{o\}) \text{ for } i < n.$$

There is a twisted version of intersection (co)homology with values in a local system $L$ defined on a Zariski dense nonsingular open subset of the variety. Intersection cohomology with twisted coefficients appears in the statement of the Decomposition Theorem.

## 1.3 Families of smooth projective varieties

By *family of projective manifolds* we mean a proper holomorphic submersion $f : X \to Y$ that factors through some product $Y \times \mathbb{P}^N$. By a result of Ehresman, such a map is also a $C^\infty$ fiber bundle.

If $f : X \to Y$ is a $C^\infty$ fiber bundle with smooth compact fiber $F$, let $\underline{H}^j(F)$ denote the local system on $Y$ whose fiber at the point $y \in Y$ is $H^j(f^{-1}(y))$. There are the associated *Leray spectral sequence*

$$E_2^{i,j} = H^i(Y; \underline{H}^j(F)) \Longrightarrow H^{i+j}(X) \tag{1}$$

and the *monodromy representation*

$$\rho_i : \pi_1(Y, y_0) \to GL(H^i(F)). \tag{2}$$

Even if $Y$ is simply connected, the Leray spectral sequence can be nontrivial, for example, the Hopf fibration $f : S^3 \to S^2$. The following degeneration and semisimplicity statements are due to Deligne [52, 55].

**Theorem 1.5** *Suppose $f : X \to Y$ is a family of projective manifolds. Then*

1. *The Leray spectral sequence* (1) *degenerates at the $E_2$-page and induces an isomorphism $H^i(X) \cong \oplus_{a+b=i} H^a(Y; \underline{H}^b(F))$*
2. *The representation* (2) *is semisimple: it is a direct sum of irreducible representations.*

Part 1. gives a rather complete description of the cohomology of $X$. Part 2. is remarkable because the fundamental group of $Y$ can be infinite.

**Remark 1.6** The isomorphism induced by the $E_2$-degeneration is not canonical, for one is merely splitting the Leray filtration and using the fact that the graded pieces are the entries of the $E_2$ page. This is a recurrent theme also in the Decomposition Theorem.

**Example 1.7** Except for the case of smooth families, the Leray spectral sequence is very seldom degenerate. If $f : X \to Y$ is a resolution of the singularities of a projective variety $Y$ whose cohomology has a mixed Hodge structure which is not pure, then $f^*$ cannot be injective, and this prohibits degeneration.

The following is the classical Global Invariant Cycle Theorem. We shall come back to this later in §1.7, where we give some context, generalizations and references.

**Theorem 1.8** *Suppose $f : X \to Y$ is a family of projective manifolds. Then*

$$H^i(F_{y_0})^{\pi_1(Y, y_0)} = \mathrm{Im}\,\{H^i(X) \longrightarrow H^i(F_{y_0})\},$$

*i.e. the monodromy invariants are restrictions from the total space of the family.*

Although the classical Lefschetz-Hodge theorems described in §1.1 and the results described in this section appear to be very different from each other, the Decomposition Theorem forms a single, beautiful common generalization which holds even for singular varieties.

## 1.4 Provisional statement of the Decomposition Theorem.

We state a provisional, yet suggestive form of the Decomposition Theorem.

**Theorem 1.9** (**Decomposition Theorem for intersection cohomology groups**) *Let $f : X \to Y$ be a proper map of varieties with $X$ nonsingular. There exist finitely many triples $(Y_a, L_a, d_a)$ made of locally closed smooth irreducible algebraic subvarieties $Y_a$, semisimple local systems $L_a$ on $Y_a$ and integer numbers $d_a$, such that for every Euclidean open set $U \subseteq Y$*

$$H^r(f^{-1}U) \simeq \bigoplus_a IH^{r-d_a}(U \cap \overline{Y}_a, L_a). \tag{3}$$

**Remark 1.10** The triples $(Y_a, L_a, d_a)$ are essentially unique, independent of $U$, and they are described in [44, 46]. The decomposition map (3) is not uniquely defined. In the case of a projective map endowed with an $f$-ample line bundle, there is a distinguished choice [50] which is well-behaved from the point of view of Hodge theory. Setting $U = Y$ we get a formula for $H(X)$.

**Remark 1.11** If $X$ is singular, then there is no analogous formula for $H(X)$. Intersection cohomology turns out to be precisely the topological invariant apt to deal with singular maps. The notion of intersection cohomology is needed even when domain and codomain are nonsingular.

**Remark 1.12** The Decomposition Theorem, coupled with the Relative Hard Lefschetz Theorem 1.15, yields a more precise description of the properties of the $Y_a$ and of the local systems $L_a$. For instance, if $\mathcal{L}$ is an ample bundle on a projective $Y$, then every summand $IH(\overline{Y}_a, L_a)$ satisfies the Hard Lefschetz Theorem with respect to cupping with $c_1(\mathcal{L})$, i.e.

$$c_1(\mathcal{L})^k : IH^{dim Y_a - k}(\overline{Y}_a, L_a) \longrightarrow IH^{dim Y_a + k}(\overline{Y}_a, L_a)$$

is an isomorphism. The Hodge-theoretic versions [144, 46] also yield the associated Hodge-Riemann Bilinear Relations, i.e. the analogue of Theorem 5.1.(3) in intersection cohomology.

## 1.5 Crash Course on the derived category

We do not know of a general method for proving the decomposition (3) without passing through the analogous decomposition taking place in the appropriate derived category. The language and theory of homological algebra, specifically derived categories, $t$-structures and perverse sheaves, plays an essential role in all the known proofs of the Decomposition Theorem [8, 144, 44, 46].

We now collect the few facts about derived categories needed in order to understand the statement of the Decomposition Theorem. We amplify and complement this list in §5. Standard references are [79, 107, 103, 16].

**1. Complexes of sheaves.** Most of the constructions in homological algebra involve complexes. For example, if $Z$ is a smooth manifold, in order to compute the cohomology of the constant sheaf $\mathbb{R}_Z$ we replace it by the complex of sheaves of differential forms and take the complex of its global sections. More generally, to define the cohomology of a sheaf $A$ on a topological space $Z$, we choose an injective, or flabby, resolution, for instance the one defined by Godement,

$$0 \longrightarrow A \longrightarrow I^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{i-1}} I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

then consider the complex of abelian groups

$$0 \longrightarrow \Gamma(I^0) \xrightarrow{d^0} \cdots \xrightarrow{d^{i-1}} \Gamma(I^i) \xrightarrow{d^i} \Gamma(I^{i+1}) \longrightarrow \cdots$$

and finally take its cohomology. The derived category is a formalism developed in order to work systematically with complexes of sheaves with a notion of morphism far more flexible than that of morphism of complexes; for instance, two different resolutions of the same sheaf are isomorphic in the derived category. Let $Z$ be a topological space. We consider sheaves of $\mathbb{Q}$-vector spaces on $Z$. A bounded *complex of sheaves* $K$ is a diagram

$$\cdots \longrightarrow K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \longrightarrow \cdots$$

with $K^i = 0$ for $|i| \gg 0$ and satisfying $d^i \circ d^{i-1} = 0$. The *shifted* complex $K[n]$ is the complex with $K[n]^i = K^{n+i}$. Complexes of sheaves form an Abelian category and we may form the *cohomology sheaf* $\mathcal{H}^i(K) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1})$.

**2. Quasi-isomorphisms and resolutions.** A morphism $K \to L$ of complexes of sheaves is a *quasi-isomorphism* if it induces isomorphisms $\mathcal{H}^i(K) \cong \mathcal{H}^i(L)$ of cohomology sheaves. An *injective (flabby, fine) resolution* of a complex $K$ is a quasi-isomorphism $K \to I$ of $K$ with a complex with injective (flabby, fine) components. The cohomology groups $H(Z, K)$ of $K$ are defined to be the cohomology groups of the complex of global sections $\Gamma(I)$ of $I$. Note that this definition extends to complexes the definition of cohomology of a single sheaf given above, as soon as one identifies sheaves with the complexes of sheaves concentrated in degree 0.

A quasi-isomorphism $K \to L$ induces isomorphisms on the cohomology, $H^i(U, K) \cong H^i(U, L)$ of any open set $U \subset Z$ and these isomorphisms are compatible with the maps induced by inclusions and with Mayer Vietoris sequences.

**3. The derived category.** The bounded *derived category* $D^b(Z)$ is a category whose objects are the bounded complexes of sheaves, but whose morphisms have been cooked up in such a way that every quasi-isomorphism $S \to T$ becomes an isomorphism in $D^b(Z)$ (i.e., it has a unique inverse morphism). In this way different complexes of sheaves that realize the same cohomology theory (such as the complex of singular cochains and the complex of differential forms on a smooth manifold) have become isomorphic in $D^b(Z)$. The definition of the morphisms in the derived category is done in two steps: in the first step homotopic morphisms are identifed while the second more important step is to formally add inverses to quasi-isomorphisms, in a way that is strongly reminiscent of the localization of a ring with respect to a multiplicative system.

**4. Derived functors.** The main feature of the derived category is the possibility of defining derived functors: As we already said, if $I$ is a complex of injective (flabby, or even fine) sheaves, the cohomology $H^i(Z, I)$ is the cohomology of the complex of abelian groups

$$\cdots \longrightarrow \Gamma(Z, I^{i-1}) \longrightarrow \Gamma(Z, I^i) \longrightarrow \Gamma(Z, I^{i+1}) \longrightarrow \cdots$$

which can be considered as an object, denoted $R\Gamma(Z, I)$ of the derived category of a point. However, if the complex is not injective, as the example of the constant sheaf on a smooth manifold shows, this procedure gives the wrong answer, as the complexes of global sections of two quasi-isomorphic complexes are not necessarily quasi-isomorphic. Since every complex $K$ has an injective resolution $K \to I$, unique up to a unique isomorphism in $D^b(Z)$, the complex $R\Gamma(K) := \Gamma(Z, I)$ (a flabby resolution can be used as well and, if there is one, also a fine one) is well defined.
A similar problem arises when $f : Z \to W$ is a continuous mapping: if $I$ is a complex of injective sheaves on $Z$ then the push forward complex $f_*(I)$ will satisfy

$$H^i(U, f_*(I)) \cong H^i(f^{-1}(U), I) \tag{4}$$

for any open set $U \subseteq W$. However if a complex $K$ is not injective, then (4) may fail, and $K$ should first be replaced by an injective resolution before pushing forward. The resulting complex of sheaves is well defined up to canonical isomorphism in $D^b(W)$, is denoted $Rf_*K$ and is called the (derived) *direct image* of $K$. Its cohomology sheaves are denoted $R^if_*K$ and are called the *i-th direct image sheaves.* We note that if $f$ maps $Z$ to a point, $Rf_*K = R\Gamma(K)$ and $R^if_*K = H^i(Z, K)$. By a similar process one defines the functor $Rf_!$, the derived direct image with proper support. There is a map of functors $Rf_! \to Rf_*$ which is an isomorphism if $f$ is proper. Under quite general hypotheses, always satisfied by algebraic maps of algebraic varieties, given a map $f : Z \to W$, there are the inverse image and extraordinary inverse image functors $f^*, f^! : D^b(W) \to D^b(Z)$; see 5.2 for their properties as well as for their relation to Verdier duality.

**5. Constructible sheaves.** From now on, suppose $Z$ is a complex algebraic variety. A subset $V \subset Z$ is *constructible* if it is obtained from a finite sequence of unions, intersections, or complements of algebraic subvarieties of $Z$. A *local system* is a locally constant sheaf with finite dimensional stalks. The complex of sheaves $K$ is *constructible* (or with constructible cohomology sheaves) if there exists a decomposition $Z = \coprod_\alpha Z_\alpha$ into finitely many constructible subsets such that each of the cohomology sheaves $\mathcal{H}^i(K)$ is locally constant along each $Z_\alpha$ with finite dimensional stalks. This implies that the limit in

$$\mathcal{H}^i_x(K) := \lim_{\to} H^i(U_x, K) \tag{5}$$

is attained by any "regular" neighborhood $U_x$ of the point $x$ (for example, one may embed (locally) $Z$ into a manifold and take $U_x = Z \cap B_\epsilon(x)$ to be the intersection of $Z$ with a sufficiently small ball centered at $x$). It also implies that $H^i(Z, K)$ is finite dimensional.

Constructibility prevents the cohomology sheaves from exhibiting Cantor-set like behavior. Most of the complexes of sheaves arising naturally from geometric construction are constructible. In what follows, all complexes of sheaves are assumed to be constructible. They are the objects in the corresponding *constructible bounded derived category* $\mathcal{D}_Z$ (strictly speaking, this is not a derived category, but a full subcategory of the derived category). This subcategory is stable with respect to the functors $Rf_*, Rf_!, f^*, f^!$ associated with an algebraic map.

**6. Intersection complexes, perverse sheaves** Let $U \subset Z$ be a nonsingular Zariski open subset and let $L$ be a local system on $U$. The *intersection complex* ([84]) $IC_Z(L)$ is a complex of sheaves on $Z$, which extends the complex $L[\dim Z]$ on $U$, uniquely determined up to quasi-isomorphism by certain *support* conditions, see 20 and 21 of §3.1. Its cohomology is the intersection cohomology, $H^i(Z, IC_Z(L)) = IH^{\dim Z+i}(Z, L)$. The intersection complexes of local systems supported on open sets of subvarieties of $Z$ are the building blocks of an abelian subcategory of $\mathcal{D}_Z$ of extreme importance in our story, the category of *perverse sheaves*. Despite the name, a perverse sheaf is a special kind of constructible complex verifying some condition on the support of its cohomology sheaves and that of its cohomology with compact support sheaves, see 3.4. The category of perverse sheaves is Artinian: every perverse sheaf is an iterated extension of finitely many simple perverse sheaves. It is important that the we use field coefficients, as this property fails for integer coefficients. The simple perverse sheaves are the intersection complexes $IC_Y(L)$ of irreducible subvarieties $Y \subset Z$ and simple local systems $L$ (defined on a nonsingular Zariski open subset of $Y$).

The (ordinary) constructible sheaves, thought of as complexes concentrated in degree 0, form an abelian subcategory of the constructible derived category $\mathcal{D}_Z$ . An object $K$ of $\mathcal{D}_Z$ is isomorphic to an object of this subactegory if and only if $\mathcal{H}^i(K) = 0$ if $i \neq 0$. There is a similar characterization of the category of perverse sheaves: every complex $K$ comes with the *perverse cohomology sheaves* ${}^{\mathfrak{p}}\!\mathcal{H}^i(K)$, each of which is a perverse sheaf that is obtained from $K$ in a canonical way. The perverse sheaves are characterized by the property that ${}^{\mathfrak{p}}\!\mathcal{H}^i(K) = 0$ if $i \neq 0$. Just as there is the *Grothendieck spectral sequence*

$$E_2^{l,m} = H^l(Z, \mathcal{H}^m(K)) \Longrightarrow H^{l+m}(Z, K),$$

there is the *perverse spectral sequence*

$$E_2^{l,m} = H^l(Z, {}^{\mathfrak{p}}\!\mathcal{H}^m(K)) \Longrightarrow H^{l+m}(Z, K).$$

**Remark 1.13** If $U$ is a nonempty, nonsingular and pure dimensional open subset of $Z$ on which all the cohomology sheaves $\mathcal{H}^i(K)$ are local systems, then the restriction to $U$ of ${}^{\mathfrak{p}}\!\mathcal{H}^m(K)$ and $\mathcal{H}^{m-\dim Z}(K)[\dim Z]$ coincide. In general, the two differ: in Example 1.23, we have ${}^{\mathfrak{p}}\!\mathcal{H}^0(Rf_*\mathbb{Q}_X[2]) = IC_Y(R^1) \oplus T_\Sigma$. This example already illustrates the non triviality of the notion of perverse cohomology.

## 1.6 Decomposition, Semisimplicity and Relative Hard Lefschetz Theorems

**Theorem 1.14** (**Decomposition and Semisimplicity Theorems**) *Let $f : X \to Y$ be a proper map. There is a noncanonical isomorphism in the constructible derived category $\mathcal{D}_Y$ of sheaves on $Y$:*

$$Rf_*IC_X \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)[-i]. \tag{6}$$

*Furthermore, the perverse sheaves ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)$ are semisimple, i.e. there is a decomposition into finitely many disjoint locally closed and nonsingular subvarieties $Y = \coprod S_\beta$ and a canonical decomposition into a direct sum of intersection complexes of semisimple local systems*

$$ {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X) = \bigoplus_\beta IC_{\overline{S_\beta}}(L_\beta). \tag{7}$$

As a consequence, the perverse Leray spectral sequence

$$H^l(Y, {}^{\mathfrak{p}}\mathcal{H}^m(f_*IC_X)) \Longrightarrow IH^{\dim X + l + m}(X, \mathbb{Q})$$

is $E_2$-degenerate. As we shall see in §2, this fact alone has striking computational and theoretical consequences.

One important application is that the intersection cohomology groups of a variety $Y$ inject, non canonically, in the ordinary singular cohomology groups of any resolution $X$ of the singularities of $Y$.

The first Chern class of a line bundle $\eta$ on $X$ yields, for every $i \geq 0$, maps $\eta^i : Rf_*IC_X \to Rf_*IC_X[2i]$ and $\eta^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(Rf_*IC_X) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)$.

**Theorem 1.15** (**Relative Hard Lefschetz Theorem**) *Assume that $f$ is projective and that $\eta$ is the first Chern class of a line bundle on $X$ whose restriction to each fiber is ample. Then we have isomorphisms*

$$\eta^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(Rf_*IC_X) \xrightarrow{\simeq} {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X) \tag{8}$$

Theorems 1.15 and 1.14 are strictly related: the Decomposition Theorem 6 follows from 1.15 and a general argument of homological algebra, just as Theorem 1.5 1. follows from the Hard Lefschetz Theorem applied to the fibres of the map $f$; furthermore, just as, in the classical approach, the Hard Lefschetz Theorem follows from the semisimplicity of the monodromy of a Lefschetz pencil, the semisimplicity Theorem is central in establishing 1.15. In the case that $f$ maps $X$ to a point the relative Hard Lefschetz Theorem boils down to the Hard Lefschetz Theorem for intersection cohomology, while, if $f$ is a family of projective manifolds, by cfr. 1.13, the Decomposition Theorem gives, after taking cohomology, the degeneration of the Leray spectral sequence, statement 1. of 1.5, and the Semisimplicity Theorem reduce to 2. of 1.5.

**Remark 1.16** These Theorems hold more generally when applied to semisimple complexes of geometric origin (see [8] and §4.1.5) or, following the work of Saito [144] when applied to $IC_X(L)$, where $L$ is the local system underlying a polarizable variation of pure Hodge structures. The case of $IC_X$ has been re-proved by de Cataldo and Migliorini [44, 46].

**Remark 1.17** The equivariant version of these results can be found in [11].

**Example 1.18** Let $X = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}$ and $Y$ be the space obtained collapsing the set $\mathbb{P}^1_{\mathbb{C}} \times \{o\}$ to a point. This is not a complex algebraic map and (7) does not hold.

**Example 1.19** Let $f : (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z} =: X \to \mathbb{P}^1$ be the fibration in elliptic curves associated with a Hopf surface. Since $\pi_1(X) \simeq \mathbb{Z}$, we have $b_1(X) = 1$ so that $X$ is not algebraic. In particular, $f$ is not projective and Deligne's Theorem, and hence the Decomposition Theorem, does not apply. In fact, $Rf_*\mathbb{Q}_X$ does not split, for if it did, then $b_1(X) = 2$.

These three theorems are cornerstones of the topology of algebraic maps. They have found many applications to algebraic geometry and to representation theory and, in our opinion, should be regarded as expressing fundamental properties of complex algebraic geometry.

## 1.7 Invariant Cycle Theorems

The following Theorem, in its local and global form, follows quite directly from the Decomposition Theorem. It generalizes previous results, which assume that $X$ is smooth. The global case was proved in Deligne, [55], 4.1.1, as an application of Mixed Hodge theory. The local case, conjectured and shown to hold for families of curves by Griffiths in [87], Conj. 8.1, was proved by Deligne in [58]. For Hodge-theoretic approaches to Theorem 1.20.(2), see [35, 155, 70, 92].

**Theorem 1.20** (**The Global and Local Invariant Cycle Theorems**) *Let $f : X \to Y$ be a proper map. Let $U \subseteq Y$ be a Zariski open subset on which the sheaf $R^i f_*(IC_X)$ is locally constant.*

1. (**Global**) *The natural restriction map*
$$IH^i(X) \longrightarrow H^0(U, R^i f_* IC_X) \qquad \textit{is surjective.}$$

2. (**Local**) *Let $u \in U$ and $B_u \subseteq U$ be the intersection with a sufficiently small Euclidean ball centered at $u$. Then the natural restriction/retraction map*
$$H^i(f^{-1}(u), IC_X) = H^i(f^{-1}(B_u), IC_X) \longrightarrow H^0(B_u, R^i f_* IC_X) \qquad \textit{is surjective.}$$

*Proof.* See [8], p.164. See also [144]. □

### 1.8 A few examples

We give three examples: the resolution of singularities of a normal surface, the resolution of the affine cone over a projective nonsingular surface and a fibration of a surface onto a curve. More details can be found in [48].

**Example 1.21** Let $f : X \to Y$ be a resolution of the singularities of a singular normal surface $Y$. Assume that we have one singular point $y \in Y$ with $f^{-1}(y) = E$ a finite union of curves on $X$. We have a canonical isomorphism (this is because $f$ is semismall [44], see 2.2)

$$Rf_* \mathbb{Q}_X[2] = IC_Y \oplus T,$$

where $T$ is a skyscraper sheaf at $y$ with fiber $T = H_2(E)$. Recall, that since $X$ is nonsingular, $IC_X = \mathbb{Q}_X[2]$.

**Example 1.22** Let $S \subseteq \mathbb{P}^N_{\mathbb{C}}$ be an embedded projective nonsingular surface and $Y \subseteq \mathbb{A}^{N+1}$ be the corresponding threefold affine cone. Let $f : X \to Y$ be the blowing up of $Y$ at the vertex $y$. We have a non canonical isomorphism

$$Rf_* \mathbb{Q}_X[3] \simeq T_{-1}[1] \oplus (IC_Y \oplus T_0) \oplus T_1[-1],$$

the $T_j$ are skyscraper sheaves at $y$ with stalks $T_1 = T_{-1} = H_4(S)$ and $T_0 = H_3(S)$.

**Example 1.23** Let $f : X \to Y$ be a projective map with connected fibers from a smooth surface $X$ onto a smooth curve $Y$. Let $\Sigma \subseteq Y$ be the finite set of critical values and $U = Y \setminus \Gamma$ be its complement. Let $R^1 = (R^1 f_* \mathbb{Q}_X)_{|U}$ be the local system on $U$ with stalk the first cohomology of the typical fiber. We have a non canonical isomorphism

$$Rf_* \mathbb{Q}_X[2] \simeq \mathbb{Q}_Y[2] \oplus (IC_Y(R^1) \oplus T_\Sigma) \oplus \mathbb{Q}_Y,$$

where $T_\Sigma$ is a skyscraper sheaf over $\Sigma$ with stalks $T_s = H_2(f^{-1}(s))/\langle [f^{-1}(s)] \rangle$ at every $s \in \Sigma$.

In all three examples the target space is a union $Y = U \coprod \Sigma$ and we have two corresponding types of summands. The summands of type $T$ consists of classes which can be represented by cycles supported over the exceptional set $\Sigma$. This is precisely the kind of statement which lies at the heart of the Decomposition Theorem. There are classes which can be represented by intersection cohomology classes of local systems on $Y$ and classes which can be represented by intersection cohomology classes of local systems supported over smaller strata, and the cohomology of $X$ is the direct sum of these two subspaces. Suggestively speaking, *it is as if the intersection cohomology relative to a stratum singled out precisely the classes which cannot be squeezed in the inverse image by $f$ of a smaller stratum.*

# 2 Applications of the Decomposition Theorem

In this section, we give, without any pretense of completeness, a sample of remarkable applications of the Decomposition Theorem. The purpose of this section is twofold. On one hand, we hope to give a sense of the broad spectrum of applications of this result. On the other hand, we want to show the theorem "in action," that is we want to apply it to concrete situations, where the data entering the statement of the theorem, e.g. the higher perverse cohomology complexes and the stratification of the map, can be determined and studied, and see how the information supplied by the theorem can be exploited. In the short §2.5 we give an exposition of the "functions-sheaves" dictionary. This dictionary constitutes a guiding principle for many applications of $l$-adic sheaves, and in particular of perverse sheaves, to geometric representation theory. We discuss two important examples in §2.6 and §2.7. For further applications and for more details, including motivation and references, about some of the examples discussed here in connection with representation theory, we suggest G. Lusztig's [119], T.A. Springer's [154], and N. Chriss and V. Ginzburg's [36]. For lack of space and competence, we will not discuss many important examples, such as the proof of the Kazhdan-Lusztig conjectures and the applications of the geometric Fourier transform.

Another important topic which we do not discuss is the recent work [140] of B.C. Ngô. For its complexity and depth, and the richness of its applications to representation theory, it would deserve a separate treatment. In [140] the Decomposition Theorem in the $l$-adic context plays a crucial role. Good part of the paper is devoted to give a geometric interpretation and an estimate on the dimensions of the subvarieties supporting the non trivial summands in the Decomposition Theorem for the Hitchin fibration restricted to an appropriate open set. This seems to be one of the first cases in which the Decomposition Theorem is studied in depth in the context of a non generically finite map.

We focus mostly on the complex case, although most of the discussion goes through over a field of positive characteristic, with constructible $\mathbb{Q}$-sheaves replaced by $l$-adic ones.

In this chapter, we use use the machinery of derived categories and functors and some results on perverse sheaves. The notions introduced in our crash course may not be sufficient to follow the (few) proofs included. We refer to §5.2, to the references quoted there and to §3.

We adopt the simplified notation $f_*, f_!$ for the derived functors $Rf_*, Rf_!$.

## 2.1 Toric varieties and combinatorics of polytopes.

Our goal in this section is to show how the Decomposition Theorem applies to two examples of toric resolutions and to thus give a feeling for the formula of R. Stanley for the "generalized $h$-vector." For the basic definitions concerning toric varieties, we refer to [74] and [142]. The recent survey [20] contains many historical details, motivation, a discussion of open problems and recent results, and an extensive bibliography. We will adopt the point of view of polytopes, which we find more appealing to intuition. We will say that a toric variety is $\mathbb{Q}$-smooth when it has only finite quotient singularities. A map of toric varieties $f : \widetilde{X} \to X$ is called a toric resolution if it is birational, equivariant with respect to the torus action, and $\widetilde{X}$ is $\mathbb{Q}$-smooth.

Let $P \subseteq \mathbb{R}^d$ be a $d$-dimensional rational convex polytope, the convex envelope of a finite set of points in $\mathbb{R}^d$ with rational coordinates, not contained in any proper affine subspace. For $i = 0, \ldots, d-1$, let $f_i$ be the number of $i$-dimensional faces of $P$. Suppose $0 \in P$. We denote by $X_P$ the projective toric variety associated with $P$. A $d$-dimensional simplex $\Sigma_d$ is the convex envelope of $d+1$ affinely independent points $v_0, \ldots, v_d$ in $\mathbb{R}^d$. $X_{\Sigma_d}$ is a $d$-dimensional projective space, eventually weighted. A polytope is said to be simplicial if its faces are simplices. The following is well known:

**Proposition 2.1** *A toric variety $X_P$ is $\mathbb{Q}$-smooth if and only if $P$ is simplicial.*

If $P$ is a simplicial $d$-dimensional polytope with "face vector" $(f_0, \ldots f_{d-1})$, then, following Stanley, one can associate its "$h$-polynomial"

$$h(P,t) = (t-1)^d + f_0(t-1)^{d-1} + \ldots + f_{d-1}. \tag{9}$$

A simplicial toric variety has a decomposition as a disjoint union of locally closed subsets, each isomorphic to the quotient of an affine space by a finite commutative group. This decomposition can be used to compute rational cohomology, and we have the following proposition, see [74], 5.2 for a detailed proof:

**Proposition 2.2** *Let $P$ be a simplicial rational polytope, with "$h$-polynomial" $h(P,t) = \sum_0^d h_k(P)t^k$. Then*

$$h_k(P) = \dim H^{2k}(X_P, \mathbb{Q}).$$

Poincaré Duality and the Hard Lefschetz Theorem imply the following

**Corollary 2.3**

$$h_k(P) = h_{d-k}(P) \quad \text{for } 0 \leq k \leq d, \qquad h_{k-1}(P) \leq h_k(P) \quad \text{for } 0 \leq k \leq d/2.$$

Corollary 2.3 amounts to a set of non trivial relations among the face numbers $f_i$. Exploiting more fully the content of the Hard Lefschetz theorem, it is possible to characterize the vectors $(f_0, \ldots f_{d-1})$ occurring as face vectors of some simplicial polytope; see [20], Theorem 1.1.

The inequality $h_{k-1}(P) \leq h_k(P)$ implies that the polynomial

$$g(P,t) = h_0 + (h_1 - h_0)t + \ldots + (h_{[d/2]} - h_{[d/2]-1})t^{[d/2]} \tag{10}$$

has positive coefficients and uniquely determines $h$. The coefficient $g_l = h_l - h_{l-1}$ is the dimension of the primitive cohomology of $X_P$ in degree $l$.

**Example 2.4** Let $\Sigma_d$ be the $d$-dimensional simplex. We have $f_0 = d+1 = \binom{d+1}{1}, \dots, f_i = \binom{d+1}{i+1}$ and

$$h(\Sigma_d, t) = (t-1)^d + \binom{d+1}{1} + \dots + \binom{d+1}{i+1}(t-1)^{d-i-1} + \dots \binom{d+1}{d} = 1 + t + \dots + t^d,$$

so that $h_i = 1$ and $g(\Sigma_d, t) = 1$, consistently with the fact that $X_{\Sigma_d} = \mathbb{P}^d$.
Let $C_2$ be the square, convex envelope of the four points $(\pm 1, 0), (0, \pm 1)$. We have $f_0 = 4, f_1 = 4$, $h(C_2, t) = (t-1)^2 + 4(t-1) + 4 = t^2 + 2t + 1$, and $g(C_2, t) = 1 + t$. In fact, $X_{C_2} = \mathbb{P}^1 \times \mathbb{P}^1$.
Similarly, for the octahedron $O_3$, convex envelope of $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$, we have $f_0 = 6, f_1 = 12, f_2 = 8$, $h(O_3, t) = t^3 + 3t^2 + 3t + 1$ and $g(O_3, t) = 2t + 1$. This is in accordance with the Betti numbers of $X_{O_3} = (\mathbb{P}^1)^3$.

If the polytope is not simplicial, so that the toric variety is not $\mathbb{Q}$-smooth, neither Poincaré Duality, nor the Hard Lefschetz Theorem necessarily hold for the cohomology groups. They hold for the Intersection Cohomology groups. One is led to look for a "generalized" $h$-polynomial $h(P, t) = \sum_0^d h_k(P) t^k$, where $h_k(P) := \dim IH^{2k}(X_P, \mathbb{Q})$. A priori, it is not clear if such a polynomial is a combinatorial invariant, i.e. that it can be defined only in terms of the partially ordered set of faces of the polytope $P$. Remarkably, this turns out to be true. In contrast, the cohomology of a singular toric variety is not a purely combinatorial invariant, but depends also on some geometric data of the polytope, e.g. the measures of the angles between the faces of the polytope.

We now give the combinatorial definitions of the $h$ and $g$-polynomial for a not necessarily simplicial polytope.

**Definition 2.5** Suppose $P$ is a polytope of dimension $d$ and that the polynomials $g(Q, t)$ and $h(P, t)$ have been defined for all convex polytopes $Q$ of dimension less than $d$. We set

$$h(P, t) = \sum_{F < P} g(F, t)(t-1)^{d-1-\dim F},$$

where the sum is extended to all proper faces of $P$ including the empty face $\emptyset$, for which $g(\emptyset, t) = h(\emptyset, t) = 1$ and $\dim \emptyset = -1$. The polynomial $g(P, t)$ is defined from $h(P, t)$ as in (10).

We note that these definitions coincide with the previous ones given in (9) and (10) if $P$ is simplicial, since $g(\Sigma, t) = 1$; see Ex. 2.4.

**Example 2.6** Let $C_i$ be the $i$-dimensional cube. For $i > 2$ it is not simplicial, and the $k$-dimensional faces of $C_i$ are $C_k$. We compute the $h$-polynomial of $C_3$. There are 8 faces of dimension 0 and 12 faces of dimension 1 which are of course simplicial; there are 6 faces of dimension 2, for which we have already computed $g(C_2, t) = 1 + t$. It follows that $h(C_3, t) = (t-1)^3 + 8(t-1)^2 + 12(t-1) + 6(1+t) = 1 + 5t + 5t^2 + t^3$ and $g(C_3, t) = 1 + 4t$.

We compute $h(C_4,t)$ : there are 16 faces of dimension 0, 32 faces of dimension 1, which are all simplicial, 24 faces of dimension 2, which are equal to $C_2$, and finally 8 faces of dimension 3, which are equal to $C_3$. We have $h(C_4,t) = (t-1)^4 + 16(t-1)^3 + 32(t-1)^2 + 24(1+t)(t-1) + 8(1+4t) = t^4 + 12t^3 + 14t^2 + 12t + 1$.
In these examples one sees that the $h$-polynomials verify Corollary 2.3.

In fact, we have the following

**Theorem 2.7** ([71]) *Let $P$ be a rational polytope. Then*

$$h(P,t) = \sum_{F<P} g(F,t)(t-1)^{d-1-\mathrm{dimF}} = \sum \dim IH^{2k}(X_P,\mathbb{Q})t^k.$$

In particular, by Poincaré Duality and the Hard Lefschetz Theorem for Intersection Cohomology, the polynomial $h(P,t)$ satisfies the conclusions of Corollary 2.3. The Hodge-Riemann Bilinear Relations for intersection cohomology discussed in §4.3 can be used to obtain further information on the polytope.

We deduce Theorem 2.7 for the dimension of the intersection cohomology groups of a toric variety on two examples by exploiting the Decomposition Theorem for a resolution. A sketch of the general proof along these lines has been given by R. MacPherson in several talks in 1982. J.Bernstein and A.Khovanskii also developed proofs which were never published.

Given a subdivision $\widetilde{P}$ of the polytope $P$, there is a corresponding map $X_{\widetilde{P}} \to X_P$. The toric orbits of $X_P$ provide a stratification for $f$. The fibers over toric orbits are in general unions of toric varieties glued along toric subvarieties. The properties of the fibers over the various orbits can be read from the combinatorics of the subdivision, See [96] for a thorough discussion.

It is well known (cf. see [74], 2.6) that any polytope becomes simplicial after a sequence of subdivisions. We consider the examples of $C_3$ and $C_4$. We know their $h$-polynomials from Example 2.6.

**Example 2.8** The 3-dimensional cube $C_3$ has a simplicial subdivision $C_3'$ which does not add any vertex, and divides every two-dimensional face into two simplices by adding its diagonal, see the picture in [74], p.50. The resulting map $f : X_{C_3'} \to X_{C_3}$ is an isomorphism outside the six singular points of $X_{C_3}$, and the fibers over this points are isomorphic to $\mathbb{P}^1$. The $f$-vector of $C_3'$ has $f_0 = 8$, $f_1 = 18$ and $f_2 = 12$ and $h$-polynomial $h(C_3',t) = t^3 + 5t^2 + 5t + 1$ which equals the $h$-polynomial $h(C_3,t)$ computed in 2.6. This equality reflects the fact that $f$ is a small resolution in the sense of 2.13, so that $H^i(X_{C_3'}) = IH^i(X_{C_3})$.

**Example 2.9** We discuss the Decomposition theorem for the map $f : X_{\widetilde{C_3}} \to X_{C_3}$ where $\widetilde{C_3}$ is obtained by the following decomposition of $C_3$: for each of the six two-dimensional faces $F_i$, we add its barycenter $P_{F_i}$ as a new vertex, and we join $P_{F_i}$ with each vertex of $F_i$. We obtain in this way a simplicial polytope $\widetilde{C_3}$ with 14 vertices, 36 edges and 24 two-dimensional simplices. Its $h$-polynomial is $h(\widetilde{C_3}, t) = t^3 + 11t^2 + 11t + 1$. The map $f$ is an isomorphism away from the six points $p_1, \ldots, p_6$ corresponding to the two-dimensional faces of $C_3$. The fibers $D_i$ over each point $p_i$ is the toric variety corresponding to $C_2$, i.e. $\mathbb{P}^1 \times \mathbb{P}^1$, in particular $H^4(D_i) = \mathbb{Q}$, and ${}^{\mathfrak{p}}\mathcal{H}^{\pm 1}(f_* \mathbb{Q}_{X_{\widetilde{C_3}}}[3]) \simeq \oplus \mathbb{Q}_{p_i}$. The Decomposition Theorem for $f$ reads as follows:

$$f_* \mathbb{Q}_{X_{\widetilde{C_3}}}[3] \ \simeq \ IC_{C_3} \ \oplus \ (\oplus_i \mathbb{Q}_{p_i}[1]) \ \oplus \ (\oplus_i \mathbb{Q}_{p_i}[-1])$$

and

$$H^l(X_{\widetilde{C_3}}) \simeq IH^l(X_{C_3}) \text{ for } l \neq 2, 4, \qquad \dim H^l(X_{\widetilde{C_3}}) = \dim IH^l(X_{C_3}) + 6 \text{ for } l = 2, 4.$$

It follows that $\sum \dim IH^{2k}(X_{C_3}) t^k = \sum \dim H^{2k}(X_{\widetilde{C_3}}) t^k - 6t - 6t^2 = h(\widetilde{C_3}, t) - 6t - 6t^2 = t^3 + 5t^2 + 5t + 1 = h(C_3, t)$, as already computed in Examples 2.6 and 2.8.

**Example 2.10** We consider the four-dimensional cube $C_4$. We subdivide it by adding as new vertices the barycenters of the 8 three-dimensional faces and of the 24 two-dimensional faces. It is not hard to see that the resulting simplicial polytope $\widetilde{C_4}$ has $f$-vector $(48, 240, 384, 192)$ and $h(\widetilde{C_4}, t) = t^4 + 44t^3 + 102t^2 + 44t + 1$. The geometry of the map $f : X_{\widetilde{C_4}} \to X_{C_4}$ which is relevant to the Decomposition Theorem is the following. The 24 two-dimensional faces correspond to rational curves $\overline{O}_i$, closures of one-dimensional orbits $O_i$, along which the map $f$ is locally trivial and looks, on a normal slice, just as the map $X_{\widetilde{C_3}} \to X_{C_3}$ examined in the example above. The fiber over each of the 8 points $p_i$ corresponding to the three-dimensional faces is isomorphic to $X_{\widetilde{C_3}}$. Each point $p_i$ is the intersection of the six rational curves $\overline{O}_{i_j}$ corresponding to the six faces of the three-dimensional cube associated with $p_i$. The last crucial piece of information is that *the local systems arising in the Decomposition Theorem are in fact trivial.* Roughly speaking, this follows from the fact that the fibers of the map $f$ along a fixed orbit depend only on the combinatorics of the subdivision of the corresponding face. We thus have ${}^{\mathfrak{p}}\mathcal{H}^{\pm 1}(f_* \mathbb{Q}_{X_{\widetilde{C_4}}}[4])_{|O_i} \simeq \oplus_i \mathbb{Q}_{O_i}[1]$ and ${}^{\mathfrak{p}}\mathcal{H}^{\pm 2}(f_* \mathbb{Q}_{X_{\widetilde{C_4}}}[4]) \simeq \oplus_i H^6(f^{-1}(p_i)) \simeq \oplus_i H^6(\widetilde{C_3})_{p_i} \simeq \oplus_i \mathbb{Q}_{p_i}$. The Decomposition Theorem reads:

$$f_* \mathbb{Q}_{X_{\widetilde{C_4}}}[4] \ \simeq \ IC_{C_4} \ \oplus \ (\oplus_i V_{p_i}) \ \oplus \ (\oplus_i (IC_{\overline{O}_i}[1] \oplus IC_{\overline{O}_i}[-1])) \ \oplus \ (\oplus_i (\mathbb{Q}_{p_i}[2] \oplus \mathbb{Q}_{p_i}[-2])).$$

The vector spaces $V_{p_i}$ are subspaces of $H^4(f^{-1}(p_i))$, and contribute to the zero perversity term ${}^{\mathfrak{p}}\mathcal{H}^0(f_* \mathbb{Q}_{X_{\widetilde{C_4}}}[4])$. In order to determine their dimension, we compute the stalk

$$\mathcal{H}^0(f_* \mathbb{Q}_{X_{\widetilde{C_4}}}[4])_{p_i} = H^4(f^{-1}(p_i)) = H^4(\widetilde{C_3}).$$

As we already computed in 2.9, $\dim H^4(\widetilde{C_3}) = 11$. By the support condition $\mathcal{H}^0(\mathcal{IC}_{C_4}) = 0$ and, since $\mathcal{IC}_{\overline{O}_i} = \mathbb{Q}_{\overline{O}_i}[1]$, we get

$$11 = \dim \mathcal{H}^0(f_* \mathbb{Q}_{X_{\widetilde{C_4}}}[4])_{p_i} = \dim V_{p_i} \oplus (\oplus_{\overline{O}_j \ni p_i} \mathcal{H}^{-1}(\mathcal{IC}_{\overline{O}_j})) = \dim V_{p_i} + 6,$$

since only six curves $\overline{O}_j$ pass through $p_i$. Hence $\dim V_{p_i} = 5$ and finally

$$f_*\mathbb{Q}_{X_{\widetilde{C_4}}}[4] \simeq \mathcal{IC}_{C_4} \oplus (\oplus_{i=1}^8(\mathbb{Q}_{p_i}^{\oplus 5} \oplus \mathbb{Q}_{p_i}[2] \oplus \mathbb{Q}_{p_i}[-2]) \oplus (\oplus_{i=1}^{24}(\mathbb{Q}_{\overline{O}_i} \oplus \mathbb{Q}_{\overline{O}_i}[2])).$$

By taking the cohomology we get:
$\sum \dim IH^{2k}(X_{C_4})t^k = \sum \dim H^{2k}(X_{\widetilde{C_4}})t^k - 8(t+5t^2+t^3) - 24(t+2t^2+t^3) = t^4+44t^3+102t^2+44t+1-8(t+5t^2+t^3)-24(t+2t^2+t^3) = t^4+12t^3+14t^2+12t+1 = h(C_4,t)$, as computed in 2.6.

The formula for the generalized $h$-polynomial makes perfect sense also in the case that the polytope is not rational, in which case there is no toric variety associated with it. It is thus natural to ask whether the properties of the $h$-polynomial reflecting the Poincaré duality and the Hard Lefschetz theorem hold more generally for any polytope.

In order to study this sort of questions, P. Bressler and V. Lunts have developed a theory of sheaves on the poset associated with the polytope $P$, or more generally to a fan, see [23]. Passing to the corresponding derived category, they define an Intersection Cohomology complex and prove the analogue of the Decomposition Theorem for it, as well as the equivariant version.

By building on their foundational work, K. Karu, proved in [106] that the Hard Lefschetz property and the Hodge-Riemann Bilinear Relations hold for every, i.e. not necessarily rational, polytope. Different proofs, each one shedding new light on interesting combinatorial phenomena, have then been given by Bressler-Lunts in [24] and by Barthel-Brasselet-Fieseler-Kaup in [2].

Another example of application of methods of intersection cohomology to the combinatorics of polytopes is the solution, due to T. Braden and R. MacPherson of a conjecture of G. Kalai concerning the behavior of the $g$-polynomial of a face with respect to the $g$-polynomial of the whole polytope. See [21] and the survey [20].

## 2.2 Semismall maps.

Semismall maps occupy a very special place in the applications of the theory of perverse sheaves to geometric representation theory. Surprisingly, many maps which arise naturally from Lie-theoretic objects are semismall. In a sense which try to illustrate in the discussion of the examples below, the semismallness of a map is related to the semisimplicity of the algebraic object under consideration. We limit ourselves to proper and surjective semismall maps with a nonsingular domain.

We will call a *stratification* for $f$ a decomposition of $Y$ into finitely many locally closed nonsingular subsets such that $f^{-1}(S_k) \to S_k$ is a topologically locally trivial fibration. The subsets $S_k$ are called *strata.*

The following easy observation makes perverse sheaves enter this picture.

**Proposition 2.11** *Let $X$ be a connected nonsingular $n$-dimensional variety, and $f : X \to Y$ be a proper surjective map of varieties. Let $Y = \coprod_{k=0}^n S_k$ be a stratification for $f$. Let $y_k \in S_k$ and set $d_k := \dim f^{-1}(y_k) = \dim f^{-1}(S_k) - \dim S_k$. The following are equivalent:*

*(1)* $f_*\mathbb{Q}_X[n]$ *is a perverse sheaf on* $Y$*;*
*(2)* $\dim X \times_Y X \leq n$*;*
*(3)* $\dim S_k + 2d_k \leq \dim X$*, for every* $k = 0, \ldots, n$*.*

*Sketch of proof.* The equivalence of (2) and (3) is clear. (3) is equivalent to the conditions of support for (1), which being self-dual it then also satisfies the conditions of co-support.

□

**Definition 2.12** A proper and surjective map $f$ satisfying one of the equivalent properties in Proposition 2.11 is said to be semismall.

A semismall map $f : X \to Y$ must be finite over an open dense stratum in $Y$ in view of property (3). Hence, semismall maps are generically finite. The converse is not true, e.g. the blowing-up of a point in $\mathbb{C}^3$.

**Remark 2.13** If the stronger inequalities $\dim S_k + 2d_k < \dim X$ is required to hold for every non-dense subset, then the map is said to be small. In this case, $f_*\mathbb{Q}_X[n]$ satisfies the support and co-support conditions for intersection cohomology ( 20,21 of §3.1,). Hence, if $Y_o \subseteq Y$ denotes a nonsingular dense open subset over which $f$ is a covering, then we have that $f_*\mathbb{Q}_X[n] = IC_Y(L)$, where $L$ is the local system $f_*\mathbb{Q}_{X_{|Y_o}}$.

Before considering the special features of the Decomposition theorem for semismall maps, we give some examples.

**Example 2.14** Surjective maps between surfaces are always semismall. A surjective map of threefolds is semismall iff no divisor $D \subseteq X$ is contracted to a point on $Y$.

A great wealth of examples of semismall maps is furnished by contractions on (holomorphic) symplectic varieties, which we now describe. A nonsingular quasi-projective complex variety is called symplectic if there is a 2-form $\omega \in \Gamma(X, \Omega^2_X)$ which is closed and nondegenerate, that is $d\omega = 0$, and $\omega^{\frac{\dim X}{2}}$ does not vanish at any point. The following is proved in [104]:

**Theorem 2.15** *Let* $X$ *be a quasi-projective symplectic variety, and* $f : X \to Y$ *a generically finite proper map. Then* $f$ *is semismall.*

**Example 2.16** (**The Hilbert scheme of points on a surface**) (See [137]). Let $X = (\mathbb{C}^2)^{[n]}$ be the Hilbert scheme of $\mathbb{C}^2$. Its points parameterize subschemes $Z$ of length $n$ in $\mathbb{C}^2$ or, equivalently, quotient rings $\mathbb{C}[X,Y]/I = \Gamma(Z, \mathcal{O})$ such that $\dim_{\mathbb{C}} \mathbb{C}[X,Y]/I = n$. The Artinian ring $\mathbb{C}[X,Y]/I$ is the product of local Artinian rings $\mathbb{C}[X,Y]/I_k$ associated with points $x_k \in \mathbb{C}^2$. Set $n_k = \dim_{\mathbb{C}} \mathbb{C}[X,Y]/I_k$. Then $n = \sum_k n_k$. The 0-cycle $|Z| := \sum_k n_k x_k$ is called the support of the subscheme $Z$. It is a point in the symmetric product $(\mathbb{C}^2)^{(n)} = (\mathbb{C}^2)^n/\mathcal{S}_n$. The map $\pi : (\mathbb{C}^2)^{[n]} \to (\mathbb{C}^2)^{(n)}$, associating with $Z$ its support $|Z|$, is well defined and proper. It is an isomorphism precisely on the set $(\mathbb{C}^2)^{(n)}_{reg}$ corresponding to cycles $x_1 +$

$\ldots + x_n$ consisting of $n$ distinct points. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be coordinates on $(\mathbb{C}^2)^n$. The form $\sum_k dx_k \wedge dy_k$ on $(\mathbb{C}^2)^n$ is $\mathcal{S}_n$-invariant and descends to a closed and nondegenerate form on $(\mathbb{C}^2)^{(n)}_{reg}$. A local computation shows that its pullback by $\pi$ extends to a symplectic form on $(\mathbb{C}^2)^{[n]}$. In particular $\pi$ is semismall (this can be also verified directly). The subvariety $(\mathbb{C}^2)^{[n]}_0$ of subschemes supported at 0 is called the punctual Hilbert scheme of length $n$. Its points are the $n$-dimensional quotient rings of $\mathbb{C}[X,Y]/(X,Y)^{n+1}$. Its geometry has been studied in depth, see [98], [25]. It is irreducible, of dimension $n-1$, and has a decomposition in affine spaces. Clearly, $(\mathbb{C}^2)^{[n]}_0 \simeq (\pi^{-1}(nx))_{red}$, for every $x \in \mathbb{C}^2$. Similarly, if $|Z| := \sum_k n_k x_k$ with $x_i, \neq x_j$ for all $i \neq j$, then $(\pi^{-1}(|Z|))_{red} \simeq \prod_i (\mathbb{C}^2)^{[n_i]}_0$. The construction can be globalized, in the sense that, for any nonsingular surface $S$, the Hilbert scheme $S^{[n]}$ is nonsingular and there is a map $\pi : S^{[n]} \to S^{(n)}$ which is semismall, and locally, in the analytic topology, isomorphic to $\pi : (\mathbb{C}^2)^{[n]} \to (\mathbb{C}^2)^{(n)}$. There also exists a version of $S^{[n]}$ for a symplectic manifold $S$ of real dimension four, which was defined and investigated by C.Voisin in [163]. Two related examples, still admitting a semismall contraction, are the nested Hilbert scheme $S^{[n,n+1]}$, whose points are couples $(Z, Z') \in S^{[n]} \times S^{[n+1]}$ such that $Z \subseteq Z'$, and the parabolic Hilbert scheme, see [49] and its Appendix for details.

**Example 2.17** (**The Nilpotent cone resolution**) (See [36], [154]). Let $G$ be a semisimple connected linear algebraic group with Lie algebra $\mathfrak{g}$, $T$ be a maximal torus, and $B$ be a Borel subgroup containing $T$. The cotangent space of the associated flag variety $\widetilde{\mathcal{N}} := T^*G/B$ is endowed with a canonical (exact) symplectic form. We recall that an element $x \in \mathfrak{g}$ is nilpotent if the endomorphism $[x, -] : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Let $\mathcal{N} \subseteq \mathfrak{g}$ the cone of nilpotent elements of $\mathfrak{g}$. It can easily be shown, see [36], that

$$\widetilde{\mathcal{N}} = \{(x, \mathfrak{b}) : \mathfrak{b} \text{ is a Borel subalgebra of } \mathfrak{g} \text{ and } x \in \mathcal{N} \cap \mathfrak{b}.\}$$

The map $p : \widetilde{\mathcal{N}} \to \mathcal{N} \subseteq \mathfrak{g}$, defined as $p(x, \mathfrak{b}) = x$, is surjective, since every nilpotent element is contained in a Borel subalgebra, generically one-to-one, since a generic nilpotent element is contained in exactly one Borel subalgebra, proper, since $G/B$ is complete and semismall. The map $p$ is called the Springer resolution. For example, if $G = SL_2$, the flag variety $G/B = \mathbb{P}^1$ and the cotangent space is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$. The contraction of its zero-section is isomorphic to the cone $z^2 = xy$ in $\mathbb{C}^3$. If $H, X, Y$ denotes the usual basis of $\mathfrak{sl}_2$, the matrix $zH + xX - yY$ is nilpotent precisely when $z^2 = xy$.

**Example 2.18** (**Quiver varieties**) See [138] for more details. A host of symplectic varieties and semismall maps is furnished by the quiver construction. It is a variation of the Atiyah-Drinfeld-Hitchin-Manin construction of instantons. We only sketch the basic idea of the construction in the algebraic category. We start with a graph $\Gamma$ without loops. An oriented edge is an edge $e$ plus an ordering $(out(e), in(e))$ of its two vertices. Let $V$ be the set of vertices and $H$ be the set of oriented edges. If $h \in H$, denote by $\overline{h}$ the same edge with opposite orientation. It is possible to choose a subset $\Omega \subseteq H$ in such a way that $\Omega \cap \overline{\Omega} = \emptyset, \Omega \cup \overline{\Omega} = H$, and there is no sequence $h_1, \ldots, h_m \in \Omega$ such that

$in(h_i) = out(h_{i+1})$ for $1 \le i \le m-1$, and $in(h_m) = out(h_1)$. Associate with each vertex $v$ a pair of complex vector spaces $V_v, W_v$. Set

$$M := (\bigoplus_{h\in H} \mathrm{Hom}(V_{out(h)}, V_{in(h)})) \bigoplus (\bigoplus_{v\in V} \mathrm{Hom}(V_v, W_v) \oplus \mathrm{Hom}(W_v, V_v)).$$

An element in $M$ is denoted $(B_h, i_v, j_v)$. The alternating bilinear form $\omega$ on $M$, defined by: $\omega((B_h, i_v, j_v), (B'_h, i'_v, j'_v)) = \sum_{h\in H} \mathrm{Tr}(\epsilon(h) B_h B'_h) + \sum_{v \in V} \mathrm{Tr}(i_v j'_v - i'_v j_v))$, with $\epsilon(h) = \pm 1$ according to $h \in \Omega$ or $h \in \overline{\Omega}$, endows $M$ with the structure of a holomorphic symplectic variety which can be identified with the cotangent space of

$$M_\Omega := (\bigoplus_{h\in\Omega} \mathrm{Hom}(V_{out(h)}, V_{in(h)}) \bigoplus (\bigoplus_{v\in V} \mathrm{Hom}(W_v, V_v))$$

with its canonical symplectic structure. Let $G := \prod GL(V_v)$. There is an Hamiltonian action of the group $G := \prod GL(V_v)$ on $M$, defined by

$$(g_v)((B_h, i_v, j_v) = (g_{in(h)} B_h g^{-1}_{out(h)}, g_v i_v, j_v g_v^{-1}),$$

with moment map $\mu : M \to \mathfrak{g} = \oplus \mathfrak{gl}(V_v)$. Set $\mathfrak{X} := \mu^{-1}(0)$, which clearly is an affine $G$-invariant subvariety of $M$, and denote by $\mathbb{C}[\mathfrak{X}]$ its coordinate ring. Every character $\chi : G \to \mathbb{C}^*$ defines a graded ring of semi-invariants: $\mathbb{C}[\mathfrak{X}]^\chi_n := \{f \in \mathbb{C}[\mathfrak{X}] : gf = \chi(g)^n f\}$. The graded ring $\mathbb{C}[\mathfrak{X}]^\chi := \oplus_{n\ge 0} \mathbb{C}[\mathfrak{X}]^\chi_n$ is a $\mathbb{C}[\mathfrak{X}]^G$-algebra, and we set

$$\mathfrak{M}^\chi := \mathrm{Proj}(\mathbb{C}[\mathfrak{X}]^\chi), \qquad \mathfrak{M}^0 := \mathrm{Spec}(\mathbb{C}[\mathfrak{X}]^G),$$

so that there is a map $\pi : \mathfrak{M}^\chi \to \mathfrak{M}^0$. The conditions under which $\mathfrak{M}^\chi$ is a nonempty holomorphic symplectic nonsingular variety and the map $\pi$ is a semismall resolution can be made explicit. The construction of the quiver varieties $\mathfrak{M}$ recovers many previously known examples, such as 1) the resolution of "DuVal singularities," i.e. of quotients $\mathbb{C}^2/\Gamma$, with $\Gamma$ a finite subgroups of $SU(2)$, 2) the moduli spaces of instantons on them, and 3) the cotangent space of the flag varieties of type $A$ with their map on the nilpotent cone, as in Example 2.17, as well as slices to the strata of this map and their resolutions . A slight variation, i.e. with a graph with one loop, gives the Hilbert scheme $\mathbb{C}^{2[n]}$.

With the graph $\Gamma$ is associated a Cartan matrix and therefore, by the Serre relations, a Kac-Moody algebra. Inspired by previous work of Lusztig, [121], Nakajima exploited the endomorphism algebra, as described in §2.2.2, associated with contractions of quivers with graph $\Gamma$, to give a geometric construction of representations of Kac-Moody algebras, see [138].

**Remark 2.19** (**Smoothing and nearby cycle for semismall resolutions**) The semismall maps $f : X \to Y$ of Examples 2.16 (for $S = \mathbb{C}^2$), 2.17 and 2.18 have a further peculiar property: there exists a *smoothing* $\phi : \mathcal{Y} \to U \subseteq \mathbb{C}^k$, such that $\phi^{-1}(0) = Y$ and $\phi^{-1}(t) = Y_t$ is, for generic $t$, nonsingular and diffeomorphic to the resolution $X$. By Remark 3.22, there is a continuous retraction map $r : Y_t \to Y$, and

$$\Psi(\mathbb{Q}_{Y_t}[n]) = r_* \mathbb{Q}_{Y_t}[n] \simeq f_* \mathbb{Q}_X[n].$$

The smoothing of Example 2.16 has been explicitly identified and related to interesting phenomena in the theory of integrable systems by G.Wilson in [166]. The one in 2.17, which we discuss below in Ex. 2.20 in the simplest case where $G = SL_2$, plays a big role in the geometric description of the group algebra $\mathbb{Q}[W]$, and is discussed in §2.4. The construction of quivers comes with a smoothing, which corresponds to a change of linearization in the GIT quotient.

**Example 2.20** In the case discussed at the end of Example 2.17, one considers the family of affine quadrics $Y_t \subseteq \mathbb{C}^3$ of equation $z^2 = xy+t$ for $t \in \mathbb{C}$. It is well known that, for $t \neq 0$, $Y_t$ is diffeomorphic, but not isomorphic, to $T^*\mathbb{P}^1$, and that after the base change $t \to t^2$, the family $z^2 = xy + t^2$ admits a simultaneous (small) resolution, whose fibre at $t = 0$ is the map $T^*\mathbb{P}^1 \to Y_0$. The generalization of this construction is at the heart of the Springer correspondence, which we describe in §2.4. The role of the Galois group $\mathbb{Z}/2 = \mathcal{S}_2$ of the covering $t \to t^2$ will be played by the Weyl group of $G$.

We now investigate two specific features of the Decomposition Theorem for semismall maps, namely, the determination of the strata and local systems, and the algebraic properties of the endomorphism algebra. We will consider in more detail the consequences of the Decomposition Theorem for Examples 2.16 and 2.17 in the following sections.

### 2.2.1 Semismall maps: strata and local systems

In the case of a semismall map, it is possible to identify precisely the strata $S_k$ contributing to the Decomposition Theorem with a non trivial summand $IC_{\overline{S}_k}(L_k)$ as well as the local systems $L_k$, which turn out to have finite monodromy.

**Definition 2.21** Let $X, Y, S_k$ and $d_k$ be as in Proposition 2.11. A stratum $S_k$ is said to be *relevant* if $\dim S_k + 2d_k = \dim X$.

Since $\dim Y = \dim X$, a relevant stratum has even codimension. Let $S_k$ be a relevant stratum, and $y_k \in S_k$. Let $\Sigma$ be a local trasversal slice to $S_k$ at $y_k$, given for example by intersecting a small ball at $y_k$ with the complete intersection of $\dim S_k$ general hyperplane sections in $Y$ passing through $y_k$. The restriction $f_| : f^{-1}(\Sigma) \to \Sigma$ is still semismall and $d_k = \dim f^{-1}(y_k) = (1/2)\dim f^{-1}(\Sigma)$. The following chain of maps:

$$H_{2d_k}(f^{-1}(y_k)) = H^{BM}_{2d_k}(f^{-1}(y_k)) \to H^{BM}_{2d_k}(f^{-1}(\Sigma)) \simeq H^{2d_k}(f^{-1}(\Sigma)) \to H^{2d_k}(f^{-1}(y_k)),$$

where the first map is the push-forward with respect to a closed inclusion and the second is the restriction, defines the refined intersection pairing (cf. §5.2.1) associated with the relevant stratum $S_k$

$$I_k : H_{2d_k}(f^{-1}(y_k)) \times H_{2d_k}(f^{-1}(y_k)) \longrightarrow \mathbb{Q}.$$

A basis of $H_{2d_k}(f^{-1}(y_k))$ is given by the classes of the $d_k$-dimensional irreducible components $E_1, \ldots, E_l$ of $f^{-1}(y_k)$. The intersection pairing $I_k$ is then represented by the intersection matrix $(E_i \cdot E_j)$ of these components, computed in $f^{-1}(\Sigma)$.

Let $U = \coprod_{l>k} S_l$ and $U' = U \cup S_k$. Denote by $i : S_k \to U' \longleftarrow U : j$ the corresponding imbeddings. The refined intersection map $H_{2d_k}(f^{-1}(y_k)) \to H^{2d_k}(f^{-1}(y_k))$ is then

$$\mathcal{H}^{-\dim S_k}(i^! f_* \mathbb{Q}_{U'}[n]) \to \mathcal{H}^{-\dim S_k}(i^* f_* \mathbb{Q}_{U'}[n]).$$

By Remark 3.31, the non-degeneracy of $I_k$ is equivalent to the existence of a unique isomorphism:

$$f_* \mathbb{Q}_{U'}[n] \simeq j_{!*} f_* \mathbb{Q}_{U'}[n] \oplus \mathcal{H}^{-\dim S_k}(i^! f_* \mathbb{Q}_{U'}[n])[\dim S_k]. \tag{11}$$

The Decomposition Theorem is equivalent to the fact that the intersection forms $I_k$ are nondegenerate.

**Remark 2.22** The Hodge Theoretic version of the Decomposition Theorem gives a more precise statement: the $I_k$ *are definite bilinear forms,* positive if $d_k$ is even, negative if it is odd.

We denote by $I_{rel}$ the set of relevant strata. For $k \in I_{rel}$, let $y_k \in S_k$ and let $E^k_1, \dots, E^k_{l_k}$ be the irreducible $d_k$-dimensional components of $f^{-1}(y_k)$. The monodromy of the $E^k_i$'s defines a group homomorphism $\rho_k : \pi_1(S_k, y_k) \to \mathcal{S}_{l_k}$ and, correspondingly, a $\mathbb{Q}$ local system $L_k$. The semisimplicity of $L_k$ in this case follows immediately from the fact that the monodromy factors through a finite group. Denoting by $Irr(\pi_1(S_k))$ the set of irreducible representations of $\pi_1(S_k, y_k)$, we have an isotypical decomposition $L_k = \oplus_{\chi \in Irr(\pi_1(S_k))} L^\chi_k$. The local systems $L^\chi_k$ are the tensor product of the irreducible local system $L^\chi$ associated with the representation $\chi$ with a vector space $V^\chi_k$, whose dimension is the multiplicity of the representation $\chi$ in $\rho_k$. With this notation, let us give the statement of the Decomposition theorem in the case of semismall maps:

**Theorem 2.23** *There is a canonical isomorphism in $\mathcal{P}_Y$ :*

$$f_* \mathbb{Q}_X[n] \simeq \bigoplus_{k \in I_{rel}} IC_{\overline{S}_k}(L_k) = \bigoplus_{\substack{k \in I_{rel} \\ \chi \in Irr(\pi_1(S_k))}} IC_{\overline{S}_k}(L^\chi_k) \simeq \bigoplus_{\substack{k \in I_{rel} \\ \chi \in Irr(\pi_1(S_k))}} IC_{\overline{S}_k}(L^\chi) \otimes V^\chi_k. \tag{12}$$

*Sketch of Proof.* By(11), only relevant strata give non trivial contributions to the Decomposition Theorem. The stalk of the local system associated with one such stratum $S_k$ at the point $y_k$ is $H_{2d_k}(f^{-1}(y_k))$ A basis for this is given by the $d_k$-dimensional irreducible components $E_1, \dots, E_l$. In particular, the local system has finite monodromy. □

### 2.2.2 Semismall maps: the semisimplicity of the endomorphism algebra

Since $\mathcal{D}_Y$ is an additive category, the $\mathbb{Q}$-vector space $\mathrm{End}_{\mathcal{D}_Y}(f_* \mathbb{Q}_X[n])$ is endowed naturally with an algebra structure. We show that if $f$ is a semismall map, then this algebra is semisimple; see [36] and [37] for details.

Let $f : X \to Y$ be any proper map with $X$ nonsingular. Let $p_{ij} : X \times_Y X \times_Y X \to X \times_Y X$ denote the projection on the $ij$-factor. For $Z, Z' \in H^{BM}_{2n}(X \times_Y X)$, the composition $Z \circ Z' := p_{13*}(p^*_{12}(Z) \cap p^*_{23}(Z))$, where the notation $\cap$ denotes the refined intersection product in $X \times_Y X$, defines an algebra structure on $H^{BM}_{2n}(X \times_Y X)$. By Verdier Duality, there is an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathrm{End}_{\mathcal{D}_Y}(f_* \mathbb{Q}_X[n]) \simeq H^{BM}_{2n}(X \times_Y X).$$

It can be proved that this is in fact an isomorphism of algebras, see [37], Lemma 2.23.

Let $f$ be a semismall. Example 3.21 implies that

$$\mathrm{Hom}_{\mathcal{D}_Y}(IC_{\overline{S}_k}(L_k), IC_{\overline{S}_l}(L_l)) = 0 \text{ if } k \neq l \quad \text{ and } \quad \mathrm{End}_{\mathcal{D}_Y}(IC_{\overline{S}_k}(L_k)) = \mathrm{End}(L_k).$$

Schur's Lemma implies that

$$\mathrm{End}(L_k) = \bigoplus_{\chi \in Irr(\pi_1(S_k))} \mathrm{End}(L_k^\chi) \quad \simeq \bigoplus_{\chi \in Irr(\pi_1(S_k))} \mathrm{End}(V^k_\chi)$$

is a product of matrix algebras. It follows that

$$H^{BM}_{2n}(X \times_Y X) \simeq \mathrm{End}_{\mathcal{D}_Y}(f_* \mathbb{Q}_X[n]) \simeq \bigoplus_{k \in I_{rel}} \mathrm{End}_{\mathcal{D}_Y}((IC_{\overline{S}_k}(L_k)) \simeq \bigoplus_{\substack{k \in I_{rel} \\ \chi \in Irr(\pi_1(S_k))}} \mathrm{End}(V^\chi_k) \tag{13}$$

*is a semisimple algebra.*

This algebra contains in particular the idempotents giving the projection of $f_* \mathbb{Q}_X[n]$ on the irreducible summand of the canonical decomposition (12). Since, again by semismallness, $H^{BM}_{2n}(X \times_Y X)$ is the top dimensional Borel Moore homology, it is generated by the irreducible components of $X \times_Y X$. The projectors are therefore realized by algebraic correspondences.

This has been pursued in [49], where we prove a "motivic" refinement of the Decomposition Theorem in the case of semismall maps. This in accordance with the general philosophy of [37]. In particular, it is possible to construct a (relative) Chow motive corresponding to the intersection cohomology groups of singular varieties which admit a semismall resolutions.

**Remark 2.24** In the case of the resolution of nilpotent cone (cf. Example 2.17), the variety $X \times_Y X = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ is known as the Steinberg variety. Its representation theoretic relevance has been recognized well-before the Decomposition Theorem; see [156] and [36].

We now investigate the consequences of the Decomposition Theorem in the Examples 2.16 and 2.17.

## 2.3 The Hilbert Scheme of points on a surface.

We resume the notation of Example 2.16, and introduce some notation for partitions of the natural number $n$. We denote by $\mathfrak{P}_n$ the set of such partitions. Let $\nu = (\nu_1, \ldots, \nu_{l(\nu)}) \in \mathfrak{P}_n$, so that $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{l(\nu)}$ and $\sum_i \nu_i = n$. We will also write $\nu = 1^{a_1} 2^{a_2} \ldots n^{a_n}$, with

$\sum ka_k = n$, where $a_i$ is the number of times that the number $i$ appears in the partition $\nu$. Clearly $l(\nu) = \sum a_i$. We consider the following stratification of $S^{(n)}$: for $\nu \in \mathfrak{P}_n$ we set

$$S_{(\nu)} = \{0 - cycles \subseteq S^{(n)} \text{ of type } \nu_1 x_1 + \ldots + \nu_{l(\nu)} x_{l(\nu)} \text{ with } x_i \neq x_j \forall i \neq j\}.$$

Set $S_{[\nu]} = \pi^{-1}(S_{(\nu)})$ (with the reduced structure). Clearly $S_{(\nu)}$ is nonsingular of dimension $2l(\nu)$. It can be shown that $\pi : S_{[\nu]} \to S_{(\nu)}$ is locally trivial with fiber isomorphic to the product $\prod_i (\mathbb{C}^2)_0^{\nu_i}$ of punctual Hilbert schemes. In particular, *the fibers of* $\pi$ *are irreducible,* hence the local systems occurring in (12) are constant of rank one. Furthermore, the closures $\overline{S}_{(\nu)}$ and their desingularization can be explicitly determined. If $\nu$ and $\mu$ are two partitions, we say that $\mu \leq \nu$ if there exists a decomposition $I_1, \ldots, I_{l(\mu)}$ of the set $\{1, \ldots, l(\nu)\}$ such that $\mu_1 = \sum_{i \in I_1} \nu_i, \ldots, \mu_{l(\mu)} = \sum_{i \in I_{l(\mu)}} \nu_i$. Then

$$\overline{S}_{(\nu)} = \coprod_{\mu \leq \nu} S_{(\mu)}.$$

This reflects just the fact that a cycle $\sum \nu_i x_i \in S_{(\nu)}$ can degenerate to a cycle in which some of the $x_i'$s come together. If $\nu = 1^{a_1} 2^{a_2} \ldots n^{a_n}$, we set $S^{(\nu)} = \prod_i S^{(a_i)}$. The variety $S^{(\nu)}$ has dimension $2l(\nu)$, and there is a natural *finite* map $\nu : S^{(\nu)} \to \overline{S}_{(\nu)}$, which is an isomorphism when restricted to $\nu^{-1}(S_{(\nu)})$. Since $S^{(\nu)}$ has only quotient singularities, it is normal, so that $\nu : S^{(\nu)} \to \overline{S}_{(\nu)}$ is the normalization map, and $IC_{\overline{S}_{(\nu)}} = \nu_* \mathbb{Q}_{S^{(\nu)}}[2l(\nu)]$. The Decomposition Theorem (12) for $\pi : S^{[n]} \to S^{(n)}$ gives a canonical isomorphism:

$$\pi_* \mathbb{Q}_{S^{[n]}}[2n] \ \simeq \bigoplus_{\nu \in \mathfrak{P}_n} \nu_* \mathbb{Q}_{S^{(\nu)}}[2l(\nu)]. \tag{14}$$

This explicit form was given by L. Göttsche and W. Soergel in [86] as an application of M. Saito's theorem [144]. Taking cohomology, we find

$$H^i(S^{[n]}, \mathbb{Q}) = \bigoplus_{\nu \in \mathfrak{P}_n} H^{i+2l(\nu)-2n}(S^{(\nu)}, \mathbb{Q}).$$

Since $S^{(n)}$ is the quotient of the nonsingular variety $S^n$ by the finite group $\mathcal{S}_n$, its *rational* cohomology $H^i(S^{(n)}, \mathbb{Q})$ is just the $\mathcal{S}_n$-invariant part of $H^i(S^n, \mathbb{Q})$. In [122], MacDonald determines the dimension of such invariant subspace. Its result is more easily stated in terms of generating function:

$$\sum \dim H^i(S^{(n)}, \mathbb{Q}) t^i q^n = \frac{(1+tq)^{b_1(S)}(1+t^3 q)^{b_3(S)}}{(1-q)^{b_0(S)}(1-t^2 q)^{b_2(S)}(1-t^4 q)^{b_4(S)}}.$$

With the help of this formula we find "Göttsche's Formula" for the generating function of the Betti numbers of the Hilbert scheme:

$$\sum_{i,n} \dim H^i(S^{[n]}, \mathbb{Q}) t^i q^n = \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1(S)}(1+t^{2m+1}q^m)^{b_3(S)}}{(1-t^{2m-2}q^m)^{b_0(S)}(1-t^{2m}q^m)^{b_2(S)}(1-t^{2m+2}q^m)^{b_4(S)}}.$$

**Remark 2.25** Setting $t{=}{-1}$, we get the following simple formula for the generating function for the Euler characteristic:

$$\sum_{n=0}^{\infty} \chi(S^{[n]}) q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{\chi(S)}}.$$

See [39], for a simple derivation of this formula.

Göttsche's Formula appeared first in [85], following some preliminary work in the case $S = \mathbb{C}^2$ by Ellingsrud and Stromme, [68, 69]. The original proof relies on the Weil conjectures, and on a delicate counting of points over a finite field with the help of the cellular structure of the punctual Hilbert scheme following from Ellingsrud and Stromme's results.

Vafa and Witten noticed in [157] that Göttsche's Formula suggests a representation theoretic structure underlying the direct sum $\oplus_{i,n} H^i(S^{[n]})$. Namely, this space should be an irreducible highest weight module over the infinite dimensional Heisenberg-Clifford super Lie algebra, with highest weight vector the generator of $H^0(S^{[0]})$. H. Nakajima and, independently I. Grojnowski took up the suggestion in [139, 89] (see also the lecture notes [137]) and realized this structure by a set of correspondences relating Hilbert schemes of different lengths.

An elementary proof of Göttsche's formula stemming form this circle of ideas was given in [43].

The papers [45, 49] prove, in two different ways, a motivic version of the Decomposition Theorem (14) for the map $\pi : S^{[n]} \to S^{(n)}$ exhibiting an equality

$$(S^{[n]}, \Delta, 2n) = \sum_{\nu \in \mathfrak{P}_n} (S^{l(\nu)}, P_\nu, 2l(\nu))$$

of Chow motives with rational coefficients. In this formula, $P_\nu$ denotes the projector associated with the action of the group $\prod \mathcal{S}_{a_i}$ on $S^{l(\nu)}$.

## 2.4 The Nilpotent Cone and Springer Theory.

We resume the set-up of Example 2.17 relative to the Springer resolution $p : \widetilde{\mathcal{N}} = T^*G/B \to \mathcal{N}$. The nilpotent cone $\mathcal{N}$ has a natural $G$-invariant stratification, given by the orbits of the adjoint action contained in $\mathcal{N}$, i.e. by the conjugacy classes of nilpotent elements. Let $\mathrm{Conj}(\mathcal{N})$ be the set of conjugacy classes of nilpotent elements in $\mathfrak{g}$. For $[x] \in \mathrm{Conj}(\mathcal{N})$, let $x$ be a representative, and denote by $\mathfrak{B}_x := p^{-1}(x)$ the fiber over $x$ and by $S_x = Gx$ the stratum containing $x$.

**Example 2.26** Let $G = SL_n$. Each conjugacy class contains exactly one matrix which is a sum of Jordan matrices, so that the $G$-orbits are parameterized by the partitions of the integer $n$. The open dense stratum of $\mathcal{N}$ corresponds to the Jordan block of length $n$.

It can be proved (cf. [156, 151]), that every stratum $S_x$ is relevant and that all the components of $\mathfrak{B}_x$ have the same dimension $d_x$. Hence we have the local system $L_x$, whose fiber at $x$ is $H_{2d_x}(\mathfrak{B}_x)$. The stabilizer $G_x$ of $x$ acts on $\mathfrak{B}_x$ since the map $p : \widetilde{\mathcal{N}} \to \mathcal{N}$ is $G$-equivariant. By homotopy, this action factors through the finite group $\Gamma_x := G_x/G_x^0$ of the connected components of $G_x$. This action splits the local system $L_x = \oplus_{\chi \in Irr\Gamma_x} L_x^\chi$.

The Decomposition Theorem reads:

$$p_* \mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}] = \bigoplus_{\substack{x \in \mathrm{Conj}(\mathcal{N}), \\ \chi \in Irr(\Gamma_x)}} IC_{\overline{S}_x}(L_x^\chi).$$

By the discussion on the semisimplicity of the endomorphism algebra in §2.2.2,

$$H^{BM}_{\dim \widetilde{\mathcal{N}}}(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}) = \mathrm{End}_{\mathcal{D}_{\mathcal{N}}}(p_* \mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}]) = \bigoplus_{\substack{x \in \mathrm{Conj}(\mathcal{N}), \\ \chi \in Irr(\Gamma_x)}} \mathrm{End}(L_x^\chi).$$

The aim of the Springer correspondence is to get an algebra isomorphism

$$\mathbb{Q}[W] \xrightarrow{\simeq} H^{BM}_{\dim \widetilde{\mathcal{N}}}(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}).$$

Now, we now sketch, following [117] (see also [13, 14]), the construction of an action of the Weyl group $W$ on $p_* \mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}]$. Let us consider the adjoint action of $G$ on $\mathfrak{g}$. By a theorem of Chevalley, there is a map $q : \mathfrak{g} \to \mathfrak{t}/W$ defined as follows:

$$\mathfrak{g} = \mathrm{Spec} \mathbb{C}[\mathfrak{g}*] \to \mathrm{Spec} \mathbb{C}[\mathfrak{g}*]^G \simeq \mathrm{Spec} \mathbb{C}[\mathfrak{t}^*]^W = \mathfrak{t}/W.$$

and $\mathfrak{t}/W$ is an affine space. Let us denote by $\mathfrak{t}^{rs} = \mathfrak{t} \setminus \{\text{root hyperplanes }\}$, the set of regular elements in $\mathfrak{t}$, and by $\mathfrak{g}^{rs}$ the set of regular semisimple elements in $\mathfrak{g}$. The set $\mathfrak{t}^{rs}/W = \mathfrak{t}/W \setminus \Delta$ is the complement of a divisor. We have $\mathfrak{g}^{rs} = q^{-1}(\mathfrak{t}^{rs}/W)$, and the map $q : \mathfrak{g}^{rs} \to \mathfrak{t}^{rs}/W$ is a fibration with fiber $G/T$. There is the monodromy representation $\rho : \pi_1(\mathfrak{t}^{rs}/W) \to \mathrm{Aut}(H^*(G/T))$.

**Example 2.27** Let $G = SL_n$. The map $q$ associates with a zero-trace matrix the coefficients of its characteristic polynomial. The set $\mathfrak{t}^{rs}/W = \mathfrak{t}/W \setminus \Delta$ is the set of polynomials with distinct roots. The statement that the map $q : \mathfrak{g}^{rs} \to \mathfrak{t}^{rs}/W$ is a fibration boils down to the fact that a matrix commuting with a diagonal matrix with distinct eigenvalues must be diagonal, and that the adjoint orbit of such matrix is closed in $\mathfrak{sl}_n$.

The affine variety $G/T$ is diffeomorphic to $\widetilde{\mathcal{N}}$. It turns out that, after a base change by a finite map, the orbits of regular elements, isomorphic to $G/T$, and $\widetilde{\mathcal{N}}$, can be put together in a family, i.e. they appear as fibers over distinct points of a connected base.

Let us define

$$\widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}) : \mathfrak{b} \text{ is a Borel subalgebra of } \mathfrak{g} \text{ and } x \in \mathfrak{b}\}.$$

If $p : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is the projection to the first factor, $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N}) \subseteq \widetilde{\mathfrak{g}}$. Let $\widetilde{\mathfrak{g}}^{rs} = p^{-1}(\mathfrak{g}^{rs})$.

The Weyl group acts simply transitively on the set of Borel subgroups containing a regular semisimple element. This observation leads to the following:

**Proposition 2.28** *The restriction $p' : \widetilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is a Galois covering with group $W$. The map $p : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small.*

We summarize what we have discussed far in the following diagram (the map $r$ will be defined below):

$$\begin{tikzcd}
 & & & \widetilde{\mathcal{N}} \arrow[dl, hook'] \arrow[dd, "p"] & \\
\widetilde{\mathfrak{g}}^{rs} \arrow[rr, hook] \arrow[dd, "p'"'] & & \widetilde{\mathfrak{g}} \arrow[rr] \arrow[dd, "p"'] & & \mathfrak{t} \arrow[dd] \\
 & G/T \arrow[rr, "r"] \arrow[dl, hook'] & & \mathcal{N} \arrow[dl, hook', "i"] & \\
\widetilde{\mathfrak{g}}^{rs}/W = \mathfrak{g}^{rs} \arrow[rr, hook] & & \mathfrak{g} \arrow[rr, "q"] & & \mathfrak{t}/W
\end{tikzcd}$$

Let $L = p'_* \mathbb{Q}_{\widetilde{\mathfrak{g}}^{rs}}$ be the local system associated with the $W$-covering. By its very definition, $L$ is endowed with an action of the Weyl group $W$. By the functoriality of the construction of intersection cohomology, this Weyl group action extends to $IC_{\mathfrak{g}}(L)$. Since $p$ is small, by Remark 2.13, $IC_{\mathfrak{g}}(L) = p_* \mathbb{Q}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}]$. In particular, by Proper Base Change, there is an action of $W$ on $i^* p_* \mathbb{Q}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}] = p_* \mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}]$.

A perhaps more intuitive way to realize this action is the following. We have $\mathcal{N} = q^{-1}(0)$. By Remark 3.22, there is a continuous retraction map $r : G/T \to \mathcal{N}$. Since the affine variety $G/T$ is diffeomorphic to $\widetilde{\mathcal{N}}$, we have an isomorphism:

$$r_* \mathbb{Q}_{G/T}[\dim \widetilde{\mathcal{N}}] \simeq p_* \mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}].$$

As we have already observed, the monodromy of the fibration $q : \mathfrak{g}^{rs} \to \mathfrak{t}^{rs}/W$ gives an action of $\pi_1(\mathfrak{t}^{rs}/W)$ on $r_* \mathbb{Q}_{G/T}[\dim \widetilde{\mathcal{N}}]$. There is an exact sequence of groups:

$$0 \to \pi_1(\mathfrak{t}^{rs}) \to \pi_1(\mathfrak{t}^{rs}/W) \to W \to 0$$

and the existence of the simultaneous resolution $\widetilde{\mathfrak{g}}$ shows that the monodromy factors through an action of $W$.

The above discussion of the endomorphism algebra yields an algebra homomorphism

$$\mathbb{Q}[W] \longrightarrow \mathrm{End}_{\mathcal{D}_{\mathcal{N}}}(p_* \mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}]) \quad = \bigoplus_{[x] \in \mathrm{Conj}(\mathcal{N})} \mathrm{End}(L_x) \quad = \quad H^{BM}_{\dim \widetilde{\mathcal{N}}}(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}})$$

which is in fact an algebra isomorphism.

We thus have a geometric construction of the representations of the Weyl group as an algebra of (relative) correspondences on $\widetilde{\mathcal{N}}$, and a basis given by the irreducible components of $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$. One can show in particular that all the irreducible representations of a Weyl group are defined over $\mathbb{Q}$, and appear as direct summands in the $W$-module $H_{2d_x}(\mathfrak{B}_x)$ for some $[x] \in \mathrm{Conj}(\mathcal{N})$. We refer to the original papers [152], [13], [14], and the book [36] for the proofs of these results, which involve a considerable amount of Lie theory.

**Remark 2.29** Since the action of $W$ on $H_{2d_x}(\mathfrak{B}_x)$ commutes with the monodromy action of $\Gamma_x$, the $W$-module $H_{2d_x}(\mathfrak{B}_x)$ is in general not irreducible. In the case $G = SL_n$, the local systems $L_x$ turn out to be trivial, and every irreducible representation has a natural realization as $H_{2d_x}(\mathfrak{B}_x)$. The irreducible representations of $\mathcal{S}_n$ are thus parameterized by the partitions of the integer $n$.

### 2.5 The functions-sheaves dictionary and geometrization.

Although we will treat the case of complex algebraic varieties, the two following applications are inspired by the Grothendieck philosophy ([91], see also [113] §1.1) of the *dictionnarie fonctions-faisceaux*, according to which the "interesting" functions on the set of $\mathbb{F}_q$ points of a variety $X_0$ defined over a finite field $\mathbb{F}_q$ are associated with objects of $D^b_c(X_0, \overline{\mathbb{Q}}_l)$, i.e. complexes of $l$-adic sheaves. In particular, perverse sheaves have been playing a prominent role in applications.

The idea of switching the attention from functions and algebraic identities among them to sheaves and their relations is at the basis of the philosophy of *geometrization*. In §2.6 we show that the Kazhdan-Lusztig polynomials associated to a Weyl group of a Lie group may be interpreted, via the functions-sheaves dictionary, as functions associated to the Intersection cohomology complex of Schubert varieties. Similarly, in §2.7 we treat the case of a variant of the Kazhdan-Lusztig polynomials for an affine Weyl group which turn out to be associated, still via the functions-sheaves dictionary, to intersection cohomology of some subvarieties of the *affine Grassmannian*.

This leads to a *geometrization* of the theory of the spherical Hecke algebra and the Satake isomorphism: the isomorphism between the spherical Hecke algebra of bi-invariant functions and the character ring of the Langland dual is refined to an isomorphism between a Tannakian category of perverse sheaves on the affine Grassmannian associated with a reductive group $G$ and the Tannakian category of representations of the Langlands dual ${}^LG$, in such a way that the intersection cohomology complex of orbits in the affine grassmannian of $G$, which are parametrized by dominant weights of ${}^LG$, correspond to the irreducible representations with that dominant weight.

The most dramatic occurrence of the functions-sheaves dictionary, and one of the reasons for the importance of perverse sheaves in representations theory, is the geometrization of the notion of *automorphic form* in the geometric Langlands program; for details, see for instance [73], 3.3 or [76]. Coarsely speaking, an (unramified) automorphic form is a function on the "adelic quotient" $GL_n(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$, where $F$ is the field of rational functions of an algebraic curve $X$ defined over a finite field $\mathbb{F}_q$, $A_F$ is the ring of adèles of $F$, and $\mathcal{O} = \prod_{x \in X} \mathcal{O}_x$. The function must also satisfy some other property, such as that of being an eigenvector for the unramified Hecke algebra. A theorem of A.Weil gives an interpretation of the adelic quotient as the set of points of the moduli stack of vector bundles on $X$. Hence, by the function-sheaves dictionary, an automorphic form should correspond to a perverse sheaf on this moduli stack, and the important condition that the automorphic form be a Hecke eigenvector can also be interpreted geometrically introducing the notion of a Hecke eigensheaf.

We use the terminology introduced in §4.1.

**Definition 2.30** For $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_l)$ we define $t_{K_0} : X_0(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l$ as

$$t_{K_0}(x) = \sum_i (-1)^i \mathrm{Trace}(Fr : \mathcal{H}^i_x(K) \to \mathcal{H}^i_x(K)),$$

where $Fr$ is the Frobenius endomorphism of $\mathcal{H}^i_x(K)$.

**Example 2.31** If $i : Y_0 \to X_0$ is a closed imbedding, and $K = i_*\overline{\mathbb{Q}}_l$, the associated function $t_{i_*\overline{\mathbb{Q}}_l}$ is the characteristic function of $Y_0(\mathbb{F}_q)$. Twisting a complex $K_0$ corresponds to multiplying by $q^{-1}$ the corresponding function.

The function thus associated with $K_0$ satisfies several properties:

**Additivity.** If $K_0 \longrightarrow L_0 \longrightarrow M_0 \xrightarrow{[1]}$ is a distinguished triangle in $D^b_c(X_0, \overline{\mathbb{Q}}_l)$, then $t_{L_0} = t_{K_0} + t_{M_0}$.

**Multiplicativity.** We have, for $K_0, L_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_l)$, $t_{K_0 \otimes L_0} = t_{K_0} t_{L_0}$.

**Compatibility with pull-back.** If $f : X_0 \to Y_0$ is a map of $\mathbb{F}_q$-schemes and $K_0 \in D^b_c(Y_0, \overline{\mathbb{Q}}_l)$, then $t_{f^*K_0} = t_{K_0} \circ f$.

**Grothendieck Trace formula.** If $f : X_0 \to Y_0$ is a map of $\mathbb{F}_q$-schemes and $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_l)$, then, for $y \in Y(\mathbb{F}_q)$,

$$t_{f_!K_0}(y) = \sum_{x \in f^{-1}(y)} t_{K_0}(x).$$

**Remark 2.32** In the case that the cohomology sheaves of $K_0$ are all in even degree and the eigenvalues of Frobenius (not just their absolute values!) on $\mathcal{H}^{2i}_x(K_0)$ are equal to $q^i$, so that, in particular, $K_0$ is pure of weight 0, then the function $t_{K_0}$ is just the "Poincaré polynomial" $t_{K_0}(x) = \sum_i \dim \mathcal{H}^{2i}_x(K_0) q^i$. This is the case in the two examples we discuss in §2.6, 2.7.

## 2.6 Schubert varieties and Kazhdan-Lusztig polynomials.

We now discuss a topological interpretation of the Hecke Algebra of the Weyl group of a semisimple linear algebraic group, and in particular of the Kazhdan-Lusztig polynomials. The connection between the Kazhdan-Lusztig polynomials and the Intersection Cohomology of Schubert varieties was worked out by D. Kazhdan and G. Lusztig, ([109, 110]), following discussions with R. MacPherson. This connection played an important role in the development of the theory of perverse sheaves. We quickly review the basic definitions in the more general framework of Coxeter groups, see [97] and the recent [12] for more details on this beautiful subject. Let $(W,S)$ be a Coxeter group.

**Example 2.33** Let $W = \mathcal{S}_{n+1}$, the symmetric group. The the set of transpositions $s_i = (i, i+1)$ yields a set of generators $S = \{s_1, \ldots, s_n\}$.

On $W$ are defined the Bruhat order $\leq$ and the length function $l : W \to \mathbb{N}$. A basic object associated with $(W,S)$ is the Hecke algebra $\mathfrak{H}$. It is a free module over the ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ with basis $\{T_w\}_{w \in W}$ and ring structure

$$T_w T_{w'} = T_{ww'} \text{ if } l(ww') = l(w) + l(w'), \qquad T_s T_w = (q-1)T_w + qT_{sw} \text{ if } l(sw) < l(w).$$

**Example 2.34** Let $G_q$ be a Chevalley group over the finite field with $q$ elements, $B_q$ be its Borel subgroup, and $W$ be the Weyl group. In [101], Iwahori proved that the free $\mathbb{Z}$-module generated by the characteristic functions of the double $B_q$-cosets, endowed with the convolution product, satisfies the two defining relations of the Hecke algebra. The survey [38] gives a useful summary of the properties of this algebra and its relevance to the representation theory of groups of Lie-type.

**Example 2.35** Let $\mathcal{K}$ be a local field, with ring of integers $\mathcal{O}$, and let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$, with residue field $k = \mathcal{O}/\mathfrak{p}$ of cardinality $q$. Let $G$ be split and reductive over $\mathcal{K}$ and let $W^{\text{aff}}$ be its affine Weyl group. There is a "reduction mod-$\mathfrak{p}$" map $\pi : G(\mathcal{O}) \to G(k)$. Let $B' := \pi^{-1}(B)$ be the inverse image of a Borel subgroup of $G(k)$. For instance, if $G = SL_2$ with the usual choice of positive root, and $\mathcal{K} = \mathbb{Q}_p$, then the "Iwahori subgroup" $B'$ consists of matrices in $SL_2(\mathbb{Q}_p)$ whose $(2,1)$ entry is a multiple of $p$. Iwahori and Matsumoto, [102] proved that the algebra of locally constant functions on $G$ which are invariant with respect to the right and the left action of $B'$, endowed with the convolution product, is the Hecke algebra for $W^{\text{aff}}$. More precisely, the double $B'$-cosets are parameterized $W^{\text{aff}}$ and the basis $T_w$ of their characteristic functions satisfies the two defining relations of the Hecke algebra. The "spherical version," consisting of functions which are bi-invariant with respect to a different subgroup, will be quickly discussed in §2.7, in connection with the geometric Satake isomorphism.

It follows from the second defining relation of the Hecke algebra that $T_s$ is invertible for $s \in S : T_s^{-1} = q^{-1}(T_s - (q-1)T_e)$. This implies that $T_w$ is invertible for all $w$.

The algebra $\mathfrak{H}$ admits two commuting involutions $\iota$ and $\sigma$, defined by

$$\iota(q^{1/2}) = q^{-1/2}, \quad \iota(T_w) = T_{w^{-1}}^{-1} \qquad \text{and} \qquad \sigma(q^{1/2}) = q^{-1/2}, \quad \sigma(T_w) = (-1/q)^{l(w)} T_w.$$

The following is proved in [109]:

**Theorem 2.36** *There exists a unique $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis $\{C_w\}$ of $\mathfrak{H}$ with the following properties:*

$$\iota(C_w) = C_w \qquad C_w = (-1)^{l(w)} q^{l(w)/2} \sum_{v \leq w} (-q)^{-l(v)} P_{v,w}(q^{-1}) T_v \tag{15}$$

*with $P_{v,w} \in \mathbb{Z}[q]$ of degree at most $1/2(l(w) - l(v) - 1)$, if $v < w$, and $P_{w,w} = 1$.*

The polynomials $P_{v,w}$ are called the Kazhdan-Lusztig polynomials of $(W,S)$.

**Remark 2.37** For $s \in S$, we have that $C_s = q^{-1/2}(T_s - qT_e)$ satisfies (15), hence $P_{s,s} = P_{e,s} = 1$. A direct computation shows that if $W = \mathcal{S}_3$, then $P_{v,w} = 1$ for all $v, w$.. In contrast, if $W = \mathcal{S}_4$, then $P_{s_1s_3,s_1s_3s_2s_3s_1} = P_{s_2,s_2s_1s_3s_2} = 1 + q$.

**Remark 2.38** By using the involution $\sigma$ defined above, one obtains a slightly different basis $C'_w$, satisfying $\iota(C'_w) = C'_w$:

$$C'_w = (-1)^{l(w)}\sigma(C_w) = (-1)^{l(w)}(-q^{1/2})^{-l(w)} \sum_{v \leq w} (-q)^{l(v)} P_{v,w}(q)(-1/q)^{l(v)} T_v =$$

$$= (q^{1/2})^{l(w)} \sum_{v \leq w} P_{v,w}(q) T_v.$$

For instance, for $s \in S$, we have $C'_s = q^{-1/2}(T_s + T_e)$. As we will see, the basis $C'_w$ affords a simple geometric interpretation. For future reference we note the following, which follows from a direct computation: Let $T = \sum_w (\sum_i h^i_w q^{i/2}) T_w \in \mathfrak{H}$.

$$C'_s T = \sum_{sw>w} (\sum_i (h^{i+1}_w + h^{i-1}_{sw}) q^{i/2}) T_w + \sum_{sw<w} (\sum_i (h^{i+1}_{sw} + h^{i-1}_w) q^{i/2})) T_w. \tag{16}$$

In this section $W$ will be the Weyl group of a linear algebraic group and $S$ the set of reflections defined by a choice of a set of simple roots. More precisely, let $G$ be a semisimple linear algebraic group, $B$ be a Borel subgroup, $T \subseteq B$ be a maximal torus, $W = N(T)/T$ be the Weyl group, and $S$ be the set of generators determined by the choice of $B$. If $w \in W$, then we denote a representative of $w$ in $N(T)$ by the same letter.

**Example 2.39** Let $G = SL_{n+1}$, $B$ be the subgroup of upper triangular matrices, $T$ be the subgroup of diagonal matrices. Then $W \simeq \mathcal{S}_{n+1}$, and the choice of $B$ correspond to $S = \{s_1, \ldots s_n\}$ as in Example 2.33.

The flag variety $X = G/B$ parameterizes the Borel subgroups via the map $gB \to gBg^{-1}$. The $B$-action on $X$ gives the "Bruhat decomposition" $X = \coprod_{w \in W} X_w$. The Schubert cell $X_w$ is the $B$-orbit of $wB$. It is well known, see [15], that $X_w \simeq \mathbb{C}^{l(w)}$ and $\overline{X}_w = \coprod_{v \leq w} X_v$, where $\leq$ is the Bruhat ordering. Hence the Schubert variety $\overline{X}_w$ is endowed with a natural $B$-invariant cell decomposition.

**Example 2.40** Clearly $X_e = \overline{X}_e$ is the point $B$, and $\overline{X}_{w_0} = X$, if $w_0$ denotes the longest element of $W$. If $s \in S$ then $\overline{X}_s \simeq \mathbb{P}^1$. For instance, in the case of Example 2.39, if $\{o\} \subseteq \mathbb{C}^0 \subseteq \mathbb{C}^1 \subseteq \ldots \subseteq \mathbb{C}^n$ is the flag determined by the canonical basis of $\mathbb{C}^n$, then $\overline{X}_{s_i}$ parameterizes the flags $\{o\} \subseteq V_1 \subseteq \ldots \subseteq V_{n-1} \subseteq \mathbb{C}^n$ such that $V_k = \mathbb{C}^k$ for all $k \neq i$. One such flag is determined by the line $V_i/V_{i-1} \subseteq V_{i+1}/V_{i-1}$. If $l(w) \geq 2$ the Schubert variety $\overline{X}_w$ is, in general, singular. The flags $\mathbb{V} = \{o\} \subseteq V_1 \subseteq \ldots \subseteq V_{n-1} \subseteq \mathbb{C}^n$ in a Schubert cell $X_w$ can be described in terms of dimension of the intersections $V_i \cap \mathbb{C}^j$ as follows:

$$X_w = \{\mathbb{V} : \dim V_i \cap \mathbb{C}^j = w_{ij}\} \text{ where } w_{ij} = \sharp\{k \leq i \text{ such that } w(k) \leq j\}\}.$$

Since $B$ acts transitively on any Schubert cell, it follows that $\dim \mathcal{H}^i(IC_{\overline{X}_w})_x$ depends only on the cell $X_v$ containing the point $x$.

We will set, for $v \leq w$, $h^i(\overline{X}_w)_v := \dim \mathcal{H}^i_x(IC_{\overline{X}_w})$ for $x$ any point in $X_v$. We then define, for $v \leq w$, the Poincaré polynomial $\widetilde{P}_{v,w}(q) = \sum_i h^{i-l(w)}(\overline{X}_w)_v q^{i/2}$. The following surprising fact holds:

**Theorem 2.41** ([110]) *We have $P_{v,w}(q) = \widetilde{P}_{v,w}(q)$. In particular, if $i + l(w)$ is odd, then $\mathcal{H}^i(IC_{\overline{X}_w}) = 0$, and the coefficients of the Kazhdan-Lusztig polynomials $P_{v,w}(q)$ are non negative.*

**Remark 2.42** Theorem 2.41 implies that $P_{v,w} = 1$ for all $v \leq w$ iff $IC_{\overline{X}_w} = \mathbb{Q}_{\overline{X}_w}[l(w)]$. This happens, for instance, for $SL_3$ (cf. 2.37). The Schubert varieties of $SL_3$ are in fact smooth.

**Remark 2.43** In the same paper [109] in which the polynomials $P_{v,w}$ are introduced, Kazhdan and Lusztig conjecture a formula, involving the values $P_{v,w}(1)$, for the multiplicities of the Jordan-Hölder sequences of Verma modules. The proofs of these conjectures, due independently to Beilinson-Bernstein and Brylinski-Kashiwara, make essential use of the geometric interpretation 2.41 of the Kazhdan-Lusztig polynomials to translate the representation theoretic problem into a geometric one. See [154], §3 for a sketch of the proof and for references.

**Remark 2.44** Since $\dim X_v = l(v)$, the support conditions (20) of § 3.1 for Intersection Cohomology imply that if $v < w$, then $\mathcal{H}^{i-l(w)}(IC_{\overline{X}_w})_v = 0$ for $i - l(w) \geq -l(v)$. It follows that the degree of $\widetilde{P}_{v,w}(q)$ is a most $1/2(l(w) - l(v) - 1)$, as required by the definition of the Kazhdan-Lusztig polynomials. Furthermore, as $(IC_{\overline{X}_w})_{|X_w} = \mathbb{Q}_{X_w}[l(w)]$, we have $P_{w,w} = 1$

The original proof of Theorem 2.41, given in [110], is inspired to the "functions-sheaves dictionary" briefly discussed in 2.5, and does not use the Decomposition Theorem, but the purity of intersection cohomology in the $l$-adic context and the Lefschetz Trace Formula, [91]. Remark 2.44 implies that the polynomials $\widetilde{P}_{v,w}$ satisfy the first property (15) on the degree. It thus remains to show the invariance under the involution $\iota$. Kazhdan and Lusztig directly show that $\mathcal{H}^i(IC_{\overline{X}_w}) = 0$ if $i + l(w)$ is odd, and that the Frobenius map acts on $\mathcal{H}^{2i-l(w)}(IC_{\overline{X}_w})$ with eigenvalues equal to $q^i$, so that, up to a shift, $\widetilde{P}_{v,w}(q) = t_{IC_{\overline{X}_w}}(x)$, if $x \in X_v(\mathbb{F}_q)$, see §2.32. Once this is shown, the invariance under the involution $\iota$ turns out to be equivalent to the Poincaré Duality Theorem for intersection cohomology, §3.2, 1.

Another proof of the result of Kazhdan and Lusztig has been worked out by T.Haines in [93]. The statement about the eigenvalues of Frobenius is proved by using the decomposition theorem applied to the Demazure resolution of the Schubert varieties, explained below, and the geometric input that the fibres of the resolution have a decomposition as a union of affine spaces. This approach has the advantage that it deals at the same time with the flag variety and with the (infinite dimensional) affine flag variety.

An alternative approach to prove Theorem 2.41 which we now discuss is due to MacPherson and gives a topological description of the Hecke algebra. We follow the presentation in [154]. The paper [120] contains another approach, again based on the purity of $l$-adic intersection cohomology.

Let $A$ be a $B$-equivariant complex of sheaves, e.g. $A = \oplus_{\substack{w\in W\\ l\in\mathbb{Z}}} IC_{\overline{X}_w}[l]$. Define

$$h(A) = \sum_w (\sum_i h^i(A)_w q^{i/2}) T_w \in \mathfrak{H},$$

where, as before, $h^i(A)_w := \dim \mathcal{H}^i(A)_x$, for $x$ any point in $X_w$.

**Example 2.45** In Example 2.40, $\overline{X}_s$ is nonsingular and one-dimensional, $IC_{\overline{X}_s} = \mathbb{Q}_{\overline{X}_s}[1]$, and $h(IC_{\overline{X}_s}) = q^{-1/2}(T_s + T_e) = C'_s$.

It follows immediately from the definition of $h$ that $h(IC_{\overline{X}_w}) = q^{l(w)/2}\sum_{v\leq w}\widetilde{P}_{v,w}(q)T_v$, hence Theorem 2.41 is equivalent to the statement $h(IC_{\overline{X}_w}) = C'_w$.

To give a topological interpretation of the Hecke algebra product $h(IC_{\overline{X}_w})h(IC_{\overline{X}_{w'}})$, it is more convenient to work on $X\times X$, endowed with the diagonal $G$-action and the two projections $p_1, p_2 : X\times X \to X$. The following follows from the Bruhat decomposition:

**Proposition 2.46** *Let $O_w$ be the $G$-orbit of $(B, wB)$ in $X\times X$. Then $X\times X = \coprod_{w\in W} O_w$ and $p_1 : O_w \to X$ is a locally trivial fibration in the Zariski topology with fiber $X_w$. The closure $\overline{O}_w = \coprod_{v\leq w} O_v$, and $p_1 : \overline{O}_w \to X$ is a locally trivial fibration in the Zariski topology with fiber $\overline{X}_w$.*

**Example 2.47** Let $G, B, T, W, S$ be as in Example 2.39. A pair of flags

$$\mathbb{V} = \{o\} \subseteq V_1 \subseteq \ldots \subseteq V_{n-1} \subseteq \mathbb{C}^n, \qquad \mathbb{W} = \{o\} \subseteq V_1 \subseteq \ldots \subseteq V_{n-1} \subseteq \mathbb{C}^n,$$

is in $O_w$ iff $\dim V_i \cap W_j = w_{ij}$, where $w_{ij}$ is as in Example 2.40.

Note that, by Proposition 2.46, the cohomology sheaves of $IC_{\overline{X}_w}$ and $IC_{\overline{O}_w}$ differ only by a fixed shift:

$$\mathcal{H}^i(IC_{\overline{O}_w})_v = \mathcal{H}^{i+\delta}(IC_{\overline{X}_w})_v, \qquad \text{where } \delta = \dim X = \text{ number of positive roots.}$$

This suggests, for a complex of sheaves $A$ on $X\times X$, constructible with respect to the decomposition in $G$-orbits, defining

$$\hat{h}(A) = \sum_w (\sum_i h^{i-\delta}(A)_w q^{i/2}) T_w \in \mathfrak{H},$$

so that $h(IC_{\overline{X}_w}) = \hat{h}(IC_{\overline{O}_w})$.

Let $\mathcal{C}$ be the category of complexes of sheaves $A \in \mathcal{D}_{X\times X}$ whose cohomology sheaves are constant along the $G$-orbits, and such that either $\mathcal{H}^i(A) = 0$ for all odd $i$, or $\mathcal{H}^i(A) = 0$ for all even $i$. As we show below, $IC_{\overline{O}_w} \in \mathcal{C}$.

First we define a "convolution product". Let

$$\begin{tikzcd}
 & X\times X\times X\times X \arrow[dl, "p_{12}"'] \arrow[dr, "p_{34}"] & & X\times X\times X \arrow[ll, "i"'] \arrow[d, "q"] \\
X\times X & & X\times X & X\times X
\end{tikzcd}$$

with

$$p_{12}(x_1,x_2,x_3,x_4) = (x_1,x_2), \qquad p_{34}(x_1,x_2,x_3,x_4) = (x_3,x_4),$$

$$q(x_1,x_2,x_3) = (x_1,x_3), \qquad i(x_1,x_2,x_3) = (x_1,x_2,x_2,x_3).$$

Given $A, A' \in \mathcal{C}$, we define

$$A \star A' = q_*(i^*(p_{12}^*A \otimes p_{34}^*A')).$$

While proving 2.41, that is $\hat{h}(IC_{\overline{O}_w}) = C'_w$, we will show that the convolution $\star$ is the geometric counterpart of the product in the Hecke Algebra. More precisely, the following will be proved: Let $C'$ be the full subcategory of $C$ consisting of complexes of sheaves of the form $\oplus_{w,i} IC_{\overline{O}_w} \otimes V_w^i[-i]$, and let Let $K(C')$ be its Grothendieck group. Then:

**Theorem 2.48** *The mapping $\hat{h}$ extends to a mapping $\hat{h} : K(C') \to \mathfrak{H}$. The convolution $\star$, induces an algebra structure on the Grothendieck group $K(C')$, and the mapping $\hat{h}$ becomes an isomorphism of algebras.*

.

We first prove that the convolution product correspond to is the product in the Hecke algebra. It suffices to prove this fact for the convolution with $IC_{\overline{O}_s}$. We already noticed, 2.45, that $\hat{h}(IC_{\overline{O}_s}) = C'_s$.

Since $\overline{O}_s \to X$ is, by Pr. 2.46 and Ex. 2.40, a $\mathbb{P}^1$-fibration over $X$, it follows that $IC_{\overline{O}_s} = \mathbb{Q}_{\overline{O}_s}[1+\delta] \in \mathcal{C}$.

**Proposition 2.49** *Let $A \in \mathcal{C}$. Then*

$$IC_{\overline{O}_s} \star A \in \mathcal{C}, \quad \text{and} \quad \hat{h}(IC_{\overline{O}_s} \star A) = \hat{h}(IC_{\overline{O}_s})\hat{h}(A) = C'_s\hat{h}(A).$$

*Proof.* Let us compute $\dim \mathcal{H}^i(IC_{\overline{O}_s} \star A)_w$. We pick a point $p \in O_w$, e.g. $p = (B, wB)$. Since $IC_{\overline{O}_s} = \mathbb{Q}_{\overline{O}_s}[1+\delta]$, we have

$$p_{12}^* IC_{\overline{O}_s} = \mathbb{Q}_{\overline{O}_s \times X \times X}[1+\delta]$$

and

$$IC_{\overline{O}_s} \star A = q_*(i^*((p_{34}^*A')_{|\overline{O}_s \times X \times X}))[1+\delta].$$

Since

$$q^{-1}(p) \cap i^{-1}(\overline{O}_s \times X \times X) = \{(B, x, wB) \text{ such that } x \in \overline{X}_s\} \simeq \mathbb{P}^1,$$

we find

$$\mathcal{H}^i(IC_{\overline{O}_s} \star A)_w = \mathcal{H}^i(IC_{\overline{O}_s} \star A)_p = H^{i+\delta+1}(Y, A_{|Y}),$$

where $Y = \overline{X}_s \times \{wB\} \subseteq X \times X$. The complex $A_{|Y}$ is constant on an open set $U \simeq \mathbb{C}$. Notice that, since $U$ is contractible, $A_{|U} \simeq \oplus \mathcal{H}^i(A_{|U})[-i]$ and $\mathcal{H}^i(A_{|U})$ is a constant sheaf. Let $u \in U$ and $u_0 = Y \setminus U$. From the direct sum decomposition above and the fact that $A \in \mathcal{C}$, it follows that the long exact sequence $\ldots \to \mathcal{H}^{i-1}(A)_{u_0} \to H^i_c(U, A_{|U}) \to H^i(Y, A_{|Y}) \to \mathcal{H}^i(A)_{u_0} \to \ldots$ splices-up into short exact sequences $0 \to H^i_c(U, A_{|U}) \to H^i(Y, A_{|Y}) \to \mathcal{H}^i(A)_{u_0} \to 0$. Poincaré Duality gives

$$\dim H^i(Y, A_{|U}) = \dim \mathcal{H}^{i-2}(A)_u + \dim \mathcal{H}^i(A)_{u_0}. \tag{17}$$

We distinguish two cases:

1. $sw > w$. In this case $Y \cap O_{sw} = U$ and $Y \cap O_w = u_0$, and (17) gives

$$h^i(IC_{\overline{O}_s} \star A)_w = \dim H^{i+\delta+1}(Y, A) = h^{i+\delta+1}(A)_w + h^{i+\delta-1}(A)_{sw}.$$

2. $sw < w$. In this case $Y \cap O_w = U$ and $Y \cap O_{sw} = u_0$, and

$$h^i(IC_{\overline{O}_s} \star A)_w = \dim H^{i+\delta+1}(Y, A) = h^{i+\delta+1}(A)_{sw} + h^{i+\delta-1}(A)_w.$$

In view of (16), this ends the proof. □

To extend the computation of Proposition 2.49 from $IC_{\overline{O}_s}$ to a general $IC_{\overline{O}_w}$, we use a beautiful construction, the Bott-Samelson variety, [17, 64], which gives a $G$-equivariant resolution of $\overline{O}_w$ in terms of a minimal expression $w = s_1 \ldots s_l$ of $w$.

Let $\widetilde{O}_w = \{(\mathbb{V}_1, \ldots, \mathbb{V}_{l+1}) \in X^{l+1}$ be such that $(\mathbb{V}_i, \mathbb{V}_{i+1}) \in \overline{O}_{s_i}$ for $i = 1, \ldots l\}$. The sequence of maps

$$\widetilde{O}_w = \widetilde{O}_{s_1 \ldots s_l} \to \widetilde{O}_{s_1 \ldots s_{l-1}} \to \widetilde{O}_{s_1} \to X$$

exhibits $\widetilde{O}_w$ as an iterated $\mathbb{P}^1$-fibration over $X$, so that $\widetilde{O}_w$ is nonsingular.

The map $\pi : \widetilde{O}_w \to \overline{O}_w \subseteq X \times X$, defined by $\pi((\mathbb{V}_1, \ldots, \mathbb{V}_{l+1})) = (\mathbb{V}_1, \mathbb{V}_{l+1})$, is a $G$-equivariant resolution of $\overline{O}_w$, and an isomorphism over $O_w$.

*Proof of Theorem 2.41.* It follows from the definition of the product $\star$ that

$$\pi_* \mathbb{Q}_{\widetilde{O}_w}[\delta + l(w)] = IC_{\overline{O}_{s_1}} \star \ldots \star IC_{\overline{O}_{s_l}}.$$

Thus, by Proposition 2.49,

$$\hat{h}(\pi_* \mathbb{Q}_{\widetilde{O}_w}[\delta + l(w)]) = q^{l(w)}(T_{s_1} + T_e) \ldots (T_{s_l} + T_e) = C'_{s_1} \ldots C'_{s_l}.$$

Since $\iota(C'_{s_i}) = C'_{s_i}$, it follows that $\iota(\hat{h}(\pi_* \mathbb{Q}_{\widetilde{O}_w}[\delta + l(w)])) = \hat{h}(\pi_* \mathbb{Q}_{\widetilde{O}_w}[\delta + l(w)])$.

We apply the Decomposition Theorem to the map $\pi$. Since $\pi$ is birational, the perverse cohomology sheaves ${}^{\mathfrak{p}}\mathcal{H}^i(\pi_*\mathbb{Q}_{\widetilde{O}_w}[\delta+l(w)])$ are supported, for $i \neq 0$, on $\overline{O}_w \setminus O_w = \coprod_{v<w} O_v$. Thus, for some finite dimensional $\mathbb{Q}$-vector spaces $V_v^i$,

$$ {}^{\mathfrak{p}}\mathcal{H}^i(\pi_*\mathbb{Q}_{\widetilde{O}_w}[\delta+l(w)]) = \oplus_{v<w}(\oplus_{i\in\mathbb{Z}} IC_{\overline{O}_v} \otimes V_v^i). $$

We have

$$ \pi_*\mathbb{Q}_{\widetilde{O}_w}[\delta+l(w)] \simeq IC_{\overline{O}_w} \oplus (\oplus_{v<w}(\oplus_{i\in\mathbb{Z}} IC_{\overline{O}_v} \otimes V_v^i[-i])), $$

and, by applying $\hat{h}$, we find

$$ C'_{s_1}\dots C'_{s_l} = \hat{h}(\pi_*\mathbb{Q}_{\widetilde{O}_w}[\delta+l(w)]) = \hat{h}(IC_{\overline{O}_w}) + \sum_{v<w} P_v(q)\hat{h}(IC_{\overline{O}_v}), \tag{18} $$

where $P_v(q) = \sum \mathrm{dim} V_v^j q^{j/2}$. Verdier Duality implies

$$ D_{\overline{O}_w}\, {}^{\mathfrak{p}}\mathcal{H}^i(\pi_*\mathbb{Q}_{\widetilde{O}_w}[\delta+l(w)]) \simeq {}^{\mathfrak{p}}\mathcal{H}^{-i}(\pi_*\mathbb{Q}_{\widetilde{O}_w}[\delta+l(w)]). $$

It follows that $\mathrm{dim} V_v^i = \mathrm{dim} V_v^{-i}$, thus $P_v(q) = P_v(q^{-1})$.
We work by induction on the length of $w$ and assume that $h(IC_{\overline{O}_v}) = C'_v$; we have already proved the case $l(v) = 1$ in Example 2.45. We deduce from (18) that

$$ \iota(\hat{h}(IC_{\overline{O}_w})) = \iota(C'_{s_1}\dots C'_{s_l}) - \sum_{v<w} \iota(P_v(q)C'_v = C'_{s_1}\dots C'_{s_l} - \sum_{v<w} P_v(q^{-1})C'_v) = \hat{h}(IC_{\overline{O}_w}). $$

Since $\hat{h}(IC_{\overline{O}_w}) = h(IC_{\overline{X}_w}) = \sum_{v<w} \widetilde{P}_{v,w}(q)T_v$, and, as we have noticed in Remark 2.44, $\mathrm{deg}\widetilde{P}_{v,w} \leq 1/2(l(w)-l(v)-1)$, we conclude that $\widetilde{P}_{v,w} = P_{v,w}$. □

The map $\hat{h}$, from the set of complexes of sheaves of the form $\oplus_{w,i} IC_{\overline{O}_w} \otimes V_w^i[-i]$ to the Hecke algebra $\mathfrak{H}$, can now be completed by formally adding "differences" of such complexes of sheaves, to get an isomorphism of algebras. This yields MacPherson's topological construction of the Hecke algebra $\mathfrak{H}$.

## 2.7 The Geometric Satake isomorphism.

We now discuss, without proofs, an affine analogue of the constructions described in §2.6, culminating in a geometrization of the spherical Hecke algebra and the Satake isomorphism. In this case, the Schubert cells will be replaced by subvarieties $\overline{\mathrm{Orb}}_\lambda$, parameterized by $\lambda \in X_\bullet(T)$, of an ind-scheme $\mathcal{GR}_G$.

Let us first recall, following the clear exposition [90], the basic statement of the classical Satake isomorphism, [149].

Let $\mathcal{K}$ be a local field and $\mathcal{O}$ be its ring of integers, and denote by $\pi$ a generator of the maximal ideal, e.g. $\mathcal{K} = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$, and $\pi = p\mathbb{Z}_p$. Denote by $q$ the cardinality of the residue field. We let $G$ be a reductive linear algebraic group *split* over $\mathcal{K}$, and denote by $G(\mathcal{K})$ the set of $\mathcal{K}$-points and by $K = G(\mathcal{O})$, the set of $\mathcal{O}$-points, a compact subgroup of $G(\mathcal{K})$. Similarly to Examples 2.34, 2.35, the spherical Hecke algebra $\mathcal{H}(G(\mathcal{K}), G(\mathcal{O}))$ is defined to be the set of $K{-}K$-invariant locally constant $\mathbb{Z}$-valued functions on $G(\mathcal{K})$ endowed with the convolution product $f_1 * f_2(x) = \int_G f_1(g) f_2(g^{-1}x) dg$. Here, $dg$ denotes the Haar measure, normalized so that the volume of $K$ is 1. Let $X_\bullet(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ be the free abelian groups of co-characters of a maximal torus $T$. It carries a natural action of the Weyl group $W$. The choice of a set of positive roots singles out a positive chamber $X_\bullet(T)^+$, which is a fundamental domain for the action of $W$. Every $\lambda \in X_\bullet(T)$ defines an element $\lambda(\pi) \in K$, and one has the following decomposition:

$$G = \coprod_{\lambda \in X_\bullet(T)^+} K\lambda(\pi)K.$$

The characteristic functions $C_\lambda$ of the double cosets $K\lambda(\pi)K$, for $\lambda \in X_\bullet(T)^+$, give a $\mathbb{Z}$-basis of $\mathcal{H}(G, K)$. We have, $\mathcal{H}(T(\mathcal{K}), T(\mathcal{O})) \simeq \mathbb{Z}[X_\bullet(T)]$.

**Example 2.50** Let $G = GL_n$. With the usual choice of positive roots, an element $\lambda \in X_\bullet(T)^+$ is of the form $diag(t^{a_1}, \dots t^{a_n})$, with $a_1 \geq a_2 \geq \dots \geq a_n$. The above decomposition boils down to the fact that, by multiplying it on the left and on the right by elementary matrices, a matrix can be reduced to a diagonal form, cfr. [19], VII.21 Cor.6.

The Satake isomorphism is an algebra isomorphism $\mathcal{S}$ of $\mathcal{H}(G(\mathcal{K}), G(\mathcal{O})) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$ with $\mathbb{Z}[X_\bullet(T)]^W \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$. The ring $\mathbb{Z}[X_\bullet(T)]^W$ is isomorphic to the ring of representations of the Langlands dual ${}^LG$ of $G$, i.e. the reductive group whose root datum is the co-root datum of $G$ and whose co-root datum is the root datum of $G$, see [153] for a very nice description of these notions. The Satake isomorphism can therefore be stated as

$$\mathcal{S} : \mathcal{H}(G(\mathcal{K}), G(\mathcal{O})) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \xrightarrow{\simeq} \mathrm{Repr}({}^LG) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

**Remark 2.51** The $\mathbb{Z}$-module $\mathrm{Repr}({}^LG)$ has a basis $[V_\lambda]$ parameterized by $\lambda \in X_\bullet(T)^+$, where $V_\lambda$ is the irreducible representation with highest weight $\lambda$. It may be tempting to associate $[V_\lambda]$ with the characteristic function $C_\lambda$ of the double coset $K\lambda(\pi)K$. However, this would not work. There exist integers $d_\lambda(\mu)$, defined for $\mu \in X_\bullet(T)^+$, with $\mu < \lambda$ such that the more complicated formula

$$\mathcal{S}^{-1}([V_\lambda]) = q^{-\rho(\lambda)}(C_\lambda + \sum_{\substack{\mu \in X_\bullet(T)^+ \\ \mu < \lambda}} d_\lambda(\mu) C_\mu), \tag{19}$$

where $\rho = (1/2) \sum_{\alpha > 0} \alpha$, holds instead.

The Satake isomorphism is remarkable in the sense that it relates $G$ and ${}^LG$. A priori, it is very unclear that the two should be related at all, beyond the defining exchanging property. The isomorphism gives, in principle, a recipe to construct the Langlands dual of $G$, through its representation ring, from the datum of the ring of functions on the double coset space $K \backslash G / K$.

A striking application of the theory of perverse sheaves is the "geometrization" of this isomorphism. The whole subject was started by the important work of Lusztig [118] (and [117] for the type $A$ case). In this work, it is shown that the Kazhdan-Lusztig polynomials associated with a group closely related to $W^{\text{aff}}$ are the Poincaré polynomials of intersection cohomology sheaves of singular varieties $\overline{\text{Orb}}_\lambda$, for $\lambda \in X_\bullet(T)$, inside an ind-scheme $\mathcal{GR}_G$ which is defined below, and coincide with the weight multiplicities $d_\lambda(\mu)$ of the representation $V_\lambda$ appearing in formula (19). As a consequence, he showed that $\dim IH(\overline{\text{Orb}}_\lambda) = \dim V_\lambda$ and that the tensor product operation $V_\lambda \otimes V_\nu$ correspond to a "convolution" operation $IC_{\overline{\text{Orb}}_\lambda} \star IC_{\overline{\text{Orb}}_\nu}$.

The geometric significance of Lusztig's result was clarified by the work of Ginzburg [80] and Mirković-Vilonen [133]. We quickly review the geometry involved, according to the paper [133]. We work over the field of complex numbers. The analogue of the coset space $G(\mathcal{K})/G(\mathcal{O}))$ of §2.6 is the affine Grassmannian, which we now introduce; see [10] for a thorough treatment. Let $G$ be a linear algebraic group. As a $\mathbb{C}$-scheme, $G(\mathbb{C}[[t]])$ is a group scheme, not of finite type, representing the functor $R \to G(R[[t]])$ from $\mathbb{C}$-algebras to groups. On the other hand, $G(\mathbb{C}((t)))$ is only a ind-group scheme, i.e its functor of points $R \to G(R((t)))$, from $\mathbb{C}$-algebras to groups, is the direct limit of functors of points of $\mathbb{C}$-group schemes which we now describe. If $r : G \to SL_n(\mathbb{C})$ is a faithful representation, then the ind-structure is defined by the representable sub-functors

$$G_N(R) = \{g \in G(R((t))) \text{ such that } r(g), r(g^{-1}) \text{ have a pole of order at most } N\}.$$

The quotient $\mathcal{GR}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ is a ind-scheme called the "affine Grassmannian".

Now, we describe some of its properties, according to [10]. The proofs of these properties are contained in [5, 114].

**Remark 2.52** Let $G = SL_n(\mathbb{C})$. The closed points of the ind-scheme $\mathcal{GR}_{SL_n(\mathbb{C})}$ correspond to special lattices in the $\mathbb{C}((t))$-vector space $V = \mathbb{C}((t))^n$. A special lattice is a $\mathbb{C}[[t]]$-module $M \subseteq V$ such that $t^N\mathbb{C}[[t]]^n \subseteq M \subseteq t^{-N}\mathbb{C}[[t]]^n$ for some $N$, and $\bigwedge^n M = \mathbb{C}[[t]]$. The action of $SL_n(\mathbb{C}((t)))$ on the set of special lattices is transitive, and $SL_n(\mathbb{C}[[t]])$ is the stabilizer of the lattice $M = \mathbb{C}[[t]]^n$.

**Remark 2.53** The affine Grassmannian has a modular interpretation. Roughly speaking, it parameterizes couples $(E, \beta)$, where $E$ is a $G$-torsor on the "formal disc" $Spec\,\mathbb{C}[[t]]$ and $\beta$ is a trivialization of the restriction of $E$ to the "formal punctured disc" $Spec\,\mathbb{C}((t))$.

**Remark 2.54** The set of points of the affine Grassmannian $\mathcal{GR}_T$ of a torus $T$ is easily seen to be $X_\bullet(T)$. The scheme structure is somewhat subtler, as explained in the following Remark.

**Remark 2.55** The ind-group $G(\mathbb{C}((t)))$ and the affine Grassmannian $\mathcal{GR}_T$ can be non reduced. This happens for instance, if $G = \mathbb{G}_m$. Roughly speaking the reason is the following: $G(\mathbb{C}((t)))$ is the inductive limit of the functors $G_{\geq -N} : \{\mathbb{C} - algebras\} \to \{groups\}$ defined as $G_{\geq -N}(R) = R((t))^*_{\geq -N}$, where $R((t))^*_{\geq -N}$ denotes the units $u \in R((t))$ such that both $u$ and $u^{-1}$ have a pole of order at most $N$. If a $\mathbb{C}$-algebra $R$ has nilpotents, the Laurent series $a_{-N}t^{-N} + \dots a_{-1}t^{-1} + a_0$ belongs to $R((t))^*_{\geq -N}$ if $a_0$ is invertible and $a_{-N}, \dots a_{-1}$ are nilpotent. This implies that the functors $G_{\geq -N}$ are represented by nonreduced schemes. Proposition 2.56.4 essentially states that this is the only way the affine Grassmannian can have nilpotents.

Set $\mathcal{K} = \mathbb{C}((t))$, and $\mathcal{O} = \mathbb{C}[[t]]$.

**Proposition 2.56** ([10], 4.5) *We have the following five facts.*
*1)* $\mathcal{GR}_G$ *is an inductive limit of algebraic varieties of finite type.*
*2) The projection* $G(\mathcal{K}) \xrightarrow{\pi} \mathcal{GR}_G$ *is locally trivial in the Zariski topology.*
*3)* $\mathcal{GR}_G$ *is an inductive limit of complete algebraic varieties if and only if* $G$ *is reductive.*
*4)* $\mathcal{GR}_G$ *and* $G(\mathcal{K})$ *are reduced if and only if* $\mathrm{Hom}(G, \mathbb{C}^*)) = 0$.
*5) There is a natural bijection* $\pi_0(\mathcal{GR}_G) \to \pi_1(G)$.

From now on, we assume that $G$ is a connected reductive linear algebraic group, so that $\mathcal{GR}_G$ is of ind-finite type and ind-proper: $\mathcal{GR}_G = \lim_n \mathcal{GR}_{G,n}$, where $\mathcal{GR}_{G,n} \subseteq \mathcal{GR}_{G,n+1}$ are closed imbeddings, the closed subschemes $\mathcal{GR}_{G,n}$ are $G(\mathcal{O})$-invariant and the action of $G(\mathcal{O})$ on this closed subsets factors through a finite dimensional quotient. Next , we describe the structure of the $G(\mathcal{O})$-orbits. The imbedding $T \subseteq G$ of the maximal torus gives a map $\mathcal{GR}_T \to \mathcal{GR}_G$. By Remark 2.54, we can identify $X_\bullet(T)$ with a subset of $\mathcal{GR}_G$. We still denote by $\lambda$ the point of the affine Grassmannian corresponding to $\lambda \in X_\bullet(T)$, and we denote its $G(\mathcal{O})$-orbit by $\mathrm{Orb}_\lambda \subseteq \mathcal{GR}_G$ (cf. [10]).

**Proposition 2.57** ([10], 5.3) *There is a decomposition* $\mathcal{GR}_G = \coprod_{\lambda \in X_\bullet(T)^+} \mathrm{Orb}_\lambda$. *Furthermore, every orbit* $\mathrm{Orb}_\lambda$ *has the structure of a vector bundle over a rational homogeneous variety, it is connected and simply connected,*

$$\mathrm{dim}\mathrm{Orb}_\lambda = 2\rho(\lambda) \qquad \textit{and} \qquad \overline{\mathrm{Orb}}_\lambda = \coprod_{\mu \leq \lambda} \mathrm{Orb}_\mu.$$

Proposition 2.57 implies that (cf. 2.6) that the category $\mathcal{P}_{G(\mathcal{O})}$ of perverse sheaves which are constructible with respect to the decomposition in $G(\mathcal{O})$-orbits is generated by the intersection cohomology complexes $IC_{\overline{\mathrm{Orb}}_\lambda}$. Lusztig has proved in [118] that the cohomology sheaves $\mathcal{H}^i(IC_{\overline{\mathrm{Orb}}_\lambda})$ are different from zero only in one parity. Together with the fact that the dimensions of all $G(\mathcal{O})$-orbits in the same connected component of $\mathcal{GR}_G$ have the same parity, this implies that $\mathcal{P}_{G(\mathcal{O})}$ is a semisimple category. Its objects are automatically $G(\mathcal{O})$-equivariant perverse sheaves. The group $\mathrm{Aut}(\mathcal{O})$ of automorphisms of the $\mathbb{C}$-algebra $\mathcal{O}$ acts on $\mathcal{GR}_G$. The objects of $\mathcal{P}_{G(\mathcal{O})}$ are automatically $\mathrm{Aut}(\mathcal{O})$-equivariant, [77], 2.1.3.

The Tannakian formalism, see [62], singles out the categories which are equivalent to categories of representations of affine groups schemes. The Satake isomorphism yields a recipe to re-construct ${}^LG$. The geometrization of this relation involves the abelian $\mathbb{Q}$-linear category $\mathcal{P}_{G(\mathcal{O})}$. To "reconstruct" the Langlands dual group ${}^LG$ from $\mathcal{P}_{G(\mathcal{O})}$ it is necessary to endow this latter with the structure of rigid tensor category with a "fiber functor." Essentially, this means that there must be 1) a bilinear functor $\star : \mathcal{P}_{G(\mathcal{O})} \times \mathcal{P}_{G(\mathcal{O})} \to \mathcal{P}_{G(\mathcal{O})}$ with compatible associativity and commutativity constraints, i.e. is functorial isomorphisms $A_1 \star (A_2 \star A_3) \overset{\simeq}{\to} (A_1 \star A_2) \star A_3$ and $A_1 \star A_2 \overset{\simeq}{\to} A_2 \star A_1$, and 2) an exact functor $F : \mathcal{P}_{G(\mathcal{O})} \to \mathrm{Vect}_{\mathbb{Q}}$ which is a tensor functor, i.e. there is a functorial isomorphism $F(A_1 \star A_2) \overset{\simeq}{\to} F(A_1) \otimes F(A_2)$.

With some technical inaccuracy, we state the geometric Satake isomorphism as follows:

**Theorem 2.58** *There exists a geometrically defined "convolution product"*

$$\star \;:\; \mathcal{P}_{G(\mathcal{O})} \times \mathcal{P}_{G(\mathcal{O})} \longrightarrow \mathcal{P}_{G(\mathcal{O})}$$

*with "commutativity constraints," such that the cohomology functor* $H : \mathcal{P}_{G(\mathcal{O})} \to \mathrm{Vect}_{\mathbb{Q}}$ *is a tensor functor, and* $(\mathcal{P}_{G(\mathcal{O})}, \star, H)$ *is equivalent to the category of representations of* ${}^L G$.

**Remark 2.59** In fact, Mirković and Vilonen, in [133], prove a more precise result, allowing an arbitrary ring of coefficients, e.g. $\mathbb{Z}$. As a result the category $(\mathcal{P}_{G(\mathcal{O})}, \mathbb{Z})$ determines a Chevalley scheme ${}^L G_{\mathbb{Z}}$.

**Remark 2.60** In [136], Nadler investigates a subcategory of perverse sheaves on the affine Grassmannian of a real form $G_{\mathbb{R}}$ of $G$, which still form a tensor category, and proves that it is equivalent with the category of representations of a reductive subgroup ${}^L H$ of ${}^L G$. This establishes a real version of the Geometric Satake isomorphism. As a corollary, the Decomposition Theorem is shown to hold for several *real* algebraic maps arising in Lie theory.

We discuss only two main points of the construction of [133], the definition of the convolution product and the use of the " semi-infinite" orbits to construct the weight functors. We omit all technical details and refer the reader to [133].

**The convolution product.** In the following description of the convolution product we treat the spaces involved as if they were honest schemes. See [77] for a detailed account. Let us consider the diagram:

$$\begin{tikzcd}
\mathcal{GR}_G & G(\mathcal{K}) \arrow[r, "\pi"] & \mathcal{GR}_G & \\
G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G \arrow[u, "p"] & G(\mathcal{K}) \times \mathcal{GR}_G \arrow[l, "q"'] \arrow[u] \arrow[r, "\pi \times \mathrm{Id}"] & \mathcal{GR}_G \times \mathcal{GR}_G \arrow[u, "p_1"'] \arrow[r, "p_2"] & \mathcal{GR}_G .
\end{tikzcd}$$

The map $q : G(\mathcal{K}) \times \mathcal{GR}_G \to G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G$ is the quotient map by the action of $G(\mathcal{O})$, the map $p : G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G \to \mathcal{GR}_G$ is the "action" map, $p(g, hG(\mathcal{O})) = ghG(\mathcal{O})$. If $A_1, A_2 \in \mathcal{P}_{G(\mathcal{O})}$, then $(\pi \times \mathrm{Id})^*(p_1^*(A_1) \otimes p_2^*(A_2))$ on $G(\mathcal{K}) \times \mathcal{GR}_G$ descends to $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G$, that is, there exists a unique complex of sheaves $A_1 \widetilde{\otimes} A_2$ on $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G$ with the property that $(\pi \times \mathrm{Id})^*(p_1^*(A_1) \otimes p_2^*(A_2)) = q^*(A_1 \widetilde{\otimes} A_2)$, and we set $A_1 \star A_2 := p_*(A_1 \widetilde{\otimes} A_2)$.

The following fact is referred to as "Miraculous Theorem" in [10]:

**Theorem 2.61** *If $A_1, A_2 \in \mathcal{P}_{G(\mathcal{O})}$, then $A_1 \star A_2 \in \mathcal{P}_{G(\mathcal{O})}$.*

The key reason why this theorem holds is that the map $p$ enjoys a strong form of semismallness.

First of all the complex $A_1 \widetilde{\otimes} A_2$ is constructible with respect to the decomposition

$$G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G = \coprod \mathcal{S}_{\lambda,\mu} \qquad \text{with} \qquad \mathcal{S}_{\lambda,\mu} = \pi^{-1}(\mathrm{Orb}_\lambda) \times_{G(\mathcal{O})} \mathrm{Orb}_\mu.$$

**Proposition 2.62** *The map $p : G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G \to \mathcal{GR}_G$ is stratified semismall, in the sense, that for any $\mathcal{S}_{\lambda,\mu}$, the map $p_{|\overline{\mathcal{S}}_{\lambda,\mu}} : \overline{\mathcal{S}}_{\lambda,\mu} \to p(\overline{\mathcal{S}}_{\lambda,\mu})$ is semismall. As a consequence $p_*$ sends perverse sheaves constructible with respect to the decomposition $\{\mathcal{S}_{\lambda,\mu}\}$, to perverse sheaves on $\mathcal{GR}_G$ constructible with respect to the decomposition $\{\mathrm{Orb}_\lambda\}$.*

**Remark 2.63** While the "associativity constraints" of the convolution product are almost immediate from its definition, the commutativity constraints are far subtler. Their proof in [133], see also [77], involves the modular interpretation of the affine Grassmannian, 2.53, and some geometry of the moduli spaces of torsors over a curve.

**The weight functor.** The global cohomology functor $H$ is the fiber functor of the category $\mathcal{P}_{G(\mathcal{O})}$. In particular, it is a tensor functor: $H(A_1 \star A_2) \simeq H(A_1) \otimes H(A_2)$. In order to verify this, Mirković and Vilonen decompose this functor as a direct sum of functors $H_\mu$ parameterized by $\mu \in X_\bullet(T)$. This decomposition is meant to mirror the weight decomposition of a representation of ${}^L G$. It is realized by introducing certain ind-subschemes $N_\mu$ which have a "cellular" property with respect to any $A \in \mathcal{P}_{G(\mathcal{O})}$, in the sense that at most one compactly supported cohomology group does not vanish. Let $U$ be the unipotent radical of the Borel group $B$, and $U(\mathcal{K})$ be the corresponding subgroup of $G(\mathcal{K})$. The $U(\mathcal{K})$- orbits in the affine Grassmannian are neither of finite dimension nor of finite codimension. They are ind-subschemes. It can be shown that they are parameterized by $X_\bullet(T)$. If, as before, we still denote by $\lambda$ the point of the affine Grassmannian corresponding to $\lambda \in X_\bullet(T)$, and set $S_\lambda := U(\mathcal{K})\lambda$, then we have $\mathcal{GR}_G = \coprod_{\lambda \in X_\bullet(T)} S_\lambda$.

**Proposition 2.64** *For any $A \in \mathcal{P}_{G(\mathcal{O})}$, we have*

$$H^l_c(S_\lambda, A) = 0 \text{ for } l \neq 2\rho(\lambda).$$

*In particular, the functor $H^{2\rho(\lambda)}_c(S_\lambda, -) : \mathcal{P}_{G(\mathcal{O})} \to \mathrm{Vect}_{\mathbb{Q}}$ is exact, and*

$$H(\mathcal{GR}_G, -) = \bigoplus_{\lambda \in X_\bullet(T)} H^{2\rho(\lambda)}_c(S_\lambda, -).$$

**Remark 2.65** The decomposition 2.64 of the cohomology functor reflects, via the Geometric Satake isomorphism, the weight decomposition of the corresponding representation of ${}^L G$. An aspect of the Geometric Satake correspondence which we find particularly beautiful is that, up to a re-normalization, the intersection cohomology complex $IC_{\overline{\mathrm{Orb}_\lambda}}$ correspond, via the Geometric Satake isomorphism, to the irreducible representation $V(\lambda)$ of ${}^LG$ with highest weight $\lambda$. This explains (cf. Remark 2.51) why the class of $V(\lambda)$ is not easily expressed in terms of the characteristic function $C_\lambda$ of the double coset $K\lambda(\pi)K$ (which corresponds, in the function-sheaves dictionary of 2.5, to the constant sheaf on $\overline{\mathrm{Orb}_\lambda}$) and once again emphasizes the fundamental nature of Intersection Cohomology.

### 2.8 Decomposition up to homological cobordism and signature

We want to mention, without any detail, a purely topological counterpart of the Decomposition Theorem. Recall that this result holds only in the algebraic context, e.g. it fails for proper holomorphic maps of complex manifolds.

In the topological context, Cappell and Shaneson [29] introduce a notion of cobordism for complexes of sheaves and prove a general topological result for maps between Whitney stratified space with only even codimension strata that in the case of a proper algebraic map $f : X \to Y$, identifies, up to cobordism, $f_*IC_X$ with ${}^p\mathcal{H}^0(f_*IC_X)$ and its splitting as in the Decomposition Theorem.

The decomposition up to cobordism is sufficient to provide exact formulae for many topological invariants, such as Goresky-MacPherson $L$-classes and signature thus generalizing the classical Chern-Hirzebruch-Serre multiplicativity property of the signature for smooth fiber bundles with no monodromy to the case of stratified maps (see [30, 31, 150]).

In the case of complex algebraic varieties, one may also look at the MacPherson Chern classes [123], the Baum-Fulton-MacPherson Todd classes [4], the homology Hirzebruch classes [22, 33] and their associated Hodge-genera defined in terms of the mixed Hodge structures on the (intersection) cohomology groups. The papers [31, 32, 33] provide Hodge-theoretic applications of the above topological stratified multiplicative formulæ. For a survey, see [128].

These results yield topological and analytic constraints on the singularities of complex algebraic maps. In the case of maps of projective varieties, these Hodge-theoretic formulæ are proved using the Decomposition Theorem, especially the identification in [46] of the local systems appearing in the decomposition combined with the Hodge-theoretic aspects of the decomposition theorem in [50]. For non-compact varieties, the authors use the functorial calculus on the Grothendieck groups of Saito's algebraic mixed Hodge modules.

## 3 Perverse sheaves

The known proofs of the Decomposition Theorem use in an essential way the theory of perverse sheaves.

Here are some of the highlights of the theory of perverse sheaves. Not all of what follows is necessary for an understanding of the Decomposition Theorem. The reader can consult [8, 107, 65]. Recall that we are dealing with $\mathbb{Q}$-coefficients and with middle-perversity only. Perverse sheaves arise naturally from the theory of D-modules (the solution sheaf of a regular holonomic D-module is perverse). Let $Y$ be a complex algebraic variety. Like the category of sheaves, the category $\mathcal{P}_Y$ of perverse sheaves is a full subcategory of the constructible derived category $\mathcal{D}_Y$, which is Abelian, Noetherian and Artinian, i.e. every perverse sheaf is a finite iterated extensions of simple perverse sheaves. The simple perverse sheaves are intersection complexes $IC_W(L)$ associated to an irreducible subvariety and a simple local system $L$ (on a Zariski open nonsingular subset of $W$). There are the notion of the ("perverse") kernel and ("perverse") cokernel of any morphism in $\mathcal{P}_Y$, and a notion of the ("perverse") cohomology of any complex $C \in \mathcal{D}_Y$. The bounded derived category of $\mathcal{P}_Y$ is again $\mathcal{D}_Y$. Many operations work better in the category of perverse sheaves than in the category of sheaves, e.g. the Duality and vanishing cycles functors preserve perverse sheaves. The Lefschetz Theorem on Hyperplane Sections holds for perverse sheaves. Specialization over a curve takes perverse sheaves to perverse sheaves. The intersection cohomology of a projective variety satisfies the Hodge and Lefschetz theorems.

## 3.1 Intersection cohomology

The interesection cohomology complex of $Y$ is a special case of a perverse sheaf and every perverse sheaf is a finite iterated extension of intersection complexes. It seems appropriate to start a discussion of perverse sheaves with this most important example.

Given a complex $n$ dimensional algebraic variety $Y$ and a local system $L$ on a nonsingular Zariski open subset $U$, there exists a constructible complex of sheaves $IC(L) \in \mathcal{D}_Y$, unique up to canonical isomorphism in $\mathcal{D}_Y$ such that $IC(L)|U \cong L$ and:

$$\dim_{\mathbb{C}} \left\{ y \in Y | \ \mathcal{H}^i_y(IC(L)) \neq 0 \right\} < -i, \text{ if } i > -n, \tag{20}$$

$$\dim_{\mathbb{C}} \left\{ y \in Y | \ \mathcal{H}^i_{c,y}(IC(L)) \neq 0 \right\} < -i, \text{ if } i > -n, \tag{21}$$

where, for any complex $S$ of sheaves,

$$\mathcal{H}^i_{c,y}(S) = \lim_{\leftarrow} H^i_c(U_y; S)$$

is the local *compactly supported* cohomology at $x$. (As explained in the "crash course" §1.5, the above limit is attained by any regular neighborhood $U_y$ of $y$.) The intersection complex $IC(L)$ is sometimes called the *middle extension* of $L$. Its (shifted) cohomology is the intersection cohomology $IH^{n+*}(Y, L) := H^*(Y, IC(L))$. The reader can consult the papers [83, 84] and the books [16, 65].

## 3.2 The Hodge-Lefschetz theorems for intersection cohomology

Even though Intersection cohomology lacks functoriality with respect to algebraic maps (however, see [3]), the intersection cohomology groups of projective varieties enjoy the same properties of Hodge-Lefschetz-Poincaré-type as projective manifolds. We employ the set-up of §1.1, with the difference that $X$ is allowed to be singular:

1. Poincaré Duality takes the form $IH^k(X) \simeq IH^{2n-j}(X)^\vee$ and follows formally from the canonical identification $IC(L) = (IC(L^\vee))^\vee$ stemming from Poincaré-Verdier Duality; in particular, there is a non degenerate geometric pairing

$$IH^i(X) \times IH^{2n-i}(X) \longrightarrow \mathbb{Q}, \qquad (a,b) \longmapsto a \cdot b;$$

2. the restriction map $IH^i(X) \to IH^i(D)$ is an isomorphism for $i < n-1$ and injective for $i = n-1$; see [6], Lemma 3.3 and [82];

3. the cup product map $c_1(H)^i : IH^{n-i}(X) \to IH^{n+i}(X)$ is an isomorphism; see [8, 144, 46];

4. there is a canonical Hodge structure on $IH(X)$; see [144, 46];

5. the Hodge-Riemann Bilinear relations hold, i.e. the signature properties of the bilinear form on $IH^{n-i}(X)$ given, via by $(a,b) \mapsto a \cdot (c_1(H)^i \cap b)$ are the same as in the nonsingular case; see [144, 46].

### 3.3 Examples of intersection cohomology

**Example 3.1** Let $E^{n-1} \subseteq \mathbb{P}^N$ be a projective manifold, $Y^n \subseteq \mathbb{A}^{N+1}$ be the associated affine cone. The link $L$ of $Y$ at the vertex $o$ of the cone, i.e. the intersection of $Y$ with a sufficiently small Euclidean ball centered at $o$, is an oriented compact smooth manifold of real dimension $2n-1$ and is a $S^1$-fibration over $E$. The cohomology groups of $L$ are

$$H_{2n-1-j}(L) = H^j(L) = P^j(E), \;\; 0 \leq j \leq n-1, \qquad H^{n-1+j}(L) = P^{n-j}(E), \;\; 0 \leq j \leq n.$$

where $P^j(E) \subseteq H^j(E)$ is the subspace of primitive vectors for the given embedding of $E$, i.e. the kernel of cupping with the appropriate power of the first Chern class of $\mathcal{O}_E(-E)$. The Poincaré intersection form on $L$ is non degenerate, as usual and also because of the Hodge-Riemann Bilinear Relations on $E$.
The intersection cohomology groups of $Y$ are

$$IH^j(Y) = P^j(E), \;\; 0 \leq j \leq n-1, \qquad IH^j(Y) = 0, \;\; n \leq j \leq 2n.$$

The intersection cohomology with compact supports of $Y$ are

$$IH_c^{2n-j}(Y) = H_j(L), \;\; 0 \leq j \leq n-1, \qquad IH_c^{2n-j}(Y) = 0, \;\; n \leq j \leq 2n.$$

Poincaré Duality holds: $IH^j(Y) = IH_c^{2n-j}(Y)^\vee$.

**Example 3.2** Let $Y$ be the projective cone over a nonsingular curve $C \subseteq \mathbb{P}^N$ of genus $g$. The cohomology groups are

$$H^0(Y) = \mathbb{Q}, \quad H^1(Y) = 0, \quad H^2(Y) = \mathbb{Q}, \quad H^3(Y) = \mathbb{Q}^{2g}, \quad H^4(Y) = \mathbb{Q}.$$

The intersection cohomology groups are:

$$IH^0(Y) = \mathbb{Q}, \quad IH^1(Y) = \mathbb{Q}^{2g}, \quad IH^2(Y) = \mathbb{Q}, \quad IH^3(Y) = \mathbb{Q}^{2g}, \quad IH^4(Y) = \mathbb{Q}.$$

Note the failure of Poincaré Duality in cohomology and its restoration via intersection cohomology. There is a canonical resolution $f : X \to Y$ of the singularities of $Y$ obtained by blowing up the vertex of $Y$. The Decomposition Theorem yields a splitting exact sequence of perverse sheaves on $Y$:

$$0 \longrightarrow IC_Y \longrightarrow f_*\mathbb{Q}_X[2] \longrightarrow H^2(C)[0] \longrightarrow 0.$$

**Example 3.3** Let $f : X \to Y$ be the space obtained by contracting to a point $v \in Y$, the zero section $C \subseteq \mathbb{P}^1 \times C =: X$. This example is analogous to the one in Example 3.2, except that the map $f$ is not a map of complex algebraic varieties. The cohomology groups are

$$H^0(Y) = \mathbb{Q}, \quad H^1(Y) = 0, \quad H^2(Y) = \mathbb{Q}, \quad H_3(Y) = \mathbb{Q}^{2g}, \quad H_4(Y) = \mathbb{Q}.$$

The intersection cohomology groups are:

$$IH^0(Y) = \mathbb{Q}, \quad IH^1(Y) = \mathbb{Q}^{2g}, \quad IH_2(Y) = 0, \quad IH^3(Y) = \mathbb{Q}^{2g}, \quad IH_4(Y) = \mathbb{Q}.$$

Note the failure of Poincaré Duality in cohomology and its restoration via intersection cohomology. There is a natural epimorphism of perverse sheaves $\tau : f_*\mathbb{Q}_X[2] \longrightarrow H^2(C)[0]$. There are non splitting exact sequences in $\mathcal{P}_Y$ :

$$0 \longrightarrow \operatorname{Ker}\tau \longrightarrow f_*\mathbb{Q}_X[2] \longrightarrow H^2(C)[0] \longrightarrow 0, \qquad 0 \longrightarrow IC_Y \longrightarrow \operatorname{Ker}\tau \longrightarrow \mathbb{Q}_v[0] \longrightarrow 0.$$

The complex $f_*\mathbb{Q}_X[2]$ is obtained by two-step-extension procedure. The intersection cohomology complexes $IC_Y$ and $\mathbb{Q}_v$ of $Y$ and $v \in Y$ appear in this process, but not as direct summands. The conclusion of the Decomposition Theorem does not hold for this map $f$.

**Example 3.4** Let $Y$ be the projective cone over the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \simeq Q \subseteq P^3$. The odd cohomology is trivial. The even cohomology is as follows:

$$H^0(Y) = 0, \quad H^2(Y) = \mathbb{Q}, \quad H^4(Y) = \mathbb{Q}^2, \quad H^6(Y) = \mathbb{Q}.$$

The intersection cohomology groups are the same as the cohomology groups, except that $IH^2(Y) = \mathbb{Q}^2$. Note the failure of Poincaré Duality in homology and its restoration via intersection homology. There are at least two different and interesting resolutions of the singularities of $Y$: the ordinary blow up of the vertex $o \in Y$ $f : X \to Y$ which has fiber $f^{-1}(o) \simeq Q$, and the blow up of any line on the cone through the origin $f' : X' \to Y$ which has fiber ${f'}^{-1}(o) \simeq \mathbb{P}^1$. The Decomposition Theorem yields (cf. Example 1.22)

$$f_*\mathbb{Q}_X[3] = IC_Y \oplus \mathbb{Q}_o[1] \oplus \mathbb{Q}_o[-1], \qquad f'_*\mathbb{Q}_{X'}[3] = IC_Y.$$

**Example 3.5** Let $E$ be the rank two local system on the punctured complex line $\mathbb{C}^*$ defined by the automorphism of $e_1 \mapsto e_1$, $e_2 \mapsto e_1 + e_2$. It fits into the non trivial extension

$$0 \longrightarrow \mathbb{Q}_{\mathbb{C}^*} \longrightarrow E \xrightarrow{\phi} \mathbb{Q}_{\mathbb{C}^*} \longrightarrow 0.$$

Note that $E$ is self-dual. If we shift the extension by [1] we have an exact sequence of perverse sheaves in $\mathcal{P}_{\mathbb{C}^*}$. Let $j : \mathbb{C}^* \to \mathbb{C}$ be the open immersion. The complex $IC_{\mathbb{C}}(E) = R^0 j_* E[1]$ is a single sheaf in cohomological degree $-1$ with generic stalk $\mathbb{Q}^2$ and stalk $\mathbb{Q}$ at the origin $0 \in \mathbb{C}$. In fact this stalk is given by the space of invariants which is spanned by the single vector $e_1$. There is the monic map $\mathbb{Q}_{\mathbb{C}}[1] \to IC_{\mathbb{C}}(E)$. The cokernel $K'$ is the nontrivial extension, unique since $\mathrm{Hom}(\mathbb{Q}_{\mathbb{C}}, \mathbb{Q}_{\{0\}}) = \mathbb{Q}$,

$$0 \longrightarrow \mathbb{Q}_{\{0\}} \longrightarrow K' \longrightarrow \mathbb{Q}_{\mathbb{C}}[1] \longrightarrow 0.$$

Note that while $IC_{\mathbb{C}}(E)$ has no subobjects and no quotients supported at $\{0\}$, it has a subquotient supported at $\{0\}$. We shall meet this example again later (Example 3.20) in the context of the non exactness of the intermediate extension functor.

**Example 3.6** Let $\Delta \subseteq \mathbb{C}^n$ be the subset $\Delta = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \prod x_i = 0\}$. The datum of $n$ commuting endomorphisms $T_1, \ldots, T_n$ of a $\mathbb{Q}$-vector space $V$ defines a local system $L$ on $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \Delta$ whose stalk at some base point $p$ is identified with $V$ and $T_i$ is the monodromy along the path "turning around the divisor $x_i = 0$." The vector space $V$ has a natural structure of $\mathbb{Z}^n = \pi_1((\mathbb{C}^*)^n, p)$-module. The complex which computes the group cohomology $H^\bullet(\mathbb{Z}^n, V)$ of $V$ can be described as follows: Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{Q}^n$, and, for $I = (i_0, \ldots, i_k)$, set $e_I = e_{i_0} \wedge \ldots \wedge e_{i_k}$. We define

$$C^k = \bigoplus_{0 < i_0 < \ldots < i_k < n} V \otimes e_I, \qquad D(v \otimes e_I) = \sum N_i(v) \otimes e_i \wedge e_I,$$

with $N_i := T_i - I$. Since $(\mathbb{C}^*)^n$ has no higher homotopy groups, we have the quasi isomorphism $(j_* L)_0 \overset{qis}{\simeq} (C^\bullet, D)$. Let

$$\widetilde{C^k} = \bigoplus_{0 < i_0 < \ldots < i_k < n} N_I V \otimes e_I,$$

where $N_I := N_{i_0} \circ \ldots \circ N_{i_k}$. It is clear that $(\widetilde{C^\bullet}, D)$ is a subcomplex of $(C^\bullet, D)$. It turns out that we have a quasi isomorphism $IC(L)_0 \overset{qis}{\to} (\widetilde{C^\bullet}, D)$. The particularly important case in which $L$ underlies a polarized variation of Hodge structures has been investigated in depth in [34] and [108].

## 3.4 Definition and first properties of perverse sheaves

Let $K \in \mathcal{D}_Y$. We say that $K$ satisfies the *support condition* if

$$\dim \mathrm{Supp}\, \mathcal{H}^{-l}(K) \leq l, \quad \forall\, l \in \mathbb{Z}.$$

We say that $K$ satisfies the *co-support condition* if the Verdier dual $K^\vee$ satisfies the conditions of support.

**Definition 3.7** A perverse sheaf on $Y$ is a complex $K$ in $\mathcal{D}_Y$ that satisfies the conditions of support and co-support.

Note that a complex $K$ is perverse iff $K^\vee$ is perverse. Note also that the defining conditions of intersection complex of §3.1 are a stricter versions of the (co-)support conditions. It follows at once that intersection complexes are special perverse sheaves.

Figure 1 below shows the support and cosupport conditions for Intersection cohomology

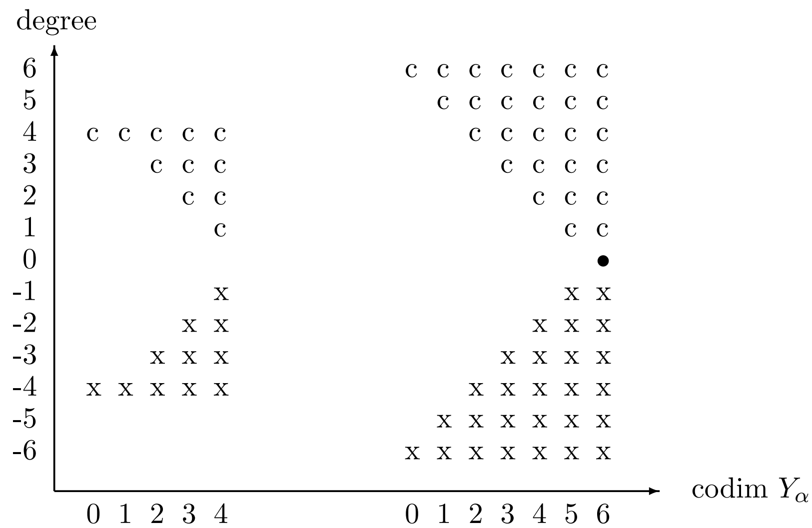


Figure 1: support conditions for $IC$ (left) and for a perverse sheaf (right)

on a variety of dimension 4 (left) and a perverse sheaf on a variety of dimension 6 (right). The symbol "c" means that compactly supported cohomology can be non-zero at that place, while the symbol "x" means that compactly supported cohomology can be non-zero at that place. Note that the $\bullet$ symbol shows that, for a perverse sheaf, there is a place at which both compactly supported and ordinary cohomology can be non-zero. The corresponding map $\mathcal{H}^i_{c,y}(\ ) \to \mathcal{H}^i_y(\ )$ rules the splitting behaviour of the perverse sheaf, as explained in §3.11.1.

Denote by $\mathcal{P}_Y$ the full subcategory of $\mathcal{D}_Y$ whose objects are perverse sheaves. Denote by ${}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y$ (${}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y$, resp.) the full subcategory of $\mathcal{D}_Y$ with objects the complexes satisfying the conditions of support (co-support, resp.). Clearly, ${}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y \cap {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y = \mathcal{P}_Y$.

**Theorem 3.8** *The datum of the conditions of (co)support together with the associated full subcategories* $({}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y, {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y)$ *yields a t-structure on* $\mathcal{D}_Y$*, called the middle perversity t-structure, with heart the category of perverse sheaves* $\mathcal{P}_Y$*.*

The resulting truncation and cohomology functors are denoted

$$ {}^{\mathfrak{p}}\tau_{\leq i}: \mathcal{D}_Y \longrightarrow {}^{\mathfrak{p}}\mathcal{D}^{\leq i}_Y, \quad {}^{\mathfrak{p}}\tau_{\geq i}: \mathcal{D}_Y \longrightarrow {}^{\mathfrak{p}}\mathcal{D}^{\geq i}_Y, \quad {}^{\mathfrak{p}}\mathcal{H}^0 = {}^{\mathfrak{p}}\tau_{\geq 0}\, {}^{\mathfrak{p}}\tau_{\leq 0},\ {}^{\mathfrak{p}}\mathcal{H}^i = {}^{\mathfrak{p}}\mathcal{H}^0 \circ [i]\ : \mathcal{D}_Y \longrightarrow \mathcal{P}_Y. $$

The key point is to show the existence of ${}^{\mathfrak{p}}\tau_{\geq 0}$ and ${}^{\mathfrak{p}}\tau_{\leq 0}$. The construction of these perverse truncation functors involves only the four functors $f^*, f_*, f_!, f^!$ for open and closed immersions and standard truncation. See [8], or [107]. Complete and brief summaries can be found in [47, 48].

Middle-perversity, is very well-behaved with respect to Verdier duality:

$$ {}^{\mathfrak{p}}\mathcal{H}^i \circ D = D \circ {}^{\mathfrak{p}}\mathcal{H}^{-i}, \qquad {}^{\mathfrak{p}}\tau_{\leq i} \circ D = D \circ {}^{\mathfrak{p}}\tau_{\geq -i}, \qquad {}^{\mathfrak{p}}\tau_{\geq i} \circ D = D \circ {}^{\mathfrak{p}}\tau_{\leq -i}. $$

Moreover, $D : \mathcal{P}_Y \to \mathcal{P}_Y$ is an equivalence.

The heart of the perverse $t$-structure is the category $\mathcal{P}_Y$ of perverse sheaves on $Y$. It is not difficult to show, by using the truncation functors, that $\mathcal{P}_Y$ is an abelian category. It is Noetherian so that, by Verdier Duality, it is also Artinian. Note that the category of constructible sheaves is Noetherian, but not Artinian.

Beilinson [6] has proved that the bounded derived category of perverse sheaves $D^b(\mathcal{P}_Y)$ is equivalent to $\mathcal{D}_Y$. Nori [141] has proved that the bounded derived category of the category of constructible sheaves on $Y$ is equivalent to $\mathcal{D}_Y$. This is an instance of the striking phenomenon that a category can arise as a derived category in fundamentally different ways.

Perverse sheaves, just like ordinary sheaves, form a stack ([8], 3.2). This is not the case for the objects and morphisms of $\mathcal{D}_Y$; e.g. a non trivial extension of vector bundles yields a morphism in the derived category that restricts to zero on the open sets of a suitable covering.

**Example 3.9** Let $Y$ be a point. The standard and perverse $t$-structure coincide. A complex $K \in \mathcal{D}_{pt}$ is perverse iff it is isomorphic in $\mathcal{D}_{pt}$ to a complex concentrated in degree zero iff $\mathcal{H}^j(K) = 0$ for every $j \neq 0$.

**Example 3.10** If $Y$ is a variety of dimension $n$, then the complex $\mathbb{Q}_Y[n]$ satisfies trivially the conditions of support. If $n = \dim Y = 0, 1$, then $\mathbb{Q}_Y[n]$ is perverse. On a surface $Y$ with isolated singularities, $\mathbb{Q}_Y[2]$ is perverse iff the singularity is unibranch, e.g. if the surface is normal. If $(Y, y)$ is a germ of a threefold isolated singularity, then $\mathbb{Q}_Y[3]$ is perverse iff the singularity is unibranch and $H^1(Y \setminus y) = 0$.

**Example 3.11** The direct image $f_*\mathbb{Q}_X[n]$ via a proper semismall map $f : X \to Y$, where $X$ is a nonsingular $n$-dimensional nonsingular variety, is perverse (cf. Proposition 2.11); e.g. a generically finite map of surfaces is semismall. For an interesting non splitting perverse sheaf arising from a non algebraic semismall map see Example 3.3.

Perverse sheaves are stable under the following functors: intermediate extension, nearby and vanishing cycle, see 3.9.

Let $i : Z \to Y$ be the closed immersion of a subvariety of $Y$. One has the functor $i_* : \mathcal{P}_Z \to \mathcal{P}_Y$. This functor is fully faithful, i.e. it induces a bijection on the Hom-sets. It is costumary, e.g. in the statement of the Decomposition Theorem, to drop the symbol $i_*$.

Let $Z$ be an irreducible closed subvariety of $Y$ and $L$ be a simple local system on a non-empty Zariski open subset of $Z_{reg}$. Recall that a simple object in an abelian category is one without trivial subobjects. The complex $IC_Z(L)$ is a simple object of the category

$\mathcal{P}_Y$. Conversely, every simple object of $\mathcal{P}_Y$ has this form. This follows from the following proposition [8], which yields a direct proof of the fact that $\mathcal{P}_Y$ is Artinian.

**Proposition 3.12** (**Composition series**) *Let $P \in \mathcal{P}_Y$. There is a finite decreasing filtration*

$$P = Q_1 \supseteq Q_2 \supseteq \ldots \supseteq Q_\lambda = 0,$$

*where the quotients $Q_i/Q_{i-1}$ are simple perverse sheaves on $Y$.*
*Every simple perverse sheaf if of the form $IC_{\overline{Z}}(L)$, where $Z \subseteq Y$ is an irreducible and nonsingular subvariety and $L$ is a simple local system on $Z$.*

As it is usual in this kind of situation, e.g. the Jordan-Hölder Theorem for finite groups, the filtration is not unique, but the constituents of $P$, i.e. the non trivial simple quotients, and the length $\lambda$ of $P$, i.e. the cardinality of the set of constituents, are uniquely determined.

## 3.5 The perverse filtration

The theory of $t$-structures (cf. [54], Appendix) endows the cohomology groups $H(Y,K)$ with a canonical filtration $P$, called the *perverse filtration*, $P^pH(Y,K) = \mathrm{Im}\,\{H(Y, {}^{\mathfrak{p}}\tau_{\leq -p}K) \to H(Y,K)\}$, which is the abutment of the *perverse spectral sequence* $H^{2p+q}(Y, {}^{\mathfrak{p}}\mathcal{H}^{-p}(K)) \Longrightarrow H^*(Y,K)$. Similarly, for cohomology with compact supports.

In [51], we give a geometric description of the perverse filtration on the cohomology and on the cohomology with compact supports of a constructible complex on a quasi projective variety. The description is in terms of restriction to generic hyperplane sections and it is somewhat unexpected, especially if one views the constructions leading to perverse sheaves as transcendental and hyperplane sections as more algebro-geometric.

Let $Y_* = \{Y \supseteq Y_{-1} \supseteq \ldots \supseteq Y_{-n}\}$ be a sequence of closed subvarieties; we call this data a $n$-flag. Basic sheaf theory endows $H(Y,K)$ with the flag filtration $F$, abutment of the spectral sequence $E_1^{p,q} = H^{p+q}(Y_p, Y_{p-1}, K_{|Y_p}) \Longrightarrow H^*(Y,K)$. We have $F^pH(Y,K) = \mathrm{Ker}\,\{H(Y,K) \to H(Y_{p-1}, K_{|Y_{p-1}})\}$.

For an arbitrary $n$-flag, the perverse and flag filtrations are unrelated. If $Y$ is *affine* of dimension $n$ and the $n$-flag is obtained using $n$ hyperplane sections in sufficiently general position, then

$$P^pH^j(Y,K) = F^{p+j}H^j(Y,K). \tag{22}$$

If $Y$ is quasi projective, there is an analogous description using two sufficiently general $n$-flags. If we have a map $f : X \to Y$ and $C \in \mathcal{D}_X$, then the corresponding spectral sequence and filtrations on $H(X,C) = H(Y, f_*C)$ are called perverse Leray and there is an analogous description of the perverse Leray filtration.

## 3.6 Perverse cohomology

There is the functor ${}^{\mathfrak{p}}\mathcal{H}^0 : \mathcal{D}_Y \to \mathcal{P}_Y$ sending a complex $K$ to its iterated truncation ${}^{\mathfrak{p}}\tau_{\leq 0}\,{}^{\mathfrak{p}}\tau_{\geq 0}K$. This functor is cohomological. In particular, given a triangle $K' \to K \to$

$K'' \to K'[1]$, one has a long exact sequence

$$\ldots \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^j(K') \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^j(K) \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^j(K'') \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^{j+1}(K') \longrightarrow \ldots$$

Kernels and cokernels in $\mathcal{P}_Y$ can be seen via perverse cohomology. Let $f : K \to K'$ be an arrow in $\mathcal{P}_Y$. View it in $\mathcal{D}_Y$, cone it and obtain a distinguished triangle

$$K \longrightarrow K' \longrightarrow \mathrm{Cone}(f) \xrightarrow{[1]} .$$

Take the associated long exact sequence of perverse cohomology

$$0 \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^{-1}(\mathrm{Cone}(f)) \longrightarrow K \xrightarrow{f} K' \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^0(\mathrm{Cone}(f)) \longrightarrow 0.$$

One verifies that $\mathcal{P}_Y$ is abelian by setting

$$\mathrm{Ker}\, a := {}^{\mathfrak{p}}\!\mathcal{H}^{-1}(\mathrm{Cone}(f)), \qquad \mathrm{Coker}\, a := {}^{\mathfrak{p}}\!\mathcal{H}^0(\mathrm{Cone}(f)).$$

**Example 3.13** Consider the natural map $a : \mathbb{Q}_Y[n] \to IC_Y$. Since $\mathbb{Q}_Y[n] \in {}^{p}\mathcal{D}_Y^{\leq 0}$, and $IC_Y$ does not admit non trivial subquotients, the long exact sequence splices-up as follows:

$${}^{\mathfrak{p}}\!\mathcal{H}^{l<0}(\mathrm{Cone}(a)) \simeq {}^{\mathfrak{p}}\!\mathcal{H}^{l<0}(\mathbb{Q}_Y[n]), \qquad 0 \to {}^{\mathfrak{p}}\!\mathcal{H}^0(\mathrm{Cone}(a)) \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^0(\mathbb{Q}_Y[n]) \longrightarrow IC_Y \to 0.$$

If $Y$ is a normal surface, then $\mathbb{Q}_Y[2]$ is perverse and we are left with the ses in $\mathcal{P}_Y$

$$0 \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^0(\mathrm{Cone}(a)) \longrightarrow \mathbb{Q}_Y[2] \xrightarrow{a} IC_Y \longrightarrow 0.$$

By taking the long exact sequence associated with $\mathcal{H}^j$, one sees that ${}^{\mathfrak{p}}\!\mathcal{H}^0(\mathrm{Cone}(a))$ reduces to a skyscraper sheaf supported at the singular points of $Y$ in cohomological degree zero and stalk computed by the cohomology of the link at $y$ : $\mathcal{H}^{-1}(IC_Y)_y = H^1(L_y)$. Note that the short exact sequence does not split, i.e. $\mathbb{Q}_X[2]$ is not a semisimple perverse sheaf.

**Example 3.14** (**Blowing up with smooth centers**) Let $X \to Y$ be the blowing up of a manifold $Y$ along a codimension $r+1$ submanifold $Z \subseteq Y$. One has an isomorphism in $\mathcal{D}_Y$ :

$$f_*\mathbb{Q}_X \simeq \mathbb{Q}_Y[0] \oplus \bigoplus_{j=1}^{r} \mathbb{Q}_Z[-2j].$$

If $r+1$ is odd (the even case is analogous and left to the reader), then

$${}^{\mathfrak{p}}\!\mathcal{H}^0(f_*\mathbb{Q}_X[n]) = \mathbb{Q}_Y[n], \qquad {}^{\mathfrak{p}}\!\mathcal{H}^j((f_*\mathbb{Q}_X[n]) = \mathbb{Q}_Z[\dim Z], \;\; 0 < |j| \leq r/2.$$

We have three sets of summands, i.e. $(j > 0, j = 0, j < 0)$. Poincaré-Verdier Duality exchanges the first and third sets and fixes the second. The Relative Hard Lefschetz Theorem identifies the first set with the third.

**Example 3.15** (**Smooth proper maps**) Let $f : X \to Y$ be a smooth projective morphism of relative dimension $d$. Deligne has proved that there is a direct sum decomposition in $\mathcal{D}_Y$:

$$f_* \mathbb{Q}_X \simeq \bigoplus_{j=0}^{2d} R^j f_* \mathbb{Q}_X[-j].$$

We have

$$ {}^{\mathfrak{p}}\!\mathcal{H}^j(f_* \mathbb{Q}_X[n]) = R^{d+j} f_* \mathbb{Q}_X[\dim Y], \quad j \in \mathbb{Z}.$$

Note again the simple form of the symmetries stemming from Duality and Hard Lefschetz.

## 3.7 $t$-exactness and the Lefschetz Hyperplane Theorem

The following prototypical Lefschetz-type result is a consequence of the left $t$-exactness of affine maps (cf. §5.2).

**Proposition 3.16** *Let $f : X \to Y$ be a proper map, $C \in {}^{\mathfrak{p}}\!\mathcal{D}^{\geq 0}_X$. Let $Z \subseteq X$ be a closed subvariety, $U := X \setminus Z$. There is the commutative diagram of maps*

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "h"'] & X \arrow[d, "f"] & Z \arrow[l, "i"'] \arrow[dl, "g"] \\ & Y. & \end{tikzcd}$$

*Assume that $h$ is affine.*
*Then*

$$ {}^{\mathfrak{p}}\!\mathcal{H}^j(f_* C) \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^j(g_* i^* C)$$

*is iso for $j \leq -2$ and monic for $j = -1$.*

*Proof.* By applying $f_! = f_*$ to the triangle $j_! j^! C \to C \to i_* i^* C \xrightarrow{[1]}$ we get the triangle

$$h_! j^* C \longrightarrow f_* C \longrightarrow g_* i^* C \xrightarrow{[1]} .$$

Since $h$ is affine, $h_!$ is left $t$-exact, so that

$$ {}^{\mathfrak{p}}\!\mathcal{H}^j(h_! j^* C) = 0 \qquad \forall j < 0.$$

The result follows by taking the long exact sequence of perverse cohomology. $\square$

It is now easy to give a proof of the Lefschetz Hyperplane Theorem in intersection cohomology.

**Theorem 3.17** (**Weak Lefschetz Theorem for intersection cohomology**) *Let $Y$ be an irreducible projective variety of dimension $n$ and $Z \subseteq Y$ be a general hyperplane section. The restriction*

$$IH^l(Y) \longrightarrow IH^l(Z) \qquad \text{is an isomorphism for } l \leq n-2 \text{ and monic for } l = n-1.$$

*Proof.* Apply Proposition 3.16 to the map to a point $f : Y \to pt$ with $C := IC_Y$. Since $Z$ is general, $i^* IC_Y[-1] = IC_Z$. $\square$

**Remark 3.18** One has the dual result for the Gysin map in the positive cohomological degree range. A similar conclusion [6], Lemma 3.3, holds, with the same proof, for any perverse sheaf on $Y$.

Another related special case of Proposition 3.16, used in [8] and in [46] as one step towards the proof of the Relative Hard Lefschetz Theorem, arises as follows. Let $\mathbb{P} \supseteq X' \to Y'$ be a proper map, $Z \subseteq X := X' \times \mathbb{P}^\vee$ be the universal hyperplane section, $Y := Y' \times \mathbb{P}^\vee$. Note that, by transversality, $i^* IC_X[-1] = IC_Z$. We have

**Theorem 3.19** (**Relative Lefschetz Hyperplane Theorem**) *The natural map*

$$ {}^{\mathfrak{p}}\mathcal{H}^j(f_* IC_X) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^{j+1}(g_* IC_Z) \quad \textit{is an isomorphism for } j \leq -2 \textit{ and monic for } j = -1. $$

## 3.8 Intermediate extensions

Let $j : U \to Y$ be a locally closed embedding $Y$ and $i : \overline{U} \setminus U =: Z \to Y$. Given a perverse sheaf $Q$ on $U$, the intermediate extension $j_{!*} : \mathcal{P}_U \to \mathcal{P}_{\overline{U}}$ is a simple operation that produces a distinguished perverse extension to $\overline{U}$ and hence to $Y$. Intersection cohomology can be viewed as an intermediate extension. A standard reference is [8].

Let $Q \in \mathcal{P}_U$. Consider the natural map $j_! Q \longrightarrow j_* Q$, and the map induced in perverse cohomology $a : {}^{\mathfrak{p}}\mathcal{H}^0(j_! Q) \to {}^{\mathfrak{p}}\mathcal{H}^0(j_* Q)$.

The *intermediate extension* of $Q \in \mathcal{P}_U$ is the perverse sheaf

$$ j_{!*} Q := \operatorname{Im}(a) \in \mathcal{P}_{\overline{U}} \subseteq \mathcal{P}_Y. $$

There is the canonical factorization in the abelian categories $\mathcal{P}_{\overline{U}} \subseteq \mathcal{P}_Y$

$$ {}^{\mathfrak{p}}\mathcal{H}^0(j_! Q) \xrightarrow{epic} j_{!*} Q \xrightarrow{monic} {}^{\mathfrak{p}}\mathcal{H}^0(j_* Q). $$

The intermediate extension $j_{!*} Q$ admits several useful characterizations. Namely:

1. it is the the unique extension of $Q \in \mathcal{P}_U$ to $\mathcal{P}_{\overline{U}} \subseteq \mathcal{P}_Y$ with no subobjects, nor quotients supported on $Z$;

2. it is the unique extension $X$ of $Q \in \mathcal{P}_U$ to $\mathcal{P}_{\overline{U}} \subseteq \mathcal{P}_Y$ such that $i^* X \in {}^{\mathfrak{p}}\mathcal{D}_Z^{\leq -1}$ and $i^! X \in {}^{\mathfrak{p}}\mathcal{D}_Z^{\geq 1}$;

There are an additional characterization and a precise formula (cf. [8], 2.1.9 and 2.1.11) both of which involve stratifications.

An intersection cohomology complex, being an intermediate extension, does not admit subobjects or quotients supported on proper subvarieties of its support.

The intermediate extension functor $j_{!*} : \mathcal{P}_U \to \mathcal{P}_Y$ is not exact in a funny way. Let $0 \to P \stackrel{a}{\to} Q \stackrel{b}{\to} R \to 0$ be exact in $\mathcal{P}_U$. Recall that $j_!$ is right $t$-exact and that $j_*$ is left $t$-exact. There is the display with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & j_!P \arrow[r] \arrow[d, "epic"] & j_!Q \arrow[r, "epic"] \arrow[d, "epic"] & j_!R \arrow[r] \arrow[d, "epic"] & 0 \\
 & j_{!*}P \arrow[d, "monic"] & j_{!*}Q \arrow[d, "monic"] & j_{!*}R \arrow[d, "monic"] & \\
0 \arrow[r] & j_*P \arrow[r, "monic"] & j_*Q \arrow[r] & j_*R \arrow[r] & \cdots
\end{tikzcd}$$

It is a simple diagram-chasing exercise to complete the middle row functorially with a necessarily monic $j_{!*}(a)$ and a necessarily epic $j_{!*}(b)$. It follows that the intermediate extension functor preserves monic and epic maps.
What fails is the exactness "in the middle:" in general $\operatorname{Ker} j_{!*}(b) / \operatorname{Im} j_{!*}(a) \neq 0$.

**Example 3.20** Let $E[1]$ be the perverse sheaf on $\mathbb{C}^*$ discussed in Example 3.5; recall that it fits in the non split short exact sequence of perverse sheaves:

$$0 \longrightarrow \mathbb{Q}[1] \stackrel{a}{\longrightarrow} E[1] \stackrel{b}{\longrightarrow} \mathbb{Q}[1] \longrightarrow 0.$$

Let $j : \mathbb{C}^* \to \mathbb{C}$ be the open immersion. We have the commutative diagram of perverse sheaves with exact top and bottom rows:

$$\begin{tikzcd}
0 \arrow[r] & j_!\mathbb{Q}[1] \arrow[r] \arrow[d, "epic"] & j_!E[1] \arrow[r] \arrow[d, "epic"] & j_!\mathbb{Q}[1] \arrow[r] \arrow[d, "epic"] & 0 \\
 & \mathbb{Q}_{\mathbb{C}}[1] \arrow[r, "monic", "j_{!*}(a)"'] \arrow[d, "monic"] & R^0j_*E[1] \arrow[r, "epic", "j_{!*}(b)"'] \arrow[d, "monic"] & \mathbb{Q}_{\mathbb{C}}[1] \arrow[d, "monic"] & \\
0 \arrow[r] & \mathbb{Q}_{\mathbb{C}}[1] \arrow[r] & j_*E[1] \arrow[r] & \mathbb{Q}_{\mathbb{C}}[1] \arrow[r] & 0.
\end{tikzcd}$$

The middle row, i.e. the one of middle extensions, is not exact in the middle. In fact, inspection of the stalks at the origin yields the non exact sequence

$$0 \longrightarrow \mathbb{Q} \stackrel{\simeq}{\longrightarrow} \mathbb{Q} \stackrel{0}{\longrightarrow} \mathbb{Q} \longrightarrow 0.$$

This failure prohibits exactness in the middle. The inclusion $\operatorname{Im} j_{!*}(a) \subseteq \operatorname{Ker} j_{!*}(b)$ is strict: $K := \operatorname{Ker} j_{!*}(b)$ is the unique non trivial extension, $\operatorname{Hom}(\mathbb{Q}_{\{0\}}, \mathbb{Q}_{\mathbb{C}}[2]) = \mathbb{Q}$,

$$0 \longrightarrow \mathbb{Q}_{\mathbb{C}}[1] \longrightarrow K \longrightarrow \mathbb{Q}_{\{0\}} \longrightarrow 0.$$

The reader can check, e.g. using the self-duality of $E$, that $K^\vee = K'$ ($K'$ as in Ex. 3.5).

Property 1., characterizing intermediate extensions, is used in the construction of composition series for perverse sheaves in Proposition 3.12. If follows that $j_{!*}Q$ is simple iff $Q$ is simple.

**Example 3.21** (**Intersection cohomology complexes with different supports**) Let $IC_{Z_i}(L_i)$, $i = 1, 2$ be intersection cohomology complexes with $Z_1 \neq Z_2$. Then

$$\mathrm{Hom}(IC_{Z_1}(L_1), IC_{Z_2}(L_2)) = 0.$$

In fact, the kernel (cokernel, resp.) of any such map would have to be either zero, or supported on $Z_1$ ($Z_2$, resp.), in which case, it is easy to conclude by looking at the supports.

Here is a nice application. Let $f : X \to Y$ be a proper and semismall map of irreducible proper varieties; see §2.2. The Decomposition Theorem yields a (canonical in this case) splitting

$$f_* IC_X = \bigoplus IC_{Z_a}(L_a).$$

Poincaré Duality on $IC_X$ yields a canonical isomorphism $e : f_*IC_X \simeq (f_*IC_X)^\vee$ which, by Example 3.21, $e$ is a direct sum map. It follows that the summands $IH(Z_a, L_a) \subseteq IH(X)$ are mutually orthogonal with respect to the Poincaré pairing.

## 3.9 Nearby and vanishing cycle functors

An important feature of perverse sheaves is their stability for the two functors $\Psi_f$, $\Phi_f$. These functors were defined in [53] in the context of étale cohomology as a generalization of the notion of vanishing cycle in the classical Picard-Lefschetz Theory. As it is explained in §3.11.2, they play a major role in the description of the possible extensions of a perverse sheaf through a principal divisor. We discuss these functors in the complex analytic setting. Let $f : X \to \mathbb{C}$ be a regular function and $X_0 \subseteq X$ be its divisor, that is $X_0 = f^{-1}(0)$. We are going to define functors $\Psi_f, \Phi_f : \mathcal{D}_X \to \mathcal{D}_{X_0}$ which send perverse sheaves on $X$ to perverse sheaves on $X_0$. We follow the convention for shifts employed in [107].

Let $e : \mathbb{C} \to \mathbb{C}$ be the map $e(\zeta) = \exp(2\pi\sqrt{-1}\zeta)$ and consider the following diagram

$$\begin{tikzcd}
X_\infty := X \times_e \mathbb{C} \arrow[r] \arrow[d] \arrow[rr, bend left, "p"] & X^* \arrow[r, "j"] \arrow[d] & X \arrow[d] & X_0 \arrow[l, "i"'] \arrow[d] \\
\mathbb{C} \arrow[r] \arrow[rr, bend right, "e"'] & \mathbb{C}^* \arrow[r] & \mathbb{C} & \{o\} \arrow[l]
\end{tikzcd}$$

For $K \in \mathcal{D}_X$, the nearby cycle functor $\Psi_f(K) \in \mathcal{D}_{X_0}$ is defined as:

$$\Psi_f(K) := i^* p_* p^* K.$$

Note that $\Psi_f(K)$ depends only on the restriction of $K$ to $X^*$. It can be shown that $\Psi_f(K)$ is constructible. Depending on the context, we shall consider $\Psi_f$ as a functor defined on $\mathcal{D}_X$, or on $\mathcal{D}_{X^*}$.

The group $\mathbb{Z}$ of deck transformations $\zeta \to \zeta + n$ acts on $X_\infty$ and therefore on $\Psi_f(K)$. We denote by $T : \Psi_f(K) \to \Psi_f(K)$ the positive generator of this action.

**Remark 3.22** (See [82], Section 6.13 for details.) Under mild hypothesis, for instance if $f$ is proper, there exists a continuous map $r : U \to X_0$ of a neighborhood of $X_0$, compatible with the stratification, whose restriction to $X_0$ is homotopic to the identity map. Denote by $r_\epsilon$ the restriction of $r$ to $f^{-1}(\epsilon)$, with $\epsilon \in \mathbb{C}$ small enough so that $f^{-1}(\epsilon) \subseteq U$. Then

$$r_{\epsilon *}(K) = \Psi_f(K)$$

In particular, let $x_0 \in X_0$, let $N$ be a neighborhood of $x_0$ contained in $U$ and let $\epsilon \in \mathbb{C}$ be as before. Then the cohomology sheaves of $\Psi_f(K)$ can be described as follows:

$$\mathcal{H}^i(\Psi_f(K))_{x_0} = H^i(N \cap f^{-1}(\epsilon), K_{|N \cap f^{-1}(\epsilon)}).$$

**Remark 3.23** Clearly, if $U \subseteq X$ is an open subset, then the restriction to $U$ of $\Psi_f(K)$ is the nearby cycle complex of the restriction $K_{|U}$ relative to the function $f_{|U}$ for $X \cap U$. On the other hand, explicit examples show that $\Psi_f(K)$ depends on $f$ and not only on the divisor $X_0$: the nearby functors associated with different defining equations of $X_0$ may differ. In particular, it is not possible to define the functor $\Psi_f$ if the divisor $X_0$ is only locally principal. Verdier has proposed in [158] an alternative functor, which he called the "specialization functor" $\mathrm{Sp}_{Y,X} : \mathcal{D}_X \to \mathcal{D}_{C_Y}$, associated with any closed imbedding $Y \to X$, where $C_Y$ is the normal cone of $Y$ in $X$. In the particular case that $Y$ is a locally principal divisor in $X$, the specialization functor is related to the nearby functor as follows: the normal cone $C_Y$ is a line bundle, and a local defining equation $f$ of $Y$ defined on an open set $V \subseteq X$ defines a section $s_f : Y \cap V \to C_{Y \cap V}$ trivializing the fibration. One has an isomorphism of functors $s_f^* \mathrm{Sp}_{Y,X} \simeq \Psi_f$.

**Example 3.24** Let $X = \mathbb{C}$ and $K$ be a local system on $\mathbb{C}^*$. Since the inverse image by $e$ of a disk centered at 0 is contractible, $\Psi_f(K)$ can be identified with the stalk at some base point $x_0$. The automorphism $T$ is just the monodromy of the local system.

The adjunction $K \to p_* p^* K$ gives a natural morphism $i^* K \to \Psi_f(K)$. The vanishing cycle complex $\Phi_f(K) \in \mathcal{D}_{X_0}$ fits in the following distinguished triangle:

$$i^* K \longrightarrow \Psi_f(K) \xrightarrow{\text{can}} \Phi_f(K)[1] \xrightarrow{[1]} . \tag{23}$$

This triangle determines $\Phi_f(K)$ only up to a non unique isomorphism. The definition of $\Phi_f$ as a functor requires more care, see [107]. The long exact sequence for the cohomology sheaves of this triangle, and Remark 3.22, show that

$$\mathcal{H}^i(\Phi_f(K))_{x_0} = H^i(N, N \cap f^{-1}(\epsilon), K).$$

Just as the nearby cycle functor, the vanishing cycle $\Phi_f(K)$ is endowed with an automorphism $T$.

We now list some of the properties of the functors $\Psi_f$ and $\Phi_f$:

1. The functors commute, up to a shift, with Verdier duality, see [99], and [27]:

$$\Psi_f(DK) = D\Psi_f(K)[2] \qquad \Phi_f(DK) = D\Phi_f(K)[2].$$

2. Dualizing the exact triangle (23) we get an exact triangle

$$i^!K \longrightarrow \Phi_f(K) \xrightarrow{\mathrm{var}} \Psi_f(K)[-1] \xrightarrow{[1]}, \tag{24}$$

with the property that

$$\mathrm{can} \circ \mathrm{var} = T - I : \Phi_f(K) \to \Phi_f(K) \qquad \mathrm{var} \circ \mathrm{can} = T - I : \Psi_f(K) \to \Psi_f(K),$$

and we have the fundamental octahedron of complexes of sheaves on $X_0$ :

$$\begin{tikzcd}
 & & i^*j_*j^*K \arrow[rrd] \arrow[lldd] & & \\
i^*K[1] \arrow[rru] \arrow[d, "{[1]}"'] & & & & i^!K[1] \arrow[llll, dashed] \arrow[lldd, dashed] \\
\Psi_f(K) \arrow[rrrr, "T-I"] \arrow[rrd, "\mathrm{can}"'] & & & & \Psi_f(K) \arrow[u] \arrow[lluu, "{[1]}"'] \\
 & & \Phi_f(K)[1] \arrow[rru, "\mathrm{var}[1]"'] \arrow[lluu, dashed] & &
\end{tikzcd}$$

3. If $K$ is a perverse sheaf on $X$, then $\Psi_f(K)[-1]$ and $\Phi_f(K)[-1]$ are perverse sheaf on $X_0$, see [82] 6.13, [8], [27], [99].

## 3.10 Unipotent nearby and vanishing cycle functors

Let $K$ be a perverse sheaf on $X \setminus X_0$. The map $j : X \setminus X_0 \to X$ is affine, so that $j_*K$ and $j_!K$ are perverse sheaves on $X$.

Let us consider the ascending chain of perverse subsheaves

$$\mathrm{Ker}\,\{(T-I)^N : \Psi_f(K)[-1] \to \Psi_f(K)[-1]\}.$$

For $N \gg 0$ this sequence stabilizes because of the Nöetherian property of the category of perverse sheaves. We call the resulting $T$-invariant perverse subsheaf the unipotent nearby cycle perverse sheaf associated with $K$ and we denote by $\Psi_f^u(K)$. In exactly the same way, it is possible to define the unipotent vanishing cycle functor $\Phi_f^u$ :

$$\Phi_f^u(K) = \mathrm{Ker}\,\{\,(T-I)^N : \Phi_f(K) \to \Phi_f(K)\,\}, \qquad \text{for } N \gg 0.$$

The perverse sheaves $\Psi_f(K)[-1]$ and $\Phi_f(K)[-1]$ are in fact the direct sum of $\Psi_f^u$ and another $T$-invariant subsheaf on which $(T-I)$ is invertible.

**Remark 3.25** The functor $\Psi_f(K)$ on a perverse sheaf $K$ can be reconstructed from $\Psi_f^u$ by applying this latter to the twists of $K$ with the pullback by $f$ of local systems on $\mathbb{C}^*$; see [6], p.47.

The cone of $(T-I) : \Psi_f(K) \to \Psi_f(K)$, which is isomorphic to $i^* j_* K$, is also isomorphic, up to a shift [1], to the cone of $(T-I) : \Psi_f^u(K) \to \Psi_f^u(K)$, and we still have the exact triangle

$$i^* j_* K \xrightarrow{[1]} \Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K) \longrightarrow$$

The long exact sequence of perverse cohomology introduced in §3.6 then gives

$${}^{\mathfrak{p}}\mathcal{H}^{-1}(i^* j_* K) = \mathrm{Ker}\,\{\, \Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K) \,\}$$

and

$${}^{\mathfrak{p}}\mathcal{H}^{0}(i^* j_* K) = \mathrm{Coker}\,\{\, \Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K) \,\}.$$

In turn, the long exact perverse cohomology sequence of the exact triangle

$$i^* j_* K \xrightarrow{[1]} j_! K \longrightarrow j_* K \longrightarrow$$

and the fact that $j_* K$ and $j_! K$ are perverse sheaves on $X$, give

$${}^{\mathfrak{p}}\mathcal{H}^{-1}(i^* j_* K) = \mathrm{Ker}\,\{\, j_! K \to j_* K \,\} = \mathrm{Ker}\,\{\, j_! K \to j_{!*} K \}.$$

and

$${}^{\mathfrak{p}}\mathcal{H}^{0}(i^* j_* K) = \mathrm{Coker}\,\{\, j_! K \to j_* K \,\} = \mathrm{Coker}\,\{\, j_{!*} K \to j_* K \,\}.$$

We thus obtain the useful formulæ

$$\mathrm{Ker}\,\{\, j_! K \to j_{!*} K \,\} \simeq \mathrm{Ker}\,\{\, \Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K) \,\}$$

$$\mathrm{Coker}\,\{\, j_{!*}(K) \to j_* K \,\} \simeq \mathrm{Coker}\,\{\, \Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K) \,\}.$$

**Remark 3.26** Let $N$ be a nilpotent endomorphism of an object $M$ of an abelian category. Suppose $N^{k+1} = 0$. By [58], 1.6, there exists a unique finite increasing filtration

$$M_\bullet : \{0\} \subseteq M_{-k} \subseteq \ldots \subseteq M_k = M$$

such that:

$$N M_l \subseteq M_{l-2} \ \text{ and } \ N^l : M_l / M_{l-1} \simeq M_{-l} / M_{-l-1}.$$

The filtration defined in this way by $T-I$ on $\Psi_f^u(K)$ is called the *monodromy weight filtration*. An important Theorem of Gabber, see Remark 4.9, characterizes this filtration in the case of $l$-adic perverse sheaves.

### 3.11 Two descriptions of the category of perverse sheaves

In this section we discuss two descriptions of the category of perverse sheaves on an algebraic variety. Although not strictly necessary for what follows, they play an important role in the theory and applications of perverse sheaves. The question is roughly as follows: suppose $X$ is an algebraic variety, $Y \subseteq X$ a subvariety, and we are given a perverse sheaf $K$ on $X \setminus Y$. How much information is needed to describe the perverse sheaves $\widetilde{K}$ on $X$ whose restriction to $X \setminus Y$ is isomorphic to K? We describe the approach developed by MacPherson and Vilonen [126] and the approach of Beilinson and Verdier [7, 159].

#### 3.11.1 The approach of MacPherson-Vilonen

We report on only a part of the description of the category of perverse sheaves developed in [126], i.e. the most elementary and the one which we find particularly illuminating.

Assume that $X = Y \coprod X \setminus Y$, where $Y$ is a closed and contractible $d$-dimensional stratum of a stratification $\Sigma$ of $X$. We have $\mathcal{P}_X^\Sigma$, i.e. the category of perverse sheaves on $X$ which are constructible with respect to $\Sigma$. Denote by $Y \xrightarrow{i} X \xleftarrow{j} X \setminus Y$ the corresponding imbeddings.

For $K \in \mathcal{P}_X^\Sigma$, the attaching triangle $i_! i^! K \longrightarrow K \longrightarrow j_* j^* K \xrightarrow{[1]}$, the support and cosupport conditions for a perverse sheaf, give the exact sequence of local systems on $Y$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^{-d-1}(i^*K) \arrow[r] & \mathcal{H}^{-d-1}(i^*j_*j^*K) \arrow[r] & \mathcal{H}^{-d}(i^!K) \arrow[d] \\ 0 & \mathcal{H}^{-d+1}(i^!K) \arrow[l] & \mathcal{H}^{-d}(i^*j_*j^*K) \arrow[l] & \mathcal{H}^{-d}(i^*K) \arrow[l] \end{tikzcd} \tag{25}$$

Note that the (trivial) local systems $\mathcal{H}^{-d-1}(i^*j_*j^*K), \mathcal{H}^{-d}(i^*j_*j^*K)$ are determined by the restriction of $K$ to $X \setminus Y$.

A first approximation to the category of perverse sheaves is given as follows:

**Definition 3.27** Let $\mathcal{P}'_X$ be the following category:
– the objects are given by: a perverse sheaf $K$ on $X \setminus Y$ constructible with respect to $\Sigma_{|X-Y}$, and an exact sequence

$$\mathcal{H}^{-d-1}(i^*j_*K) \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \mathcal{H}^{-d}(i^*j_*K)$$

of local systems on $Y$;
– given two objects $(K, \dots)$ $(L, \dots)$, the morphisms between them are defined to be morphisms of perverse sheaves $\phi : K \to L$ together with morphisms of exact sequences:

$$\begin{tikzcd} \mathcal{H}^{-d-1}(i^*j_*K) \arrow[r] \arrow[d, "\phi"] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & \mathcal{H}^{-d}(i^*j_*K)) \arrow[d, "\phi"] \\ \mathcal{H}^{-d-1}(i^*j_*L) \arrow[r] & W_1 \arrow[r] & W_2 \arrow[r] & \mathcal{H}^{-d}(i^*j_*L)). \end{tikzcd}$$

**Theorem 3.28** *The functor $\mathcal{P}_X^{\Sigma} \to \mathcal{P}'_X$, sending a perverse sheaf $\widetilde{K}$ on $X$ to its restriction to $X \setminus Y$ and to the exact sequence*

$$\mathcal{H}^{-d-1}(i^* j_* j^* \widetilde{K}) \to \mathcal{H}^{-d}(i^! \widetilde{K}) \to \mathcal{H}^{-d}(i^* \widetilde{K}) \to \mathcal{H}^{-d}(i^* j_* j^* \widetilde{K})$$

*is a bijection on isomorphism classes of objects.*

To give an idea why the theorem is true, we note that for any object $Q$ in $\mathcal{P}_X$, we have the triangle

$$i_! i^! Q \longrightarrow Q \longrightarrow j_* j^* Q \xrightarrow{[1]},$$

and $Q$ is identified by the extension map $e \in \mathrm{Hom}(j_* j^* Q, i_! i^! Q[1])$. We have $i_! = i_*$ hence

$$\mathrm{Hom}(j_* j^* Q, i_! i^! Q[1]) = \mathrm{Hom}(i^* j_* j^* Q, i^! Q[1]) = \oplus_l \mathrm{Hom}(\mathcal{H}^l(i^* j_* j^* Q), \mathcal{H}^{l+1}(i^! Q)).$$

The last equality is due to the fact that the derived category of complexes with constant cohomology sheaves on a contractible space is semisimple ($K \simeq \oplus H^i(K)[-i]$, for every $K$). By the support condition

$$\mathcal{H}^l(i^* j_* j^* Q) \simeq \mathcal{H}^{l+1}(i^! Q) \text{ for } l > -d.$$

By the co-support condition,

$$\mathcal{H}^l(i^! Q) = 0 \text{ for } l < -d.$$

There are the two maps

$$\mathcal{H}^{-d}(i^* j_* j^* Q) \longrightarrow \mathcal{H}^{-d+1}(i^! Q), \qquad \mathcal{H}^{-d-1}(i^* j_* j^* Q) \longrightarrow \mathcal{H}^{-d}(i^! Q)$$

which are not determined a priori by the restriction of $Q$ to $X \setminus Y$. They appear in the exact sequence (25) and contain the information about how to glue $j^* Q$ to $i^! Q$. The datum of this exact sequence makes it possible to reconstruct $Q \in \mathcal{P}_X$ satisfying the support and cosupport conditions.

Unfortunately the functor is not as precise on maps, as we will see. There are non zero maps between perverse sheaves which induce the zero map in $\mathcal{P}'_X$, i.e. the corresponding functor is not faithful. However, it is interesting to see a few examples of applications of this result.

**Example 3.29** Let $X = \mathbb{C}$, $Y = \{o\}$ with strata $X \setminus Y = \mathbb{C}^*$ and $Y$. A perverse sheaf on $\mathbb{C}^*$ is then of the form $L[1]$ for $L$ a local system. Let $L$ denote the stalk of $L$ at some base point, and $T : L \to L$ the monodromy. An explicit computation shows that

$$i^* j_* L[1] \simeq \mathrm{Ker}\,(T - I)[1] \oplus \mathrm{Coker}\,(T - I),$$

where $\mathrm{Ker}(T - I)$ and $\mathrm{Coker}(T - I)$ are interpreted as sheaves on $Y$. Hence a perverse sheaf is identified up to isomorphism by $L$ and by an exact sequence of vector spaces:

$$\mathrm{Ker}\,(T - I) \longrightarrow V_1 \to V_2 \longrightarrow \mathrm{Coker}\,(T - I).$$

A sheaf of the form $i_*V$ is represented by $L = 0$ and by the sequence

$$0 \longrightarrow V \xrightarrow{\simeq} V \longrightarrow 0.$$

Since $j$ is an affine imbedding, $j_*$ and $j_!$ are $t$-exact, i.e. $j_*L[1]$ and $j_!L[1]$ are perverse. The perverse sheaf $j_*L[1]$ is represented by

$$\mathrm{Ker}\,(T-I) \longrightarrow 0 \longrightarrow \mathrm{Coker}\,(T-I) \xrightarrow{Id} \mathrm{Coker}(T-I),$$

which expresses the fact that $i^!j_*L[1] = 0$.
Similarly $j_!L[1]$, which verifies $i^*j_!L[1] = 0$, is represented by

$$\mathrm{Ker}\,(T-I) \xrightarrow{Id} \mathrm{Ker}\,(T-I) \to 0 \longrightarrow \mathrm{Coker}\,(T-I).$$

The intermediate extension $j_{!*}L[1]$ is represented by

$$\mathrm{Ker}\,(T-I) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Coker}\,(T-I),$$

since, by its very definition,

$$\mathcal{H}^0(i^*j_{!*}L[1]) = \mathcal{H}^0(i^!j_{!*}L[1]) = 0.$$

Let us note another natural exact sequence given by

$$\mathrm{Ker}\,(T-I) \longrightarrow L \xrightarrow{T-I} L \longrightarrow \mathrm{Coker}\,(T-I).$$

which corresponds to Beilinson's maximal extension $\Xi(L)$, which will be described in the next section. From these presentations one sees easily the natural maps

$$j_!L[1] \longrightarrow j_{!*}L[1] \longrightarrow j_*L[1], \qquad \text{and} \qquad j_!L[1] \longrightarrow \Xi(L[1]) \longrightarrow j_*L[1].$$

**Remark 3.30** If $T$ has no eigenvalue equal to one, then the sequence has the form $0 \to V \to V \to 0$. This corresponds to the fact that a perverse sheaf which restricts to such a local system on $\mathbb{C} \setminus \{o\}$ is necessarily of the form $j_!L[1] \oplus i_*V$. Note also that $j_!L[1] = j_*L[1] = j_{!*}L[1]$.

**Remark 3.31** One can use Theorem 3.28 to deduce the following special case of a splitting criterion used in our proof of the Decomposition Theorem [46]:
*let $d = dimY$; a perverse sheaf $K \in \mathcal{P}_X$ splits as $K \simeq j_{!*}j^*K \oplus \mathcal{H}^{-d}(K)[d]$ if and only if the map $\mathcal{H}^{-d}(i^!K) \to \mathcal{H}^{-d}(i^*K)$ is an isomorphism.*
In fact, if this condition is verified, then the maps $\mathcal{H}^{-d-1}(i^*j_*j^*K) \to \mathcal{H}^{-d}(i^!K)$ and $\mathcal{H}^{-d}(i^*K) \to \mathcal{H}^{-d}(i^*j_*j^*K)$ in (25) vanish, and the exact sequence corresponding to $K$ is of the form

$$\begin{array}{ccccccc}
\mathcal{H}^{-d-1}(i^*j_*K) \longrightarrow 0 & \longrightarrow & 0 \longrightarrow \mathcal{H}^{-d}(i^*j_*K)) & & & & j_{!*}j^*K \\
 & \bigoplus & & & & & \\
 & & W \longrightarrow W & & & & \mathcal{H}^{-d}(K)[d].
\end{array}$$

The following example shows that the functor $\mathcal{P}_X^\Sigma \to \mathcal{P}'_X$ is not faithful. Consider the perverse sheaf $j_*\mathbb{Q}_{\mathbb{C}^*}[1]$. It has a non-split filtration by perverse sheaves

$$0 \to \mathbb{Q}_{\mathbb{C}}[1] \to j_*\mathbb{Q}_{\mathbb{C}^*}[1] \xrightarrow{\alpha} i_*\mathbb{Q}_0 \to 0$$

Dually, the perverse sheaf $j_!\mathbb{Q}[1]$ has a non-split filtration

$$0 \to i_*\mathbb{Q}_0 \xrightarrow{\beta} j_!\mathbb{Q}_{\mathbb{C}^*}[1] \to \mathbb{Q}_{\mathbb{C}}[1] \to 0.$$

The composition $\beta\alpha : j_*\mathbb{Q}_{\mathbb{C}^*}[1] \to j_!\mathbb{Q}_{\mathbb{C}^*}[1]$ is not zero, being the composition of the epimorphism $\alpha$ with the monomorphism $\beta$, however, it is zero on $\mathbb{C}^*$, and the map between the associated exact sequences is zero, since $i^!j_*\mathbb{Q}_{\mathbb{C}^*}[1] = 0$ and $i^*j_!\mathbb{Q}_{\mathbb{C}^*}[1] = 0$.

In the paper [126], MacPherson and Vilonen give a refinement of the construction which describes completely the category of perverse sheaves, both in the topological and complex analytic situation. The main idea here is that of *perverse link*. For an application to representation theory, see [130].

### 3.11.2 The approach of Beilinson and Verdier.

We turn to the Beilinson's approach [6], i.e. the one used by Saito in his theory of mixed Hodge modules. Beilinson approach is based on the functors $\Psi_f$ and $\Phi_f$ introduced in §3.9. In [160], Verdier obtained similar results using the specialization to the normal cone functor $\mathrm{Sp}_{Y,X}$, 3.23, which is not discussed here.

The assumption is that we have an algebraic map $f : X \to \mathbb{C}$ and $X_0 = f^{-1}(0)$ as in §3.9. We have the nearby and vanishing cycle functors $\Psi_f$ and $\Phi_f$. Let $K$ be a perverse sheaf on $X \setminus X_0$. Beilinson defines an interesting extension of $K$ to $X$ which he calls the maximal extension and denotes by $\Xi(K)$. It is a perverse sheaf restricting to $K$ on $X \setminus X_0$ and can be constructed as follows: we have the unipotent nearby and vanishing cycle functor $\Psi_f^u$ and $\Phi_f^u$ (see §3.10) and the triangle

$$i^*j_*K \xrightarrow{[1]} \Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K) \longrightarrow \cdot$$

The natural map $i^*j_*K \to \Psi_f^u(K)[1]$ defines, by adjunction, an element of

$$\mathrm{Hom}^1_{\mathcal{D}_{X_0}}(i^*j_*K, \Psi_f^u(K)) = \mathrm{Hom}^1_{\mathcal{D}_X}(j_*K, i_*\Psi_f^u(K))$$

which, in turn, defines an object $\Xi(K)$ fitting in the triangle

$$i_*\Psi_f^u(K) \longrightarrow \Xi(K) \longrightarrow j_*K \longrightarrow i_*\Psi_f^u(K)[1]. \tag{26}$$

Since $j$ is an affine morphism, it follows that $j_*K$ is perverse. The long exact sequence of perverse cohomology implies that $\Xi(K)$ is perverse as well.

Let us note that in [6] Beilinson gives a different construction of $\Xi(K)$ (and also of $\Psi_f^u(K)$ and $\Phi_f^u(K)$) which implies automatically that $\Xi$ is a functor and that it commutes with Verdier duality.

There is the exact sequence of perverse sheaves

$$0 \longrightarrow i_*\Psi_f^u(K) \xrightarrow{\beta_+} \Xi(K) \xrightarrow{\alpha_+} j_*K \longrightarrow 0$$

and, applying Verdier duality and the canonical isomorphisms $\Xi \circ D \simeq D \circ \Xi$ and $\Psi_f^u \circ D \simeq D \circ \Psi_f^u$,

$$0 \longrightarrow j_!K \xrightarrow{\alpha_-} \Xi(K) \xrightarrow{\beta_-} i_*\Psi_f^u(K) \longrightarrow 0.$$

The composition $\alpha_+\alpha_- : j_!K \to j_*K$ is the natural map, while $\beta_-\beta_+ : i_*\Psi_f^u(K) \to i_*\Psi_f^u(K)$ is $T - I$. We may now state Beilinson's results:

**Definition 3.32** Let $Gl(X,Y)$ be the category whose objects are quadruples $(K_U, V, u, v)$, where $K_U$ is a perverse sheaf on $U := X \setminus X_0$, $V$ is a perverse sheaf on $Y$, $u : \Psi_f^u(K) \to V$, and $v : V \to \Psi_f^u(K)$ such that $vu = T - I$.

**Theorem 3.33** *The functor $\gamma : F : \mathcal{P}_X \to Gl(X,Y)$ which associates to a perverse sheaf $K$ on $X$ the quadruple $(j^*K, \Phi_f^u(K), can, var)$ is an equivalence of categories. Its inverse is the functor $G : Gl(X,Y) \to \mathcal{P}_X$ associating to $(K_U, V, u, v)$ the cohomology of the complex*

$$\Psi_f^u(K_U) \xrightarrow{(\beta_+,u)} \Xi(K_U) \oplus V \xrightarrow{(\beta_-,v)} \Psi_f^u(K_U).$$

**Example 3.34** Given a perverse sheaf $K_U$ on $U = X \setminus X_0$, we determine

$$\gamma(j_!K_U) \longrightarrow \gamma(j_{!*}K_U) \longrightarrow \gamma(j_*K_U).$$

We make use of the triangles (23) and (24) discussed in §3.9 and restricted to the unipotent parts $\Psi_f^u$ and $\Phi_f^u$. Since $i^*j_!K_U = 0$, the map can : $\Psi_f^u(j_!K_U) \to \Phi_f^u(j_!K_U)$ is an isomorphism. Hence

$$\gamma(j_!K_U) = \Psi_f^u(K_U) \xrightarrow{\mathrm{Id}} \Psi_f^u(K_U) \xrightarrow{T-I} \Psi_f^u(K_U).$$

Similarly, since $i^!j_*K_U = 0$, the map $var : \Phi_f^u(j_*K_U) \longrightarrow \Psi_f^u(j_*K_U)$ is an isomorphism, and

$$\gamma(j_*K_U) = \Psi_f^u(K_U) \xrightarrow{T-I} \Psi_f^u(K_U) \xrightarrow{id} \Psi_f^u(K_U).$$

The canonical map $j_!K_U \to j_*K_U$ is represented by the following diagram, in which we do not indicate the identity maps:

$$\begin{tikzcd}
\gamma(j_!K_U) \arrow[d] & \Psi_f^u(K_U) \arrow[r] \arrow[d] & \Psi_f^u(K_U) \arrow[r, "T-I"] \arrow[d, "T-I"] & \Psi_f^u(K_U) \arrow[d] \\
\gamma(j_*K_U) & \Psi_f^u(K_U) \arrow[r, "T-I"] & \Psi_f^u(K_U) \arrow[r] & \Psi_f^u(K_U).
\end{tikzcd} \tag{27}$$

The intermediate extension $j_{!*}K_U$ corresponds to $j_{!*}K_U := \mathrm{Im}\{\, j_!K_U \to j_*K_U \,\}$, hence

$$\gamma(j_{!*}K_U) = \qquad \Psi^u_f(K_U) \xrightarrow{T-I} \mathrm{Im}(T-I) \hookrightarrow \Psi^u_f(K_U),$$

where the second map is the canonical inclusion. We can complete the diagram (27) as follows:

$$\begin{tikzcd}
\gamma(j_!K_U) \arrow[d] & \Psi^u_f(K_U) \arrow[r] \arrow[d] & \Psi^u_f(K_U) \arrow[r, "T-I"] \arrow[d, "T-I"] & \Psi^u_f(K_U) \arrow[d] \\
\gamma(j_{!*}K_U) \arrow[d] & \Psi^u_f(K_U) \arrow[r, "T-I"] \arrow[d] & \mathrm{Im}(T-I) \arrow[r, hook] \arrow[d] & \Psi^u_f(K_U) \arrow[d] \\
\gamma(j_*K_U) & \Psi^u_f(K_U) \arrow[r, "T-I"] & \Psi^u_f(K_U) \arrow[r] & \Psi^u_f(K_U).
\end{tikzcd} \tag{28}$$

The maximal extension $\Xi(K_U)$ is represented by the factorization

$$\Psi^u_f(K_U) \xrightarrow{(I,T-I)} \Psi^u_f(K_U) \oplus \Psi^u_f(K_U) \xrightarrow{p_2} \Psi^u_f(K_U)$$

where $p_2((a_1,a_2)) = a_2$ is the projection on the second factor.
Finally we note that if $L$ is a perverse sheaf on $X_0$, then, since $\Psi_f(i_*L) = 0$,

$$\gamma(i_*L) \;=\; 0 \longrightarrow L \longrightarrow 0.$$

**Remark 3.35** From the examples of $\gamma(j_{!*}K_U)$ and $\gamma(i_*L)$ discussed in Example 3.34, one can derive the following criterion (Lemme 5.1.4 in [144]) for a perverse sheaf $K$ on $X$ to split as $K \simeq j_{!*}j^*K \oplus i_*L$ :
*let $X$ be an algebraic variety and $X_0$ be a principal divisor; let $i : X_0 \to X \longleftarrow X \setminus X_0 : j$ be the corresponding closed and open imbeddings; a perverse sheaf $K$ on $X$ is of the form $K \simeq j_{!*}j^*K \oplus i_*L$ if and only if* $\Phi^u_f(K) = \mathrm{Im} : \Psi^u_f(K) \stackrel{can}{\to} \Phi^u_f(K) \bigoplus \mathrm{Ker} : \Phi^u_f(K) \stackrel{var}{\to} \Psi^u_f(K)$.
This criterion is used in [144] to establish the semisimplicity of certain perverse sheaves.

# 4 Three approaches to the Decomposition Theorem

## 4.1 The proof of Beilinson, Bernstein, Deligne and Gabber

The original proof [8] of the Decomposition Theorem uses the language of étale cohomology and the arithmetic properties of varieties defined over finite fields in an essential way.

In this section we try to introduce the reader to some of the main ideas in [8]. Here is a very brief and rough summary. There are the pure complexes on varieties over finite fields. They split over the algebraic closure as the shifted direct sum of their perverse cohomology complexes which, in turn, also split as direct sum of intersection cohomology complexes. Purity is preserved by the push-forward under a proper map. Gabber has proved that the intersection cohomology complex is pure. It follows that the Decomposition Theorem holds for $f_*IC_X$ at least after passing to the algebraic closure (of a finite field). The result is then lifted to characteristic zero by delicate "spreading-out" techniques.

Let us fix some notation. A variety over a field is a separated scheme of finite type over that field. Let $\mathbb{F}_q$ be a finite field, $\mathbb{F}$ be a fixed algebraic closure of $\mathbb{F}_q$ and $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ be the Galois group. This group is profinite, isomorphic to the profinite completion of $\mathbb{Z}$, and it admits as topological generator the geometric Frobenius $Fr := \varphi^{-1}$, where $\varphi : \mathbb{F} \to \mathbb{F}$, $t \mapsto t^q$ is the arithmetic Frobenius. Let $l \neq \mathrm{char}\,\mathbb{F}_q$ be a fixed prime number, $\mathbb{Z}_l$ be the ring of $l$-adic integers, i.e. the projective limit of the system $\mathbb{Z}/l^n\mathbb{Z}$ (abbreviated by $\mathbb{Z}/l^n$), $\mathbb{Q}_l$ be the $l$-adic numbers, i.e. the quotient field of $\mathbb{Z}_l$, and $\overline{\mathbb{Q}}_l$ be a fixed algebraic closure of $\mathbb{Q}_l$. Recall that $\mathbb{Z}_l$ is uncountable and that $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$, non canonically.

### 4.1.1 Constructible $\overline{\mathbb{Q}}_l$-sheaves

Let $X_0$ be a variety over a finite field $\mathbb{F}_q$. Unlike the case of complex varieties, one cannot expect local topological triviality of varieties and complexes with respect to a suitable stratification. However, one can still speak of constructible sheaves etc., and they still are "locally constant" (in a suitable sense) along "strata." We do not discuss this notion in detail and refer instead to [8]. There are the categories $D^b_c(X_0, \mathbb{Z}_l)$ of constructible complexes of $\mathbb{Z}_l$-adic sheaves and their variants for $\mathbb{Q}_l$, $E$ ($E$ a finite extension of $\mathbb{Q}_l$) and $\overline{\mathbb{Q}}_l$-adic sheaves. We need the variant $D^b_c(X_0, \overline{\mathbb{Q}}_l)$. All sheaves are assumed to be constructible. The construction of these categories requires a massive background. Let us try to give an idea of what these objects are. We start with the sheaves of sets with finite fibers for the étale topology on $X_0$. In the definition of sheaf as a contravariant functor from the category of open sets subject to the sheaf axioms, one replaces the category of Zariski open sets with the category of étale maps to $X_0$. Roughly speaking, an étale map is like a finite un-branched covering of an open subset. "Constructible" refers to the existence of a partition of $X_0$ so that the sheaf becomes locally constant on each part, i.e. constant on some étale covering of the part. One has the notion of sheaf of abelian groups and hence of sheaf of $\mathbb{Z}/m$-modules. There are enough injectives and one usually obtains finite abelian groups as cohomology groups. Giving an étale sheaf on the one-point variety $\mathrm{Spec}\,\mathbb{F}_q$ is the same as giving a finite discrete (in the sense of Serre) $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-module. A constructible $\mathbb{Z}_l$-adic sheaf is a special projective system $\{F_n\}_{n\geq 1}$ of constructible sheaves of $\mathbb{Z}/l^n$-modules. One does not take the projective limit sheaf, rather keeps the system and defines the cohomology of the $\mathbb{Z}_l$-adic sheaf to be the projective limit of the étale cohomology groups of the $F_n$. The resulting groups are $\mathbb{Z}_l$-modules, usually of finite type. As cohomology does not commute with projective limits, these goups are not the same as the étale cohomology groups of the projective limit sheaf, and this is good!

A $\mathbb{Q}_l$-sheaf is essentially just a $\mathbb{Z}_l$-sheaf, where the cohomology is defined by tensoring the cohomology $\mathbb{Z}_l$-modules above with $\mathbb{Q}_l$. One can repeat what above for any finite extension $E/\mathbb{Q}_l$ by replacing $\mathbb{Z}_l$ with the integral closure $Z_E$ of $\mathbb{Z}_l$ in $E$ and noting that $Z_E$ is local with maximal ideal $\mathfrak{m}$, residual characteristic $l$, so that one uses $Z_E/\mathfrak{m}^n$ instead of

$\mathbb{Z}/l^n$. One gets $Z_E$ and $E$-sheaves and cohomology.

A $\overline{\mathbb{Q}}_l$-sheaf is an object of the direct limit category over the system of categories of $E$-adic sheaves, as $E$ ranges over all finite extensions of $\mathbb{Q}_l$. A $\overline{\mathbb{Q}}_l$-sheaf is represented by an $E$-adic sheaf, for some $E$, and cohomology is defined by tensoring the cohomology of the $E$-sheaf with $\overline{\mathbb{Q}}_l$. Taking $\overline{\mathbb{Q}}_l$-sheaves to be $\mathbb{Q}_l$-sheaves with cohomology obtained by tensoring with $\overline{\mathbb{Q}}_l$ would not yield enough sheaves to compare, as we do in §4.1.6, with $\mathbb{C}$-coefficients perverse sheaves of geometric origin in the case of complex varieties.

A special role is played by the lisse $\overline{\mathbb{Q}}_l$-sheaves. They are the $\overline{\mathbb{Q}}_l$-analogue of local systems. A lisse $\mathbb{Z}_l$-sheaf is one for which the system $F_n$ is made of locally constant sheaves of $\mathbb{Z}/l^n$-modules. On a connected variety $X_0$, this corresponds to a continuous representation of the algebraic fundamental group into a finite type $\mathbb{Z}_l$-module. Continuity refers to the profinite topology on the group and to the $l$-adic topology on the module. The example of the lisse sheaf $\mathbb{Z}_l(1)$ on $\operatorname{Spec}\mathbb{F}_q$, given by the system $\mu_{l^n}$ of sheaves of $l^n$-roots of unity, shows that a lisse sheaf $\{F_n\}$ need not be constant on any étale covering, for the $F_n$ become constant only on bigger and bigger extensions of the field. Similarly, for $\mathbb{Q}_l$, $E$ and $\overline{\mathbb{Q}}_l$-sheaves. In the case of a $\mathbb{Q}_l$-sheaf of rank one on $\operatorname{Spec}\mathbb{F}_q$, keeping in mind that the Galois group is compact, continuity means that $Fr \in \operatorname{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts by units in $\mathbb{Z}_l$, i.e. there are restrictions on the representations arising in this context.

The categories $D^b_c(X_0, \mathbb{Z}_l)$ etc., are not actual derived categories. Their objects are special projective system of complexes and cohomology is defined by taking projective limits and, for $\mathbb{Q}_l$, $E$ and $\overline{\mathbb{Q}}_l$, by tensoring the result. One needs some homological restrictions on Tor groups in order to have a good theory.

Why do all this? With the ultimate goal of solving the Weil Conjectures (see [94]), Grothendieck introduced étale cohomology in order to produce a good cohomology theory with characteristic zero coefficients for algebraic varieties over finite fields. Given a complex projective manifold $X(\mathbb{C})$, one can take a defining set of complex polynomials and reduce the coefficients to a finite field $\mathbb{F}_q$. One gets a variety $X_0/\mathbb{F}_q$. The Weil Conjectures predict that the Betti numbers $b_i(X(\mathbb{C}))$ control in a precise way the number of points of $X_0$ defined over the finite extensions of $\mathbb{F}$, and viceversa.

The efforts made to attack these conjectures have deeply influenced the development of algebraic geometry and number theory. The Weil Conjectures have been solved by the efforts of several mathematicians. The last step was completed by Deligne [57].

Why consider $\mathbb{Z}_l$-sheaves? The Zariski topology typically yields trivial higher cohomology groups. Unfortunately, so does the étale topology if one uses the sheaves $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_l, \mathbb{Q}_l$. They give the "wrong" groups even for curves (cf. [72], p.118).
The étale cohomology groups with finite coefficients are well-behaved, at least if the torsion is coprime with the characteristic, but they are torsion and hence not suitable for counting. Note that $\mathbb{Z}$ not being an inverse limit in an interesting way, it is not possible to follow the procedure outlined above and produce a good étale cohomology theory with $\mathbb{Z}, \mathbb{Q}$-coefficients. Moreover, considerations of cardinality and Galois actions seem to prohibit the existence of a good theory with $\mathbb{Q}$-coefficients (cf. [129], p.8).

The categories $D^b_c(X_0, \overline{\mathbb{Q}}_l)$ are stable under the usual operations $f^*, f_*, f_!, f^!$, derived

Hom and tensor product, vanishing and nearby cycles, and Duality. Standard truncation has to be carefully defined (hence the aforementioned homological restrictions of Tor type).

One defines the middle perversity $t$-structure using the four functors $(f^*, f_*, f_!, f^!)$ and standard truncation and obtains the category $\mathcal{P}(X_0, \overline{\mathbb{Q}}_l)$ of perverse $\overline{\mathbb{Q}}_l$-sheaves on $X_0$. Recall that the theory of perverse sheaves, outlined in the complex case in §3, works equally well over any field. If $X$ is the $\mathbb{F}$-variety obtained from $X_0$ by extending the scalars to $\mathbb{F}$, then we have the category $D^b_c(X, \overline{\mathbb{Q}}_l)$ with the same stabilities and the category of perverse $\overline{\mathbb{Q}}_l$-sheaves $\mathcal{P}(X, \overline{\mathbb{Q}}_l)$.

The category of perverse sheaves is Noetherian and, with field coefficients, it is also Artinian: every object admits a finite filtration whose graded pieces are simple objects. The simple objects are intersection cohomology complexes $IC_{\overline{Z}_0}(L_0) = j_{!*}L_0[d]$ associated with an irreducible $d$-dimensional subvariety $j : Z_0 \to X_0$ for which $Z_{red}$ over $\mathbb{F}$ is smooth, and with an irreducible lisse $\overline{\mathbb{Q}}_l$-sheaf $L_0$ on $Z_0$. In particular, the simple graded pieces (constituents) of $P_0$, correspond to the constituents of $P$. However, the filtration for $P$ could split while the one for $P_0$ may fail to do so; see the important Remark 4.8.

### 4.1.2 Weights

We have just discussed the formalism of constructible sheaves in the étale context. In positive characteristic, this theory presents a feature which is absent in characteristic zero: weights, i.e. eigenvalues of Frobenius.

Let $X_0$ be a variety over the finite field $\mathbb{F}_q$. Suppression of the index $-_0$ denotes extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}$. For example, if $F_0$ is a $\overline{\mathbb{Q}}_l$-sheaf on $X_0$, then we denote its pull-back to $X$ by $F$.

To give a $\overline{\mathbb{Q}}_l$-sheaf $F_0$ on the one-point variety $\operatorname{Spec}\mathbb{F}_q$ is equivalent to giving a finite dimensional continuous $\overline{\mathbb{Q}}_l$-representation of the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. The pull-back $F$ to $\operatorname{Spec}\mathbb{F}$ is the sheaf given by the underlying $\overline{\mathbb{Q}}_l$-vector space of the representation. This is called the stalk of $F_0$ at the point.
Let $X_0(\mathbb{F}_q)$ be the finite set of closed points in $X_0$ which are defined over $\mathbb{F}_q$. This is precisely the set of closed points which is fixed under the action of the geometric Frobenius $Fr : X \to X$ which is a dense generator of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. Let $\mathbb{F}_q \subseteq \mathbb{F}_{q^n}$ be the usual degree $n$ extension. Let $X_0(\mathbb{F}_{q^n})$ be the finite set of closed points in $X_0$ which are defined over $\mathbb{F}_{q^n}$. It coincides with the fixed set for the $n$-th iterated geometric Frobenius $Fr^n$.
Let $x \in X_0(\mathbb{F}_{q^n})$. The $\overline{\mathbb{Q}}_l$-sheaf $F_0$ restricted to $x$ has stalk the $\overline{\mathbb{Q}}_l$-vector space $F_x$ on which $Fr^n$ acts as an automorphism.

**Definition 4.1** (**Punctually pure**) The $\overline{\mathbb{Q}}_l$-sheaf $F_0$ on $X_0$ is *punctually pure of weight* $w$ ($w \in \mathbb{Z}$) if, for every $x \in X_0(\mathbb{F}_{q^n})$, the eigenvalues of the action of $Fr^n$ on $F_x$ are algebraic numbers such that their complex algebraic conjugates have absolute value $q^{n\,w/2}$.

On $\operatorname{Spec}\mathbb{F}_q$, $\overline{\mathbb{Q}}_l$ has weights 0, while $\overline{\mathbb{Q}}_l(1)$ has weights $-2$.

It should be emphasized that while $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$, there is no natural isomorphism between them. However, since $\mathbb{Q} \subseteq \overline{\mathbb{Q}}_l$, it makes sense to request that the eigenvalues are algebraic.

Once the numbers are algebraic, the set of algebraic conjugates is well-defined independently of any isomorphism $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$ so that the request on the absolute value is meaningful. This is a strong request: $1+\sqrt{2}$ and $1-\sqrt{2}$ are algebraic conjugate, but have different absolute value In the case of smooth projective curves, the request is met for $H^i(X, \overline{\mathbb{Q}}_l)$, where $w = i$. The cases $i = 0, 2$ are elementary; the case $i = 1$ goes to the heart of the matter ([164]). See [94], Appendix C.2 for a discussion and references; see also [94], Ex. V.1.10.

**Definition 4.2** (**Mixed, weights**) A $\overline{\mathbb{Q}}_l$-sheaf $F_0$ on $X_0$ is *mixed* if it admits a finite filtration with punctually pure successive quotients. The *weights* of a mixed $F_0$ are the weights of the non-zero quotients.

**Definition 4.3** (**Mixed complexes**) The category $D^b_m(X_0, \overline{\mathbb{Q}}_l)$ of *mixed complexes* is the full subcategory of $D^b_c(X_0, \overline{\mathbb{Q}}_l)$ given by those complexes whose cohomology sheaves are mixed.

**Definition 4.4** (**Weights** $\leq, =, \geq$) One says that $K_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_l)$ *has weights* $\leq w$ if the cohomology sheaves $\mathcal{H}^i K_0$ are punctually pure of weigths $\leq w + i$. Denote by $D^b_{\leq w}(X_0, \overline{\mathbb{Q}}_l)$ the corresponding full subcategory.
One says that $K_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_l)$ *has weights* $\geq w$ if the Verdier dual $K_0^\vee$ has weights $\leq -w$. Denote by $D^b_{\geq w}(X_0, \overline{\mathbb{Q}}_l)$ the corresponding full subcategory.
One says that $K_0 \in D^b_m(X_0, \mathbb{Q}_l)$ is *pure of weight* $w$ if it has weights $\leq w$ and $\geq w$.

Remarkably, the categories $D^b_m(X_0, \overline{\mathbb{Q}}_l)$ are stable with respect to the usual operations $f^*, f_*, f_!, f^!$, derived Hom and tensor product, nearby and vanishing cycles, and Duality.

**Theorem 4.5** (**Stabilities: Relative Weil Conjectures**) *Let $f_0 : X_0 \to Y_0$ be a separated morphism of schemes of finite type over $\mathbb{F}_q$. Then*

$$f_{0!},\, f_0^* \;:\; D^b_{\leq w} \longrightarrow D^b_{\leq w}, \qquad\qquad f_0^!,\, f_{0*} \;:\; D^b_{\geq w} \longrightarrow D^b_{\geq w},$$

$$\otimes \;:\; D^b_{\leq w} \times D^b_{\leq w'} \longrightarrow D^b_{\leq w+w'}, \qquad\qquad R\mathcal{H}om \;:\; D^b_{\leq w} \times D^b_{\geq w'} \longrightarrow D^b_{\geq -w+w'},$$

*Verdier Duality exchanges $D^b_{\leq w}$ and $D^b_{\geq -w}$.*

*Proof.* See [58], 3.3.1, 6.2.3. □

If $f_0$ is proper, then $f_{0!} = f_{0*}$ and we have the important

**Corollary 4.6** (**Purity is stable for proper maps**) *Let $K_0$ be pure of weight $w$ and $f_0$ be proper. Then $f_{0*}K_0$ is pure of weight $w$.*

### 4.1.3 The structure of pure complexes

The Decomposition Theorem over a finite field and its algebraic closure is a statement about pure complexes. In this section we discuss the special splitting features of purity.

If $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_l)$, then the cohomology groups $H(X, K)$ on $X$ are finite dimensional $\overline{\mathbb{Q}}_l$-vector spaces with a continuous $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-action and one can speak about the weights of $H(X, K)$. Theorem 4.5 implies the following

**Corollary 4.7** (**Weights in cohomology**) *Let $K_0, L_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_l)$ have weights $\leq w$ and $\geq w'$, respectively. Then $\mathrm{Hom}^i(K, L)$ has weights $\geq i + w' - w$.*

**Remark 4.8** (**Killing extensions**) This Corollary becomes quite powerful when used in conjunction with the following exact sequences pertaining $K_0, L_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_l)$ and originating by a projective limit of spectral sequences of continuous Galois cohomology:

$$0 \longrightarrow \mathrm{Hom}^{i-1}(K, L)_{Fr} \longrightarrow \mathrm{Hom}^i(K_0, L_0) \longrightarrow \mathrm{Hom}^i(K, L)^{Fr} \longrightarrow 0. \tag{29}$$

A superscript means invariants (biggest invariant subspace), a lower-script means co-invariants (biggest quotient with trivial induced action). If $K_0$ and $L_0$ are as in Corollary 4.7 and $w = w'$, then $\mathrm{Hom}^1(K, L)^{Fr} = 0$ so that

$$\mathrm{Hom}^1(K_0, L_0) \xrightarrow{0} \mathrm{Hom}^1(K, L)$$

is the zero map. The upshot of (29) is the remarkable fact that, given the right weights, a non-trivial extension over $\mathbb{F}_q$ becomes trivial over $\mathbb{F}$.

The stabilities of Theorem 4.5 imply that $D^b_m(X_0, \overline{\mathbb{Q}}_l)$ inherits the middle perversity $t$-structure and we obtain the category $\mathcal{P}_m(X_0, \overline{\mathbb{Q}}_l)$ of *mixed perverse $\overline{\mathbb{Q}}_l$-sheaves.*

It is a fact that every mixed perverse $\overline{\mathbb{Q}}_l$-sheaf $P_0$ admits a canonical and functorial finite increasing filtration $W$ with quotients $Gr^W_i P_0$ perverse and pure of weights $i$.

**Remark 4.9** (**Canonical weigth filtration and monodromy weight filtration**) A Theorem of Gabber (see [9], §5) gives a characterization of the weight filtration in an important case. Let $S_0$ be a nonsingular curve over $\mathbb{F}_q$, $s \in S_0(\mathbb{F}_q)$, and $f : X_0 \to S_0$ be a map. Set $X_{s,0} = f^{-1}(s)$. As described in §3.9 over the complex field, we have the functors $\Psi_f, \Psi^u_f$, etc. After a Theorem of Grothendieck, for any perverse sheaf $K_0$ on $X_0 \setminus X_{s,0}$, there is a nilpotent map $N : \Psi^u_f(K_0) \to \Psi^u_f(K_0)(-1)$, the *logarithm of the monodromy* (see [58], 1.7.2 and 1.7.3). The corresponding filtration, as described in Remark 3.26, is the *monodromy filtration.* Given a mixed perverse sheaf $K_0$, its nearby functor $\Psi^u_f(K_0)$ is again mixed. When $K_0$ is pure, Gabber proved that the monodromy weight filtration of the mixed perverse sheaf $\Psi^u_f(K_0)$ equals, up to a renumbering, the canonical weight filtration. When $K_0$ is mixed, the relation between the weight filtration of $\Psi^u_f(K_0)$ and its monodromy filtration is more complicated, and involves the so called relative filtration, introduced in [58], Proposition 1.6.13.

While the notion of weight is not so well-behaved with respect to ordinary cohomology sheaves, one has the following

**Proposition 4.10** *The complex $K_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_l)$ has weights $\leq w$ (resp. $\geq w$, resp. $w$) iff the perverse cohomology complexes ${}^p\mathcal{H}^i(K_0)$ have weights $\leq w+i$ (resp. $\geq w+i$, resp. $w+i$).*

The following two lemmata show that pure complexes on $X_0$ are very special. They split completely when pulled back to $X$.

**Lemma 4.11** (**Purity and decomposition, I**) *Let $K_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_l)$ be pure of weight $w$. There is an isomorphism in $D^b_c(X, \overline{\mathbb{Q}}_l)$*

$$K \simeq \bigoplus_i {}^p\mathcal{H}^i(K)[-i].$$

*Proof.* Without loss of generality, we may assume that $w = 0$. By a simple induction using the perverse truncation distinguished triangles, we are reduced to the the case when ${}^p\mathcal{H}^i(K_0) = 0$ for all $i \neq -1, 0$. In this case we have the distinguished triangle

$${}^p\mathcal{H}^{-1}(K_0)[1] \longrightarrow K_0 \longrightarrow {}^p\mathcal{H}^0(K_0) \xrightarrow{[1]} .$$

By Proposition 4.10, ${}^p\mathcal{H}^{-1}(K_0)[1]$ and ${}^p\mathcal{H}^0(K_0)$ have weight 0. By Remark 4.8, the extension class in $\mathrm{Hom}^1({}^p\mathcal{H}^0(K_0), {}^p\mathcal{H}^{-1}(K_0)[1])$ vanishes when extending from $\mathbb{F}_q$ to $\mathbb{F}$. $\square$

One proves the following in a similar way. Recall that lisse $\overline{\mathbb{Q}}_l$-sheaves are the $\overline{\mathbb{Q}}_l$-analogue of local systems in the classical topology.

**Lemma 4.12** (**Purity and decomposition, II**) *Let $P_0 \in \mathcal{P}_m(X_0, \overline{\mathbb{Q}}_l)$ be a pure perverse $\overline{\mathbb{Q}}_l$-sheaf on $X_0$. The pull-back $P$ to $X$ splits in $\mathcal{P}(X, \overline{\mathbb{Q}}_l)$ as a direct sum of intersection cohomology complexes associated with lisse irreducible sheaves on subvarieties of $X$.*

*Proof.* [8], Théorème 5.3.8. $\square$

**Remark 4.13** Note that the pure perverse complex $P_0$ still splits according to supports into a direct sum of pure intersection cohomology complexes associated with lisse $\overline{\mathbb{Q}}_l$-sheaves; see [8], Corollaire 5.3.11. However, these pure lisse $\overline{\mathbb{Q}}_l$-sheaves do not necessarily split on $X_0$.

The following result of Gabber's shows the class of pure complexes contains the intersection cohomology complexes of pure lisse $\overline{\mathbb{Q}}_l$-sheaves. This result is complemented by Lemmata 4.11 and 4.12: pure complexes $K_0$ on $X_0$ split on $X$ as direct sums of shifts of intersection cohomology complexes arising from pure lisse sheaves on $X_0$. In general, they do not split on $X_0$.

**Theorem 4.14** (**Gabber's Purity Theorem**) *The intersection cohomology complex $IC_{X_0}$ of a connected d-dimensional variety $X_0$ is pure of weight d. More generally, if L is a pure lisse $\overline{\mathbb{Q}}_l$-sheaf of weight w on a connected, d-dimensional subvariety $j : Z_0 \to X_0$ then $IC_{\overline{Z}_0}(L) := j_{!*}L[d]$ is a pure perverse sheaf of weight $w + d$.*

*Proof.* Gabber proves this purity result in the unpublished [75]. Another proof is presented in [8] and is summarized in [26]. □

The following result generalizes Gabber's Purity Theorem and it is [8]'s key to the proof of the Decomposition Semisimplicity and Relative Hard Lefschetz Theorems over the complex numbers (see the very end of §4.1.6).

**Theorem 4.15** (**Mixed and simple is pure**) *Let $P_0 \in \mathcal{P}_m(X_0, \overline{\mathbb{Q}}_l)$ be a simple mixed perverse $\overline{\mathbb{Q}}_l$-sheaf. Then $P_0$ is pure.*

*Proof.* See [8], Cor. 5.3.4. □

### 4.1.4 The Decomposition Theorem over $\mathbb{F}$

Let $Z$ be a $\mathbb{F}$-variety and $K \in D^b_c(Z, \overline{\mathbb{Q}}_l)$. We say that we can lower the field of definition of the pair $(Z, K)$ to a finite subfield $\mathbb{F}_1 \subseteq \mathbb{F}$ if there are $(Z_1, K_1)$ defined over $\mathbb{F}_1$ such that $(Z, K)$ arises from $(Z_1, K_1)$ by extending the scalars from $\mathbb{F}_1$ to $\mathbb{F}$. In this case one can speak about weights for complexes, for cohomology etc. The resulting weights are well-defined for $(X, K)$, independently of $\mathbb{F}_1, X_1, K_1$, etc. One can always lower the field of definition of $Z$. In general, this is not possible for $K$, e.g. a one-dimensional representation with eigenvalue a non $l$-adic-unit.

**Theorem 4.16** (**Decomposition Theorem over** $\mathbb{F}$) *Let $f : X \to Y$ be a proper morphism of $\mathbb{F}$-varieties and $K \in D^b_c(X, \overline{\mathbb{Q}}_l)$. Assume that one can lower the field of definition for $(X, K)$ to a finite field and that the resulting $K_1$ is in $D^b_m(X_1, \overline{\mathbb{Q}}_l)$ and is pure on $X_1$. There is an isomorphism in $D^b_c(Y, \overline{\mathbb{Q}}_l)$*

$$f_*K \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(f_*K)[-i], \tag{30}$$

*where each ${}^{\mathfrak{p}}\mathcal{H}^i(f_*K)$ splits as a direct sum of intersection cohomology complexes associated with lisse irreducible sheaves on subvarieties of Y.*

*Proof.* Once the field of definition is lowered to a finite level to accommodate $(X, K)$, then we raise it, if necessary by means of a finite extension so that $f : X \to Y$ is defined over the resulting finite field. Purity is not lost in the process. We now apply Corollary 4.6 and Lemmata 4.11 and 4.12. □

**Remark 4.17** By virtue of the Deligne-Lefschetz splitting criterion [52] and the Relative Hard Lefschetz Theorem 4.18 below, the splitting (30) holds over $X_0$. However, the further splitting of the perverse cohomology complexes ${}^{\mathfrak{p}}\mathcal{H}^i(f_*K)_0$ may fail to hold over $X_0$.

Let $f_0 : X_0 \to Y_0$ be a morphism of $\mathbb{F}_0$-varieties, $\eta_0$ be the first Chern class of a line bundle $\eta_0$ on $X_0$. This defines a natural transformation $\eta_0 : f_{0*} \to f_{0*}[2](1)$, (here (1) is the Tate twist, lowering the weigths by two; the reader un-familiar with this notion, may ignore the twist and still get a good idea of the meaning of the statements) as well as its iterates $\eta_0^i : f_{0*} \to f_{0*}[2i](i)$, $i \geq 0$. In particular, it defines natural transformations $\eta_0^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_{0*}(-)) \to {}^{\mathfrak{p}}\mathcal{H}^{i}(f_{0*}(-))(i)$.

**Theorem 4.18** (**Relative Hard Lefschetz over** $\mathbb{F}_0$ **and** $\mathbb{F}$) *Let $P_0$ be a pure perverse sheaf on $X_0$. Assume that $f_0$ is projective and that the line bundle $\eta_0$ is $f$-ample. Then the iterated cup product operation induces isomorpisms*

$$\eta_0^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_{0*}P_0) \xrightarrow{\simeq} {}^{\mathfrak{p}}\mathcal{H}^{i}(f_{0*}P_0)(i), \qquad \forall i \geq 0.$$

*The same holds over $\mathbb{F}$ (with the understanding that $P$ should come from a $P_0$).*

*Proof.* See [8], Théorème 5.4.10. □

**Remark 4.19** The case $Y_0 = pt$, $P_0 = IC_{X_0}$, yields the Hard-Lefschetz Theorem for intersection cohomology (over $\mathbb{F}_0$ and over $\mathbb{F}$). Using the same technique "from $\mathbb{F}$ to $\mathbb{C}$" from [8], summarized in §4.1.6, one sees that Theorem 4.18 implies the Hard Lefschetz Theorem for the intersection cohomology of complex projective varieties.

The proof of the Relative Hard Lefschetz Theorem in [8] is analogous to the proof of the Hard Lefschetz Theorem for $H(X_0, \overline{\mathbb{Q}}_l)$, $X_0$ smooth and projective given in [58]: a) use a general hyperplane section and the Lefschetz Hyperplane Theorem to reduce to the case $i = 1$; b) use a Lefschetz pencil and the semisimplicity of the corresponding monodromy action to study the case $i = 1$. The differences in this case are: 1) one takes the universal hyperplane section instead of just a pencil; 2) one uses the Relative Lefschetz Hyperplane Theorem; 3) the relevant semisimple object here is a perverse sheaf. One needs to pass through $\mathbb{F}$ to ensure semisimplicity, but one can then show that if $\eta^i$ is an isomorphism, then so is $\eta_0^i$ (see Remark 4.8, for example).

### 4.1.5 The Decomposition Theorem for complex varieties

Let $X$ be a complex variety. Consider the categories $\mathcal{D}_X$ of bounded constructible complexes of sheaves of vector spaces and its full sub-category of complex perverse sheaves $\mathcal{P}_X$. Recall that every perverse sheaf admits a finite filtration with simple quotients called the constituents of the perverse sheaf.

**Definition 4.20** (**Perverse sheaves of geometric origin**) A perverse sheaf $P \in \mathcal{P}_X$ is said to be of *geometric origin* if it belongs to the smallest set such that

(a) it contains the constant sheaf $\mathbb{C}$ on a point,

and that is stable under the following operations

(b) for every map $f$, take the simple constituents of ${}^{\mathfrak{p}}\mathcal{H}^{i}(T(-))$, where $T = f^*, f_*, f_!, f^!$,
(c) take the simple constituents of ${}^{\mathfrak{p}}\mathcal{H}^{i}(- \otimes -)$, ${}^{\mathfrak{p}}\mathcal{H}^{i}(R\mathcal{H}om(-,-))$.

As a first example on a variety $Z$ one may start with the map $g : Z \to pt$, take $g^*\mathbb{C}_{pt} = \mathbb{C}_Z$, and set $P$ to be any simple constituent of one of the perverse complexes ${}^{\mathfrak{p}}\mathcal{H}^i(\mathbb{C}_Z)$. If $f : Z \to W$ is a map, one can take a simple constituent of ${}^{\mathfrak{p}}\mathcal{H}^j(f_*P)$ as an example on $W$. Another example consists of taking a simple local system of geometric origin $L$ on a connected and smooth Zariski open subvariety $j : U \to X$ and setting $P := j_{!*}L[\dim U]$. In particular, the intersection complex of a local system of geometric origin is of geometric origin. This includes the intersection cohomology complexes $IC_Y$, i.e. the case $L = \mathbb{C}_U$.

**Definition 4.21** (**Semisimple of geometric origin**) A perverse sheaf $P$ on $X$ is said to be *semisimple of geometric origin* if it is a direct sum of simple perverse sheaves of geometric origin. A complex $K \in \mathcal{D}_X$ is said to be *semisimple of geometric origin* if there is an isomorphism $K \simeq \oplus\, {}^{\mathfrak{p}}\mathcal{H}^i(K)[-i]$ in $\mathcal{D}_X$ and each perverse cohomology complex ${}^{\mathfrak{p}}\mathcal{H}^i(K)$ is semisimple of geometric origin.

We can now state the Decomposition Theorem and the Relative Hard Lefschetz Theorems as they are stated and proved in [8]. If $X$ is irreducible, then $IC_X$ is simple of geometric origin and then the two theorems apply.

**Theorem 4.22** (**Decomposition Theorem over** $\mathbb{C}$) *Let $f : X \to Y$ be a proper morphism of complex varieties. If $K \in \mathcal{D}_X$ is semisimple of geometric origin, then so is $f_*K$.*

**Theorem 4.23** (**Relative Hard Lefschetz Theorem over** $\mathbb{C}$) *Let $f : X \to Y$ be a projective morphism, $P$ a perverse sheaf on $X$ which is semisimple of geometric origin, $\eta$ the first Chern class of an $f$-ample line bundle on $X$. Then the iterated cup product operation induces isomorphism*

$$\eta^i \;:\; {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*P) \xrightarrow{\simeq} {}^{\mathfrak{p}}\mathcal{H}^i(f_*P), \qquad \forall\, i \geq 0.$$

*Proof.* See [8], Théorème 6.2.10. □

**Remark 4.24** Note that while the results are proved for sheaves of $\mathbb{C}$-vector spaces, one can deduce easily the variant for sheaves of $\mathbb{Q}$-vector spaces.

### 4.1.6 From $\mathbb{F}$ to $\mathbb{C}$

In this section we try to give an idea of how one can deduce Theorem 4.22, which is a result for sheaf cohomology on varieties over the complex numbers with the Euclidean topology, from Theorem 4.16, which is a result for the étale cohomology on varieties over the algebraic closure of a finite field.

The technique requires to "spread out" a finite amount of data over the complex numbers so that it is defined over a localization of a $\mathbb{Z}$-algebra of finite type $A \subseteq \mathbb{C}$. The spectrum $S = \operatorname{Spec} A$ has a generic point with residue field contained in $\mathbb{C}$. The closed points of $S$ have finite residue fields. We can view the data as a family of data varying over the base $S$. The fiber over the generic point carries what is essentially the initial data over $\mathbb{C}$. This initial data is related to the resulting data over the fibers over the closed points. A good analogy is the one of a flat fibration or even a fiber bundle. The initial data over $\mathbb{C}$ is in this way related to data over $\mathbb{F}_q$ and eventually over $\mathbb{F}$.

The root of this idea is the classical result that a finite system of rational polynomial equations has a solution over an algebraic number field if it has a solution modulo an infinite number of prime numbers. One can "visualize" the proof as follows. The system of equations gives a variety over $\mathbb{Q}$. This variety has a closed point iff there is a solution over a number field. The variety can be spread out over $\operatorname{Spec}\mathbb{Z}$ and the assumption implies that there is a closed point over infinitely many prime numbers. The set of solutions, spread out over $\operatorname{Spec}\mathbb{Z}$ maps dominantly over $\operatorname{Spec}\mathbb{Z}$. Since the spread-out variety has only finitely many components, one component must hit the generic point of $\operatorname{Spec}\mathbb{Z}$ which implies the conclusion.

There are several appearances of this technique in the literature, often in connection with a beautiful discovery. Here are few: Deligne-Mumford's proof [63] that the moduli space of curves of a given genus is irreducible in any characteristic, Mori's proof [135] of Hartshorne's conjecture, Deligne and Illusie's algebraic proof [61] of the Kodaira Vanishing Theorem and of the degeneration of the Hodge to de Rham spectral sequence (see the nice survey [100]).

The first step in the proof of Theorem 4.22 is the replacement of the complex coefficients in $\mathcal{D}_X$ with the isomorphic $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$, i.e. we replace $\mathcal{D}_X(\mathbb{C})$ with $\mathcal{D}_X(\overline{\mathbb{Q}}_l)$.

The category of constructible sheaves of finite sets for the Euclidean topology is equivalent to the one for the étale topology. This is essentially because the fundamental groups involved, the usal one and the algebraic one, act very similarly on finite sets. In fact, there is an equivalence of categories $D^b_c(X, \mathbb{Z}_l) \simeq \mathcal{D}_X(\mathbb{Z}_l)$. Passing to $\overline{\mathbb{Q}}_l$-coefficients we have a fully faithful embedding

$$\epsilon^* : D^b_c(X, \overline{\mathbb{Q}}_l) \longrightarrow \mathcal{D}_X(\overline{\mathbb{Q}}_l). \tag{31}$$

This embedding is not essentially surjective. Since $P$ is of geometric origin, $P$ is in the essential image. Hence, we may work with $D^b_c(X, \overline{\mathbb{Q}}_l)$ and $D^b_c(Y, \overline{\mathbb{Q}}_l)$. In fact, a splitting of $f_*P$ in $D^b_c(Y, \overline{\mathbb{Q}}_l)$ implies one in $\mathcal{D}_Y(\overline{\mathbb{Q}}_l)$, hence in $\mathcal{D}_Y(\mathbb{C})$. Summarizing, we left the Euclidean topology, are using the étale topology and are one step closer.

Let $P$ be a $\overline{\mathbb{Q}}_l$-adic perverse sheaf of geometric origin on the complex variety $X$. In what follows, we actually discuss the case of $\mathbb{Z}_l$-sheaves, but we keep the $\overline{\mathbb{Q}}_l$ notation. The variants for $\mathbb{Q}_l$, $E$ finite over $\mathbb{Q}_l$ and $\overline{\mathbb{Q}}_l$ are analogous.

By standard constructibility results, there exists a stratification $\mathcal{T}$ of $X$ and the datum $L$ of a finite family $L(T)$, one for each stratum $T$, of irreducible finite and free étale sheaves of $\mathbb{Z}/l$-modules on $T$ such that each cohomology $\overline{\mathbb{Q}}_l$-adic sheaf $\mathcal{H}^i(P)$ restricts to every stratum $T$ to a lisse $\overline{\mathbb{Q}}_l$-sheaf on $T$ such that its $\mathbb{Z}/l$-component, say $F_1$, is an étale sheaf of $\mathbb{Z}/l$-modules on $T$ which is a finite iterated extension of elements of $L(T)$. A Noetherian argument shows that then the same is true for $l^iF_n/l^{i+1}F_n$, for all $i$ and $n$. We shorten all this by saying that $P$ is $(\mathcal{T}, L)$-*constructible.*

This is a special case of what one means by an object of finite presentation on the complex variety $X$ and objects of finite presentation can be "spread-out." By definition, the object $P$ of geometric origin is obtained by a procedure which involves a finite repetition of certain operations involving a finite number of complex varieties and that starts with the constant sheaf $\overline{\mathbb{Q}}_l$ on a point.

**Example 4.25** Let $g : Z \to X$ be a morphism of complex quasi projective varieties. Consider $g_*\overline{\mathbb{Q}}_l$. We have an associated $(\mathcal{T}, L)$ on $X$. Take finitely many polynomial equations defining $g : Z \to X$, take their coefficients $a_i$ and form the algebra of finite type $A' = \mathbb{Z}[\{a_i\}]$. Let $S = \operatorname{Spec} A'$. We obtain a diagram of schemes

$$\begin{tikzcd} Z_S \arrow[rr, "f_S"] \arrow[dr] & & X_S \arrow[dl] \\ & S & \end{tikzcd}$$

which specializes to the given map $f$ when we extend the scalars from $A'$ to $\mathbb{C}$. All the closed points of $S$ have finite residue field. We can throw-in finitely more coefficients to include the equations defining the strata $T$. This procedure results in a collection $\mathcal{T}_S$. We shink $S$, by replacing $A'$ by $A := A'[1/f]$ where $f$ is a nonzero function vanishing at a suitable closed set, so that the parts $T_S$ are smooth over $S$. Finally, according to the Generic Base Change Theorem, we shrink $S$ further so that the Base Change Theorem holds over $S$ for the sheaf $\overline{\mathbb{Q}}_{l\,Z_S}$. We now collect the finite data $L_S$ for $f_{S*}\overline{\mathbb{Q}}_{l\,Z_S}$.

The procedure of Example 4.25 can be repeated as one performs the operations involving the definition of being of geometric origin.

Summarizing, we had a complex of geometric origin $P$ on $X$ and we produced one $P_S$ on $X_S$ that specializes back to $P$. We also found a pair $(\mathcal{T}_S, L_S)$ specializing back to the original one $(\mathcal{T}, L)$.

The point is that we can also specialize at a closed point, with some finite residue field $\mathbb{F}_q$ and we can also extend the scalars to an algebraic closure $\mathbb{F}$. We denote the end result by $(X_s, P_s, \mathcal{T}_s, L_s)$. To do so precisely requires an extra step involving Henselianization. We omit discussing this delicate point. The end result is that we have an equivalence of categories of $(\mathcal{T}, L)$-constructible complexes

$$D^b_c(X, \overline{\mathbb{Q}}_l)_{\mathcal{T},L} \simeq D^b_c(X_s, \overline{\mathbb{Q}}_l)_{\mathcal{T}_s,L_s}. \tag{32}$$

The validity of the Base Change Theorem over $S$ for the complexes involved is used as an ingredient to ensure that the Hom groups calculated in the two categories are the same.

Note that we could have carried $f : X \to Y$ along for the ride and, since our objective is to study $f_*P$, we could have thrown in the new necessary strata and sheaf data on $Y$.

We have now reduced the problem to the situation:

$$f_s : X_s \longrightarrow Y_s, \qquad P_s \in \mathcal{P}(X_s, \overline{\mathbb{Q}}_l)_{\mathcal{T}_s,L_s} \quad \text{(perverse and simple)}.$$

We need to relate $f_*P$ at the generic point, to $f_{s*}P_s$ at the chosen geometric closed point, via $P_S$ over $S$. This is another reason why we need to shrink $S$ in accordance to the Generic Base Change Theorem.

This situation arises from one over a finite field by extension of scalars: $f_0 : X_0 \to Y_0$, $P_0 \in \mathcal{P}(X_0, \overline{\mathbb{Q}}_l)$. The definition of geometric origin is through operations that commute with the equivalence involved. The stabilities of Theorem 4.5 and the definition of geometric origin ensure that $P_0$ is mixed: the starting point being the pure $\overline{\mathbb{Q}}_l$ over a point.

We claim that $P_0$ is pure. This is the key point. This object is the arrival point of an iterated procedure where each step is as follows: take a perverse $F_s$ coming from a pure $F_0$; apply an operation that produces a $F'_s$, coming from a mixed $F'_0$; take a simple constituent of $F'_s$, i.e. a simple subquotient $G'_s$; this simple subquotient corresponds to a simple constituent $G'_0$ of the mixed perverse sheaf $F'_0$; the subquotient $G'_0$ is the output of the step. The sub-claim is that $G'_0$ is pure: since $G'_0$ is a subquotient of the mixed perverse sheaf $F'_0$, it is mixed; the simple and mixed perverse $G'_0 \in \mathcal{P}_m(X_0, \overline{\mathbb{Q}}_l)$ is pure by Theorem 4.15. It follows that $P_0$ is pure.

Since $P_0$ is pure, we apply the Decomposition Theorem 4.16 over $\mathbb{F}$, the equivalence (32) and the fully faithful embedding (31) to conclude.

## 4.2 M. Saito's approach via mixed Hodge modules

The authors of [8] left open two questions: whether the Decomposition Theorem holds for the push forward of the intersection cohomology complex of a local system underlying a polarizable variation of pure Hodge structures and whether it holds in the Kähler context. See [8], p.165.

In his remarkable work on the subject, M. Saito has answered the first question in the affirmative in [144] and the second question in the affirmative in the case of $IC_X$ in [146]. In fact, he has developed in [145] a whole general theory of compatibility of mixed Hodge theory with the various functors and in the process has completed the extension of the Hodge-Lefschetz package to intersection cohomology.

There are at least two important new ideas in his work. The former is that the Hodge filtration is to be obtained by a filtration at level of $D$-modules. A precursor of this idea is Griffiths' filtration by the order of the pole. The latter is that the properties of his mixed Hodge modules are defined and tested using the vanishing cycle functor.

Saito's approach is deeply rooted in the theory of $D$-modules and, due to our ignorance on the subject, it will not be explained here. We refer to Saito's papers [144, 145, 146]. For a more detailed overview, see [28]. The papers [148] and [67] contain brief summaries of the results of the theory. See also [131].

Due to the importance of these results, we would like to discuss very informally Saito's achievements in the hope that even a very rough outline can be helpful to some. For simplicity only, we restrict ourselves to complex algebraic varieties.

Saito has constructed, for every variety $Y$, the abelian category $\mathrm{MHM}(Y)$ of mixed Hodge modules on $Y$. The construction is a tour-de-force which uses induction on dimension via a systematic use of the vanishing cycle functors associated with germs of holomorphic maps. It is in the derived category $D^b\mathrm{MHM}(Y)$ that Saito's results on mixed Hodge structures can be stated and proved. If one is interested only in the Decomposition and Relative Hard lefschetz Theorems, then it will suffice to work with the categories $MH(Y,w)$ below.

One starts with the abelian and semisimple category of polarizable Hodge modules of some weigth $MH(Y,w)$. Philosophically they correspond to perverse pure complexes in $\overline{\mathbb{Q}}_l$-adic theory. Recall that, on a smooth variety, the Riemann-Hilbert correspondence assigns to a regular holonomic $D$-module a perverse sheaf with complex coefficients. Roughly speaking, the simple objects are certain filtered regular holonomic $D$-modules $(\mathcal{M},F)$. The $D$-module $\mathcal{M}$ corresponds, via an extension of the Riemann-Hilbert correspondence to singular varieties, to the intersection cohomology complex of the complexification of a rational local system underlying a polarizable simple variation of pure Hodge structures of some weight (we omit the bookkeeping of weights).

Mixed Hodge modules correspond philosophically to perverse mixed complexes and are, roughly speaking, certain bifiltered regular holonomic $D$-modules $(\mathcal{M},W,F)$ with the property that the graded objects $Gr_i^W\mathcal{M}$ are polarizable Hodge modules of weight $i$. The resulting abelian category $\mathrm{MHM}(Y)$ is not semisimple. However, the extensions are not arbitrary, as they are controlled by the vanishing cycle functor. The extended Riemann-Hilbert correspondence assigns to the pair $(\mathcal{M},W)$ a filtered perverse sheaf $(P,W)$ and this data extends to a functor of $t$-categories

$$\mathfrak{r}\ :\ D^b(\mathrm{MHM}(Y))\longrightarrow\mathcal{D}_Y,$$

with the standard $t$-structure on $D^b(\mathrm{MHM}(Y))$ and the perverse $t$-structure on $\mathcal{D}_Y$. Beilinson's Equivalence Theorem [6], i.e. $\mathcal{D}_Y\simeq D^b(\mathcal{P}_Y)$, is used here, and in the rest of this theory, in an essential way.
In fact, there Ian second $t$-structure, say $\tau'$, on $D^b(\mathrm{MHM}(Y))$ corresponding to the standard one on $\mathcal{D}_Y$; see [145], Remarks 4.6.

The usual operations on $D$-modules induce a collection of operations on $D^b(\mathrm{MHM}(Y))$ that correspond to the usual operations on the categories $\mathcal{D}_Y$, i.e. $f^*,f_*,f_!,f^!$, tensor products, Hom, Verdier Duality, nearby and vanishing cycle functors (cf. [145], Th. 0.1).

In the case when $Y$ is a point, the category $\mathrm{MHM}(pt)$ is naturally equivalent to the category of graded polarizable rational mixed Hodge structures (cf. [145], p.319); here "graded" means that one has polarizations on the graded pieces of the weigth filtration. At the end of the day, the $W$ and $F$ filtrations produce two filtrations on the cohomology and on the cohomology with compact supports of a complex in the image of $\mathfrak{r}$ and give rise to mixed Hodge structures compatible with the usual operations. Note that the functor $\mathfrak{r}$ is exact and faithful, but not fully faithful (the map on Hom sets is injective, but not surjective), not even over a point: in fact, a pure Hodge structure of weigth 1 and rank 2, e.g. $H^1$ of an elliptic curve, is irreducible as a Hodge structure, but not as a vector space.

The constant sheaf $\mathbb{Q}_Y$ is in the image of the functor $\mathfrak{r}$ and Saito's theory recovers Deligne's functorial mixed Hodge theory of complex varieties [55, 56]. See [145], p. 328 and [147], Corollary 4.3.

As mentioned above, mixed Hodge modules are a Hodge-theoretic analogue of the arithmetic mixed perverse sheaves discussed in §4.1. A mixed Hodge module $(\mathcal{M}, W, F) \in MHM(Y)$ is said to be pure of weight $k$ if $Gr_i^W \mathcal{M} = 0$, for all $i \neq k$. In this case it is, by definition, a polarizable Hodge module so that a mixed Hodge module which is of some pure weight is analogous to an arithmeitc pure perverse sheaf.

Saito proves the analogue of the arithmetic Corollary 4.6, i.e. that if $f$ is proper, then $f_*$ preserves weights. Though the context and the details are vastly different, the rest of the story unfolds by analogy with the arithmetic case discussed in §4.1. A complex in $D^b(\mathrm{MHM}(Y))$ is said to be semisimple if it is a direct sum of shifted mixed Hodge modules which are simple and pure of some weight (= polarizable Hodge modules, i.e. associated with a simple variation of polarizable pure Hodge structures).

In what follows, note that the faithful functor $\mathfrak{r}$ commutes, up to natural equivalence, with the usual operations, e.g. $\mathfrak{r}(\mathcal{H}^j(M)) = {}^{\mathfrak{p}}\mathcal{H}^j(\mathfrak{r}(M))$, $f_*(\mathfrak{r}(M)) = \mathfrak{r}(f_*(M))$.

**Theorem 4.26** (**Decomposition Theorem for polarizable Hodge modules**) *Let $f : X \to Y$ be proper and $M \in D^b(MHM(X))$ be semisimple.*
*The direct image $f_*M \in D^b(MHM(Y))$ is semisimple. More precisely, if $M \in MHM(X)$ is semisimple and pure, then*

$$f_*M \simeq \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^j(f_*M)[-j]$$

*where the $\mathcal{H}^j(f_*M) \in MHM(Y)$ are semisimple and pure.*

**Theorem 4.27** (**Relative Hard Lefschetz for polarizable Hodge modules**) *Let $f : X \to Y$ be projective, $M \in MHM(X)$ be semisimple and pure and $\eta \in H^2(X, \mathbb{Q})$ be the first Chern class of an $f$-ample line bundle on $X$.*
*The iterated cup product map is an isomorphism*

$$\eta^j : \mathcal{H}^{-j}(f_*M) \xrightarrow{\simeq} \mathcal{H}^j(f_*M)$$

*of semisimple and pure mixed Hodge modules.*

The proof relies on an inductive use, via Lefschetz pencils, of Zucker's [167] results on Hodge theory for degenerating coefficients in one variable.

The intersection cohomology complex of a polarizable variation of pure Hodge structures is the perverse sheaf associated with a pure mixed Hodge module (= polarizable Hodge module). This fact is not as automatic as in the case of the constant sheaf, for it requires the verification of the conditions of vanishing-cycle-functor-type involved in the definition of the category of polarizable Hodge modules. One may view this fact as the analogue of Gabber's Purity Theorem 4.14.

M. Saito thus establishes the Decomposition and the Relative Hard Lefschetz Theorems for coefficients in the intersection cohomology complex $IC_X(L)$ of a polarizable variation of pure Hodge structures, with the additional fact that one has mixed Hodge structures on the cohomology of the summands on $Y$ and that the (non-canonical) splittings on the intersection cohomology group $IH(X, L)$ are compatible with the mixed Hodge structures of the summands. He has also established the Hard Lefschetz Theorem and the Hodge Riemann Bilinear Relations for the intersection cohomology groups of projective varieties.

Saito's results complete the verification of the Hodge-Lefschetz package for the intersection cohomology groups of a variety $Y$, thus yielding the wanted generalization of the classical results in §1.1 to singular varieties.

The perverse and the standard truncations in $\mathcal{D}_Y$ correspond to the standard and to the above-mentioned $\tau'$ truncations in $D^b(\mathrm{MHM}(Y))$, respectively. See [145], p. 224 and Remarks 4.6. It follows that the following spectral sequences associated with complexes $K \in \mathfrak{r}(D^b(\mathrm{MHM}(Y))) \subseteq \mathcal{D}_Y$ are spectral sequences of mixed Hodge structures:

1) the perverse spectral sequence;
2) the Grothendieck spectral sequence;
3) the perverse Leray spectral sequence associated with a map $f : X \to Y$
4) the Leray spectral sequence associated with a map $f : X \to Y$.

**Remark 4.28** C. Sabbah, [143] and T. Mochizuki [134] have extended the range of applicability of the Decomposition Theorem to the case of intersection cohomology complexes associated with semisimple local systems on quasi-projective varieties. They use, among other ideas, M. Saito's $D$-modules approach.

## 4.3 A proof via classical Hodge theory

Our paper [46] gives a geometric proof of the Decomposition Theorem for the push forward $f_*IC_X$ of the intersection cohomology complex via a proper map $f : X \to Y$ of complex algebraic varieties, and complements it with a series of Hodge-theoretic results. Some of these results have been obtained much earlier and in greater generality by M. Saito in [144, 145]. While our approach uses heavily the theory of perverse sheaves, it ultimately rests on classical and mixed Hodge theory. It is geometric in the sense that:

a. we identify the refined intersection forms in the fibres of the map as the agent responsible for the splitting behavior of $f_*IC_X$ and

b. we give a geometric interpretation of the perverse filtration.

In the following two sections we list the results contained in [46] and give an outline of the proofs in the key special case of a projective map $f : X \to Y$ of irreducible projective varieties with $X$ nonsingular of dimension $n$. We denote with the same symbol a line bundle, its first Chern class and the operation of cupping with it. We choose an ample line bundle $\eta$ on $X$, an ample line bundle $L$ on $Y$, and set $L' := f^*L$.

### 4.3.1 The results

1. (**Decomposition Theorem**) $f_*\mathbb{Q}_X[n]$ splits as a direct sum of shifted intersection cohomology complexes with twisted coefficients on subvarieties of $Y$ (cf. §1.6.(6).(7)).

2. (**Semisimplicity Theorem**) The summands are semisimple, i.e. the local systems (7) giving the twisted coefficients are semisimple. They are described below, following the Refined Intersection Form Theorem.

3. (**Relative Hard Lefschetz Theorem**) Cupping with $\eta$ yields isomorphisms

$$\eta^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) \simeq {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n]), \qquad \forall\, i \geq 0.$$

4. (**Hodge Structure Theorem**) The perverse $t$-structure yields the perverse filtration

$$P^p H(X) = \operatorname{Im}\{H(Y, {}^{\mathfrak{p}}\tau_{\leq -p} f_*\mathbb{Q}_X[n]) \to H(Y, f_*\mathbb{Q}_X[n])\}$$

on the cohomology groups $H(X)$. This filtration is by Hodge substructures and the perverse cohomology groups

$$H^{a-n}(Y, {}^{\mathfrak{p}}\mathcal{H}^b(f_*\mathbb{Q}_X[n]) \simeq P^{-b}H^a(X)/P^{-b+1}H^a(X) = H^a_b(X)$$

i.e. the graded groups of the perverse filtration, inherit a pure Hodge structure.

5. (**Hard Lefschetz Theorems for Perverse Cohomology Groups**) The collection of perverse cohomology groups $H^*(Y, {}^{\mathfrak{p}}\mathcal{H}^*(f_*\mathbb{Q}_X[n])$ satisfy the conclusion of the Hard Lefschetz Theorem with respect to cupping with $\eta$ on $X$ and with respect to cupping with an $L$ on $Y$, namely:

   The cup product with $\eta^i : H^*(Y, {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n])) \to H^{*+2i}(Y, {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n]))$ is an isomorphism for all $i \geq 0$.

   The cup product with $L^l : H^{-l}(Y, {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n])) \to H^l(Y, {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n]))$ is an isomorphism for all $l \geq 0$ and all $i$.

   As a consequence, we have the following characterization of the perverse filtration (where it is understood that a linear map with a non-positive exponent is defined to be the identity and that kernels and images are inside of $H^r(X)$):

$$P^p H^r(X) \;=\; \sum_{a+b=n-(p+r)} \operatorname{Ker} L'^{a+1} \cap \operatorname{Im} L'^{-b}.$$

6. (**Generalized Hodge-Riemann Bilinear Relations**) The Hard Lefschetz Theorems for Perverse Cohomology Groups yield a two-variable analogue of the primitive Lefschetz Decomposition, the $(\eta, L)$-decompostion, of the perverse cohomology groups $H^a_b(X)$. The primitive spaces are polarized by certain bilinear forms constructed on $H(X)$ via the usual Poincaré intersection form, modified by $\eta$ and $L'$, and descended to the perverse cohomology groups.

7. (**Generalized Grauert Contractibility Criterion**) Fix $y \in Y$ and $j \in \mathbb{Z}$. The natural class map, obtained by composing push forward in homology with Poincaré Duality,

$$H_{n-j}(f^{-1}(y)) \longrightarrow H^{n+j}(X)$$

is naturally filtered. The graded class map

$$H_{n-j,j}(f^{-1}(y)) \longrightarrow H^{n+j}_j(X)$$

is an injection of pure Hodge structures polarized in view of the Generalized Hodge-Riemann Relations above.

8. (**Refined Intersection Form Theorem**) The graded refined intersection form

$$H_{n-j,k}(f^{-1}(y)) \longrightarrow H^{n+j}_k(f^{-1}(y)) \quad \text{is zero for } j \neq k \text{ and an isomorphism for } j = k.$$

### 4.3.2 An outline of the proof.

We outline the proof of the above statements. We start by sketching the proof in the non-trivial toy model of a semismall map ([49]). Many important steps appear already in this case.

**1. The case of semismall maps.** We proceed by induction on $n = \dim Y$ to prove all the results in 4.3.1. As ${}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n]) = 0$ if $i \neq 0$, the Relative Hard Lefschetz is trivial and so is the perverse filtration. The first point to show is that, *from the point of view of the Hodge Lefschetz package, $L' = f^*L$ behaves as if it were ample,* even though it is not: all the theorems in 5.1 hold with $L'$ replacing $\eta$.

*The Hard Lefschetz Theorem for $L'$.* By induction, we assume that the statements in 4.3.1 hold for all semismall maps between varieties of dimension less than $n$. Let $D \subseteq Y$ be a generic hyperplane section of $L$. The map $f^{-1}(D) \to D$ is still semismall. Since $f_*\mathbb{Q}_X[n]$ is perverse, in the range $i \geq 2$, the Weak Lefschetz Theorem (3.7) reduces the Hard Lefschetz for $L'^i$ on $X$ to that for $L'^{i-1}$ on $f^{-1}(D)$. In the critical case $i = 1$, the cup product with $L'$ factors as $H^{n-1}(X) \to H^{n-1}(f^{-1}(D)) \to H^{n+1}(X)$, where the first map is injective and the second is surjective. As explained in the "inductive approach to Hard Lefschetz" paragraph of §5.1, the Hodge Riemann relations for the restriction of $L'$ to $f^{-1}(D)$ give the Hard Lefschetz Theorem for the cup product with $L'$.

*The approximation trick.* We must prove the Hodge Riemann relations for the space of primitives $P^n_{L'} = \operatorname{Ker} L' : H^n(X) \to H^{n+2}(X)$ (for use in the case $n+1$). The Hard Lefschetz Theorem just discussed implies that $\dim P^n_{L'} = b_n - b_{n-2}$ and that the decomposition $H^n(X) = P^n_{L'} \oplus L'H^{n-2}(X)$ is orthogonal with respect to the Poincaré pairing, just as if $L'$ were ample. In particular, the restriction of the Poincaré pairing $S(\alpha,\beta) = \int_X \alpha \wedge \beta$ to $P^n_{L'}$ is nondegenerate. The bilinear form $\widetilde{S}(\alpha,\beta) = S(\alpha, C\beta)$ ($C$ is the Weil operator; see §5.1) is still nondegenerate. The class $L'$ is on the boundary of the ample cone: for any positive integer $r$, the class $L' + \frac{1}{r}\eta$ is ample, and we have the classical Hodge Riemann relations on the subspace $P^n_{L_r} := \operatorname{Ker}(L' + \frac{1}{r}\eta) \subseteq H^n(X)$: the remark made above on the dimension of $P^n_{L'}$ implies that any class $\alpha \in P^n_{L'}$ is the limit of classes $\alpha_r \in P^n_{L_r}$, so that the restriction of $\widetilde{S}$ to $P^n_{L'}$ is semidefinite; since it is also nondegenerate, the Hodge Riemann bilinear relations follow.

*Decomposition and semisimplicity.* To prove the Decomposition and Semisimplicity theorems. we proceed one stratum at the time; higher dimensional strata are dealt with inductively by cutting transversally with a generic hyperplane section $D$ on $Y$, so that one is reconduced to the semismall map $f^{-1}(D) \to D$. The really significant case left is that of a zero-dimensional relevant stratum $S$. As explained in 2.2.1, the semisimplicity Theorem is equivalent to the non degeneracy of the intersection form $I : H_n(f^{-1}(y)) \times H_n(f^{-1}(y)) \longrightarrow \mathbb{Q}$, for $y \in S$. We rely on the following:

(**Weight Miracle I**) if $Z \subseteq U \subseteq X$ are inclusions with $X$ a nonsingular compact variety, $U \subseteq X$ a Zariski dense open subvariety and $Z \subseteq U$ a closed subvariety of $X$, then the images in $H^j(Z, \mathbb{Q})$ of the restriction maps from $X$ and from $U$ coincide; see [56].

Thanks to this, $H_n(f^{-1}(y))$ injects in $H^n(X)$ as a Hodge substructure. Since, for a general section $D$, we have $f^{-1}(y) \cap f^{-1}(D) = \emptyset$, we see that $H_n(f^{-1}(y))$ is contained in fact in $P^n_{L'}$. The restriction of the Poincaré pairing to $H_n(f^{-1}(y))$ is then a polarization and is hence nondegenerate. The same is thus true for $I$. As already noted in 2.23, the local systems involved have finite monodormy, hence they are obviously semisimple.

**2. The general case: extracting the semismall "soul" of a map.** The induction is on the the pair of indices $(\dim Y, r(f))$, where $r(f) = \dim X \times_Y X - \dim X$ is the *defect of semismallness* of the map $f$. To give an idea of the role played by $r(f)$ let us say that in the Decomposition Theorem §1.6.(6), the direct sum ranges precisely in the interval $[-r(f), r(f)]$. Suppose that all the statements in 4.3.1 have been proved for all proper maps $g : X' \to Y'$ with $r(g) < r(f)$, or with $r(g) = r(f)$ and $\dim Y' < \dim Y$. We set $n = \dim X$.

**a. The universal hyperplane section and Relative Hard Lefschetz.** If $g : X' \subseteq X \times \mathbb{P}^\vee \to Y' = Y \times \mathbb{P}^\vee$ is the universal hyperplane section, it can be shown that $r(g) < r(f)$ if $r(f) > 0$ and this gives us the inductive step: the statements of 4.3.1 may be assumed to hold for $g$. Just as in the classical case (cf. §5.1), the relative Lefschetz Hyperplane Theorem 3.19 implies the Relative Hard Lefschetz Theorem for $f$ except for $i = 1$, where we have the factorization of the cup product map with $\eta$

$$ {}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*\mathbb{Q}_X[n]) \xrightarrow{\rho} {}^{\mathfrak{p}}\mathcal{H}^0(g_*\mathbb{Q}_{X'}[n-1]) \xrightarrow{\gamma} {}^{\mathfrak{p}}\mathcal{H}^1(f_*\mathbb{Q}_X[n]). $$

The first map is a monomorphism and the second is an epimorphism. Just as in the proof of the Hard Lefschetz Theorem via the semisimplicity of the monodromy, an argument similar to the identification of the image of the cohomology of a variety into the cohomology of a hyperplane section with the monodromy invariants of a Lefschetz pencil, coupled with the semisimplicity (inductive assumption!) of ${}^{\mathfrak{p}}\mathcal{H}^0(g_*\mathbb{Q}_{X'}[n-1])$ imply:

**Proposition 4.29** *The image of ${}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*\mathbb{Q}_X[n])$ in ${}^{\mathfrak{p}}\mathcal{H}^0(g_*\mathbb{Q}_{X'}[n+1])$ is a split summand applied isomorphically on ${}^{\mathfrak{p}}\mathcal{H}^1(f_*\mathbb{Q}_X[n])$ by $\gamma$.*

The Relative Hard Lefschetz for $f$ follows and, by applying Theorem 5.4, we conclude that $f_*\mathbb{Q}_X[n] \simeq \oplus\, {}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n])$.

From the statements known for $g$ by induction, we get that ${}^{\mathfrak{p}}\mathcal{H}^i(f_*\mathbb{Q}_X[n])$ is a direct sum of intersection cohomology complexes of semisimple local systems for all $i \neq 0$. Moreover, for all $i \neq 0$, the associated cohomology groups verify the Hard Lefschetz theorem and the Hodge Riemann relations with respect to cupping with $L$.

What is left to investigate is the zero perversity complex ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])$. Again in analogy with the classical case, another pieces may be "shaved off" and reconduced to hyperplane sections. In fact, the analogue of the Primitive Lefschetz Decomposition Theorem 5.1.2. holds: for every for $i \geq 0$, set $\mathcal{P}^{-i} := \mathrm{Ker}\,\{\eta^{i+1} : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) \to {}^{\mathfrak{p}}\mathcal{H}^{i+2}(f_*\mathbb{Q}_X[n])\}$ and we have:

$$ {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) = \bigoplus_{r\geq 0} \eta^r \mathcal{P}^{-i-2r}, \qquad {}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n]) = \bigoplus_{r\geq 0} \eta^{i+r} \mathcal{P}^{-i-2r}. \tag{33} $$

The only remaining piece for which we have to prove the statements of 4.3.1 is the perverse sheaf $\mathcal{P}^0$ and its cohomology $H^*(Y, \mathcal{P}^0)$ which, in view of the primitive decomposition, is a summand of $H_0^{*+n}(X)$. (Note the analogy with the classical study of algebraic varieties by mean of hyperplane sections: the new cohomology classes, i.e. the ones not coming from a hyperplane section, appear only in $P^n = \mathrm{Ker}\,\{\eta : H^n(X) \to H^{n+2}(X)\}$.) We have to prove:

1. The Hodge package of §5.1 holds for $H^*(Y, \mathcal{P}^0)$ with respect to cupping with $L$.
2. $\mathcal{P}^0$ is a direct sum of twisted intersection cohomology complexes.
3. The twisting local systems are semisimple.

**b. The Hodge package for $\mathcal{P}^0$.** The main intuition behind the proof of the first two statements, inspired also by the illuminating discussion of the Decomposition Theorem contained in [124], is that $H^*(Y, \mathcal{P}^0)$ is the "semismall soul of the map $f$," that is *it behaves as the cohomology of a (virtual) nonsingular projective variety with a semismall map to $Y$*. In order to handle the group $H^*(Y, \mathcal{P}^0)$, we mimic the proof of the Decomposition Theorem for semismall maps.

One of the main difficulties in [46] is that, in order to use classical Hodge theory, we have to prove at the outset that the perverse filtration is Hodge-theoretic, i.e. that the subspaces $P^pH(X) \subseteq H(X)$ (cf. §4.3.1.(4)) are Hodge substructures. The geometric description of the perverse filtration in [51] (see §3.5) implies that this fact holds in greater generality, i.e. for any algebraic map, proper or not, to a quasi projective variety, and independently of the Decomposition Theorem. This result can be used to yield a considerable simplification of the line of reasoning in [46] for it endows, at the outset, the perverse cohomology groups $H_b^a(X)$ with a natural Hodge structure, compatible with the primitive Lefschetz decompositions stemming from (33), with respect to which the cup product maps $L : H^*(Y, \mathcal{P}^i) \to H^{*+2}(Y, \mathcal{P}^i)$ and $\eta : P^kH^*(X) \to P^{k-2}H^{*+2}(X)$ are Hodge maps of type $(1,1)$.

We start by proving 1., i.e. the Hodge package for $H^*(Y, \mathcal{P}^0)$. The argument for the Hard Lefschetz isomorphism $L^i : H^{-i}(Y, \mathcal{P}^0) \simeq H^i(Y, \mathcal{P}^0)$ is completely analogous to the one used for a semismall map: the weak Lefschetz Theorem for the perverse sheaf $\mathcal{P}^0$ and the inductive hypothesis (for a generic hyperplane section $D \subseteq Y$, we have $f' : f^{-1}(D) \to D$ and $\mathcal{P}^0$ restricts, up to a shift, to the analogous complex $\mathcal{P}'^0$ for $f'$) gives immediately the Theorem in the range $i \geq 2$ and gives a factorization of $L : H^{-1}(Y, \mathcal{P}^0) \to H^1(Y, \mathcal{P}^0)$, as the composition of the restriction to $D$ and a Gysin map. Again by the inductive hypotheses, the Poincaré pairing polarizes $\operatorname{Ker} L : H^0(D, \mathcal{P}'^0) \to H^2(D, \mathcal{P}'^0)$, and this proves the remaining case $i = 1$.

The most delicate point is to prove that the Riemann Hodge relations hold for $P^{00} := \operatorname{Ker}\{L : H^0(Y, \mathcal{P}^0) \to H^2(Y, \mathcal{P}^0)\}$. The Poincaré pairing induces a bilinear form $S(-,-)$ on $H^n(X) = H^0(f_*\mathbb{Q}_X[n])$ and on its subquotient $H^0(Y, \mathcal{P}^0)$. This is because we have the following orthogonality relation $P^1H(X) \subseteq P^0H(X)^\perp$. More is true: $S$ is nondegenerate on $P^0H^n(X)/P^1H^n(X) = H^n_0(X)$ and the $(\eta, L)$ decomposition is orthogonal so that the restriction of $S$ to the summand $P^{00}$ is nondegenerate. The Hodge Riemann relations are then proved with an "approximation trick" similar, although more involved, to the one used in the semismall case. We consider the subspace $\Lambda = \lim_r \operatorname{Ker}(L' + \frac{1}{r}\eta) \subseteq H^n(X)$. Clearly, we have $\Lambda \subseteq \operatorname{Ker} L'$ and the Hard Lefschetz Theorem implies that $\operatorname{Ker} L' \subseteq P^0H^n(X)$. The nondegenerate form $\widetilde{S}$ is semidefinite on $\Lambda/\Lambda \cap P^1H^n(X)$. It follows that it is a a polarization. A polarization restricted to a Hodge substructure is still a polarization. The Hodge Riemann relations for $P^{00}$ follow from the inclusion of Hodge structures $P^{00} \subseteq \Lambda/\Lambda \cap P^1H^n(X)$.

**c. Semisimplicity** We need to prove that $\mathcal{P}^0$ splits as a direct sum of intersection cohomology complexes of semisimple local systems. As in the case of semismall maps, higher dimensional strata are disposed-of by induction on the dimension of $Y$ and by cutting with generic hyperplane sections of $Y$. One is left to prove the critical case of a zero-dimensional stratum. Again by the splitting criterion of Remark 3.31, we have to prove that, for any point $y$ in the zero dimensional stratum, denoting by $i : y \to Y$ the closed imbedding, $\iota : \mathcal{H}^0(i^!\mathcal{P}^0) \to \mathcal{H}^0(i^*\mathcal{P}^0)$ is an isomorphism. Given the decomposition (33), $\mathcal{H}^0(i^!\mathcal{P}^0)$ is a direct summand of $H_n(f^{-1}(0))$ and $\mathcal{H}^0(i^*\mathcal{P}^0)$ is a direct summand of $H^n(f^{-1}(0))$. The map is the restriction of the refined intersection form on $f^{-1}(0)$. The Weight Miracle I is used to prove that the map $\mathcal{H}^0(i^!\mathcal{P}^0) \to H^n_0(X)$ is an injection (although the whole map $H_n(f^{-1}(0)) \to H^n(X)$ is not, in general) with image a pure Hodge structure. Since the image lands automatically in the $L'$-primitive part, we conclude that the descended intersection form polarizes this image, hence $\iota$ is an isomorphism and we have the desired splitting into intersection cohomology complexes.

We still have to establish the semisimplicity of the local systems in (7) (and hence of the ones appearing in $\mathcal{P}^0$). This is accomplished by exhibiting them as quotients of local systems associated with smooth proper maps and are hence semisimple by the Semisimplicity for Smooth Maps Theorem 5.2.

This concludes the proof.

Once these results are established for $f : X \to Y$, with $Y$ projective and nonsingular, it is easy to extend the Decomposition Theorem to the case of the direct image of the intersection cohomology complex under proper maps of algebraic varieties. One uses resolution of singularities and Chow envelopes. Similarly, for the Relative Hard Lefschetz Theorem for projective morphisms and relatively ample line bundles.

As to the other Hodge-theoretic results, they also extend, but for them to make even sense, one has to put a pure Hodge structure on the intersection cohomology groups of a projective variety $Y$. For this purpose, we need the

(**Weight Miracle II**) if $g: T \to Z$ is a map of proper varieties, $T$ nonsingular, then [56]

$$\operatorname{Ker}\{\, g^* \,:\, H^j(Z,\mathbb{Q}) \longrightarrow H^j(T,\mathbb{Q})\,\} \,=\, W_{j-1}H^j(Z,\mathbb{Q}), \qquad \forall j \geq 0.$$

Again, the critical case is the middle dimensional group $IH^n(Y)$. We take a projective resolution of the singularities $f: X \to Y$. The complex $IC_Y$ is canonically a direct summand of ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n])$ so that $IH^n(Y)$ is one of the direct summands of $H^n_0(X)$. This last group has the pure Hodge structure inherited from $H^n(X)$ by the Hodge Structure Theorem. We use Weight Miracle II and an inductive argument on the strata to show that $IH^n(Y)$ is the orthogonal complement of the sum of the remaining summands with respect to the Poincaré pairing on $H^n_0(X)$. By induction, the sum is a pure Hodge sub structure so that so is $IH^n(Y)$. This endows intersection cohomology with a pure Hodge structure which is independent of the resolution chosen.

Once this is done, all the results of this section hold with intersection cohomology replacing cohomology. In particular, this re-establishes the validity of the Hodge-Lefschetz package for intersection cohomology (due to Saito) with the difference that the Hodge structures and the polarizations involved are inherited directly from a resolution.

The method yields naturally the Purity Theorem in [46]: every stratum contributes to each perverse cohomology group a summand which inherits a pure Hodge structure and a primitive $(\eta, L)$-decomposition and polarizations.

The paper [50] shows how to choose a splitting in the Decomposition Theorem so that the resulting splitting of the intersection cohomology $IH(X)$ of the domain is of pure Hodge structures. This is a more precise result than the Purity Theorem which establishes the analogous fact for the perverse cohomology groups which are merely subquotients of $H(X)$.

These methods also allow to prove that Poincaré Duality for the intersection cohomology of projective varieties is an isomorphism of pure Hodge structures, as well as proving that the natural map $a: H^j(Y,\mathbb{Q}) \to IH^j(Y,\mathbb{Q})$ is a map of mixed Hodge structures with kernel given precisely by the cohomology classes of weights $< j$: $W_{j-1}H^j(X,\mathbb{Q})$. Note that a priori the kernel could be bigger.

The papers [51, 41] contain a geometric description of the perverse filtration and spectral sequence based on hyperplane sections. As a result of this description, the paper [41] endows the intersection cohomology groups of a quasi projective variety with a mixed Hodge structure and proves the mixed analogues of the Purity and Canonical Splitting Theorems, of Poincaré Duality on intersection cohomology, of the Hodge-theoretic splitting [50], of the fact that the natural map $a: IH(Y) \to H(Y)$ is of mixed Hodge structures.

These statements, except to our knowledge the one for the map $a$, are due to M. Saito. They are then statements concerning the mixed Hodge structures stemming from his theory of mixed Hodge modules.

The paper [41] also contains the observation that M. Saito's mixed Hodge structures in this context coincide with the ones found by us.

# 5 Appendices

## 5.1 Hard Lefschetz and mixed Hodge structures

We re-state the Hard Lefschetz Theorem and the Hodge Riemann Relations in the language of Hodge structures, which we now briefly recall.

Let $l \in \mathbb{Z}$, $H$ be a finitely generated abelian group, $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$, $H_{\mathbb{R}} = H \otimes_{\mathbb{Z}} \mathbb{R}$, $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$. A *pure Hodge structure of weight $l$* on $H$, $H_{\mathbb{Q}}$ or $H_{\mathbb{R}}$, is a direct sum decomposition $H_{\mathbb{C}} = \oplus_{p+q=l} H^{p,q}$ such that $H^{p,q} = \overline{H^{q,p}}$. The Hodge filtration is the decreasing filtration $F^p(H_{\mathbb{C}}) := \oplus_{p' \geq p} H^{p',q'}$. A *morphism of Hodge structures* $f : H \to H'$ is a group homomorphism such that $f \otimes Id_{\mathbb{C}}$ is compatible with the Hodge filtration, i.e. such that it is a filtered map. Such maps are automatically what one calls *strict*, i.e. $(\operatorname{Im} f) \cap F^i = f(F^i)$. The category of Hodge structures of weight $l$ with strict maps is abelian.
Let $C$ be the *Weil operator*, i.e. $C : H_{\mathbb{C}} \simeq H_{\mathbb{C}}$ is such that $C(x) = i^{p-q}x$, for every $x \in H^{pq}$. It is a real operator. Replacing $i^{p-q}$ by $z^p \overline{z}^q$ we get a real action $\rho$ of $\mathbb{C}^*$ on $H_{\mathbb{C}}$. A *polarization* of the real pure Hodge structure $H_{\mathbb{R}}$ is a real bilinear form $\Psi$ on $H_{\mathbb{R}}$ which is invariant under the action given by $\rho$ restricted to $S^1 \subseteq \mathbb{C}^*$ and such that the bilinear form $\widetilde{\Psi}(x,y) := \Psi(x, Cy)$ is symmetric and positive definite. If $\Psi$ is a polarization, then $\Psi$ is symmetric if $l$ is even, and antisymmetric if $l$ is odd. In any case, $\Psi$ is nondegenerate. In addition, for every $0 \neq x \in H^{pq}$, $(-1)^l i^{p-q} \Psi(x, \overline{x}) > 0$, where $\Psi$ also denotes the $\mathbb{C}$-bilinear extension of $\Psi$ to $H_{\mathbb{C}}$.

Let $\eta$ be the first Chern class of an ample line bundle on the projective $n$-fold $Y$. For every $r \geq 0$, define the space of *primitive vectors* $P^{n-r} := \operatorname{Ker} \eta^{r+1} \subseteq H^{n-r}(Y, \mathbb{Q})$.

Classical Hodge Theory states that, for every $l$, $H^l(Y, \mathbb{Z})$ is a pure Hodge structure of weight $l$, $P^{n-r}$ is a rational pure Hodge structure of weight $(n-r)$ polarized by a modification of the Poincaré pairing on $Y$.

**Theorem 5.1**

1. (**Hard Lefschetz Theorem**) *For every $r \geq 0$ one has isomorphisms*

$$\eta^r \,:\, H^{n-r}(Y, \mathbb{Q}) \,\simeq\, H^{n+r}(Y, \mathbb{Q}).$$

2. (**Primitive Lefschetz Decomposition**) *For every $r \geq 0$ there is the direct sum decomposition*

$$H^{n-r}(Y, \mathbb{Q}) \,=\, \bigoplus_{j \geq 0} \eta^j P^{n-r-2j}$$

*where each summand is a pure Hodge sub-structure of weight $n-r$ and all summands are mutually orthogonal with respect to the bilinear form $\int_Y \eta^r \wedge - \wedge -$.*

3. (**Hodge-Riemann Bilinear Relations**) *For every $0 \leq l \leq n$, the bilinear form $(-1)^{\frac{l(l+1)}{2}} \int_Y \eta^{n-l} \wedge - \wedge -$ is a polarization of the pure weight $l$ Hodge structure $P^l \subseteq$*

$H^l(Y,\mathbb{R})$. *In particular,*

$$(-1)^{\frac{l(l-1)}{2}} i^{p-q} \int_Y \eta^{n-l} \wedge \alpha \wedge \overline{\alpha} \; > \; 0, \qquad \forall \; 0 \neq \alpha \in P^l \cap H^{pq}(Y,\mathbb{C}).$$

**Inductive approach to Hard Lefschetz.** The induction is on the dimension of $Y$ and uses hyperplane sections. The Lefschetz Theorem on Hyperplane sections plus the semisimplicity of the monodromy in a Lefschetz Pencil, or the Hodge Riemann Relations for a hyperplane section, imply the Hard Lefschetz Theorem for $Y$. However, they do not imply neither semisimplicity, nor the Hodge Riemann Relations for the critical cohomology group $H^n$ in higher dimension and to make the proof work, one has to establish these facts separately. Let us discuss the classical Hard Lefschetz Theorem (§1.1) from this perspective. The case $i=0$ is trivial. The cases $i \geq 2$ follow by an easy induction on the dimension of $Y$ using the Lefschetz Hyperplane Theorem. One is left with the key case $i=1$. Given a smooth hyperplane section $D \subseteq Y$ of the $n$-fold $Y$, the cup product map $\eta := c_1(D) \wedge -$ factors as $\eta = g \circ r$:

$$H^{n-1}(Y) \xrightarrow{r} H^{n-1}(D) \xrightarrow{g} H^{n+1}(Y),$$

where $r$ is the injective restriction map and $g$ is the surjective Gysin map. It is easy to show that $\eta$ is an isomorphism iff the intersection form on $D$, restricted to the image of the restriction, is nondegenerate. While the form on $D$ is non degenerate by Poincaré Duality, there is no a priori reason why it should restrict to a non degenerate form. This is implied by the Hodge Riemann relations and also by the semisimplicity of the monodromy of a Lefschetz pencil of which $D$ is part.

The semisimplicity of monodromy, as well as the Hodge Riemann Relations on $D$, also imply that $\eta$ is an isomorphism (see [58, 40]). We give the argument for the derivation from the Hodge Riemann relations as it is used in a crucial way in our proof, see 4.3.2. Suppose in fact that there is a non-zero class $\alpha \in \mathrm{Ker}\eta$ which we may suppose of pure Hodge type. Then $r(\alpha)$ is primitive in $H^{n-1}(D)$, and, by the Hodge Riemann relations, $0 = \int_Y \eta \wedge \alpha \wedge \overline{\alpha} = \int_D \alpha \wedge \overline{\alpha} \neq 0$.

The Hard Lefschetz theorem applied to the fibres of a smooth projective morphism and Theorem 5.4 yield:

**Theorem 5.2** (**Decomposition, Semisimplicity and Relative Hard Lefschetz for proper smooth maps**) *Let $f : X^n \to Y^m$ be a smooth proper map of smooth algebraic varieties of the indicated dimensions. Then*

$$f_*\mathbb{Q}_X \; \simeq \; \bigoplus_{j \geq 0} R^j f_* \mathbb{Q}_X[-j]$$

*and the $R^j f_*\mathbb{Q}_X$ are semisimple local systems.*
*If, in addition, $f$ is projective and $\eta$ is the first Chern class of an $f$-ample line bundle on $X$, then*

$$\eta^i \, : \, R^{n-m-i} f_*\mathbb{Q}_X \; \simeq \; R^{n-m+i} f_*\mathbb{Q}_X, \; \forall i \geq 0,$$

*and the local systems $R^j f_*\mathbb{Q}_X$ underlie polarizable variations of pure Hodge structures.*

*Proof.* See [52] and [55], Théorème 4.2.6. $\square$

**Mixed Hodge structures**. The singular cohomology groups $H^j(Y,\mathbb{Z})$ of a singular variety cannot carry the structure of a pure Hodge structure of weight $j$; e.g. $H^1(\mathbb{C}^*,\mathbb{Z})$ has rank one, and pure Hodge structures of odd weight have even rank. However, they underlie a more subtle structure, the presence of which makes the topology of complex algebraic varieties even more remarkable.

**Theorem 5.3** (**Mixed Hodge structure on cohomology**) *Let $Y$ be an algebraic variety. For each $j$ there is an increasing filtration (the weight filtration)*

$$\{0\} = W_{-1} \subseteq W_0 \subseteq \ldots \subseteq W_{2j} = H^j(Y,\mathbb{Q})$$

*and a decreasing filtration (the Hodge filtration)*

$$H^j(Y,\mathbb{C}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^m \supseteq F^{m+1} = \{0\}$$

*such that the filtration induced by $F^\bullet$ on the complexified graded pieces of the weight filtration endows every graded piece $W_l/W_{l-1}$ with a pure Hodge structure of weight $l$.*
*This structure is functorial for maps of algebraic varieties and the induced maps strictly preserve both filtrations.*

## 5.2 The formalism in $\mathcal{D}_Y$

Standard references for what follows are [107, 79, 16, 103, 8]. In what follows, we freely refer to our crash-course in §1.5 and to the complete references given above.

Let $Y$ be an algebraic variety and $\mathcal{D}_Y$ be the constructible bounded derived category.

The category $\mathcal{D}_Y$ is a *triangulated category.* In particular, it is additive, so that we can form finite direct sums, and it is equipped with the translation functor $A \mapsto A[1]$. A most important feature is the presence of *distinguished triangles* (or simply triangles). A triangle is a diagram of maps $A \to B \to C \to A[1]$ in $\mathcal{D}_Y$ which is isomorphic to the one coming from the cone construction associated with an ordinary map of complexes. An essential computational tool is that the application of a cohomological functor to a distinguished triangle produces a long exact sequence. A *cohomological functor*, with values in an abelian category $\mathcal{A}$, is an additive functor $T : \mathcal{D}_Y \to \mathcal{A}$ such that $T(A) \to T(B) \to T(C)$ is exact for every triangle as above. Setting $T^i(A) := T(A[i])$, we get the long exact sequence

$$\cdots \longrightarrow T^i(A) \longrightarrow T^i(B) \longrightarrow T^i(C) \longrightarrow T^{i+1}(C) \longrightarrow \cdots$$

Using injective resolutions and the two global sections functors $\Gamma$ and $\Gamma_c$ we define the *derived global sections functors*

$$R\Gamma, R\Gamma_c \ : \ \mathcal{D}_Y \longrightarrow \mathcal{D}_{pt}$$

and, for $K \in \mathcal{D}_Y$, the finite dimensional (hyper)cohomology vector spaces of $Y$ with coefficients in $K$:

$$H^*(Y,K) := H^*(R\Gamma(Y,K)), \qquad H^*_c(Y,K) := H^*(R\Gamma_c(Y,K)) \text{ (compact supports).}$$

Given a map $f : X \to Y$ there are the *four functors*

$$Rf_*, Rf_! \; : \; \mathcal{D}_X \longrightarrow \mathcal{D}_Y, \qquad f^*, f^! \; : \; \mathcal{D}_Y \longrightarrow \mathcal{D}_X.$$

The sheaf-theoretic direct image functors $f_*, f_! : Sh_X \to Sh_Y$ are *left exact* as functors, e.g. if $0 \to F \to G \to H \to 0$ is an exact sequence of sheaves on $X$, then $0 \to f_!F \to f_!G \to f_!H$ is an exact sequence of sheaves on $Y$. The *right derived* functors $Rf_*$ and $Rf_!$ arise by applying the sheaf-theoretic direct image functors $f_*$ and $f_!$ (proper supports), term-by-term, to injective resolutions. Taking cohomology sheaves, we obtain the $i$-th right derived functors $R^if_*$ and $R^if_!$. We have equalities of sheaf-theoretic functors $R^0f_* = f_*$, $R^0f_! = f_!$. The *inverse image* functor $f^* : Sh_Y \to Sh_X$ is exact on sheaves and descends to the derived cateogory. The *exceptional inverse image functor* $f^!$ does not arise from a functor defined on sheaves.

It is costumary to employ the following simplified notation to denote the four functors $(f^*, f_*, f_!, f^!)$. In this paper, $f_*$ and $f_!$ denote the right derived functors. To avoid confusion, the sheaf-theoretic functors are denoted $R^0f_*$, $R^0f_!$.

Given maps $f : X \to Y$, $g : Y \to Z$, we have $(g \circ f)^! = f^! \circ g^!$, etc. For $g : Y \to pt$ and for $C \in \mathcal{D}_X$, we have

$$H^*(X, C) = H^*(Y, f_*C), \qquad H^*_c(X, C) = H^*(Y, f_!C).$$

**$f_!$ and $f^!$ in special cases**

If $f$ is proper, e.g. a closed immersion, then $f_! = f_*$.
If $f$ is smooth of relative dimension $d$, then $f^! = f^*[2d]$.
If $f$ is a closed embedding of pure codimension $d$ transverse to all strata of a stratification $\Sigma$ of $Y$, then $f^! = f^*[-2d]$ holds for every $\Sigma$-constructible complex. Such so-called normally nonsingular inclusions can be obtained intersecting $Y$ with general hypersurfaces.
If $f$ is an open embedding, then $f^! = f^*$.
If $f$ is a locally closed embedding, then
1) $f_!$ is the extension-by-zero functor and $f_! = R^0f_!$;
2) $f^! = f^*R\Gamma_X$, where $\Gamma_XF$, not to be confused with $f_!f^*$, is the sheaf of sections of the sheaf $F$ supported on $X$ (see [107], p.95). If, in addition, $f$ is a closed embedding, then $H(X, f^!K) = H(Y, Y \setminus X; K) = H_X(Y, K)$.

The usual Hom complex construction can de derived and we get right derived functors

$$\mathrm{RHom} : \mathcal{D}^o_Y \times \mathcal{D}_Y \longrightarrow \mathcal{D}_{pt}, \quad R\mathcal{H}om : \mathcal{D}^o_Y \times \mathcal{D}_Y \longrightarrow \mathcal{D}_Y$$

with the associated $\mathrm{Ext}^i$ and $\mathcal{E}xt^i$ functors. We have

$$\mathrm{Hom}_{\mathcal{D}_Y}(K, K') = H^0(Y, R\mathcal{H}om(K, K')) = H^0(Y, \mathrm{RHom}(K, K')).$$

The pair $(f^*, f_*)$ is an *adjoint pair* (this holds also for the sheaf-theoretic version) and so is $(f_!, f^!)$ and we have, for every $C \in \mathcal{D}_X$ and $K \in \mathcal{D}_Y$:

$$f_*R\mathcal{H}om(f^*K, C) = R\mathcal{H}om(K, f_*C), \qquad f_*R\mathcal{H}om(f_!C, K) = R\mathcal{H}om(C, f^!K).$$

Since we are working with field coefficients, the tensor product operation $\otimes$ on complexes is exact and there is no need to derive it. For $K_i \in \mathcal{D}_Y$, we have (also for $R\mathcal{H}om$):

$$\mathrm{RHom}(K_1 \otimes K_2, K_3) = \mathrm{RHom}(K_1, R\mathcal{H}om(K_2, K_3))$$

and, if the sheaves $\mathcal{H}^i(K_3)$ are locally constant:

$$R\mathcal{H}om(K_1, K_2 \otimes K_3) = R\mathcal{H}om(K_1, K_2) \otimes K_3$$

There is the *dualizing complex* $\omega_Y \in \mathcal{D}_Y$, well-defined, up to canonical isomorphism by setting $\omega_Y := \gamma^! \mathbb{Q}_{pt}$, where $\gamma : Y \to pt$. If $Y$ is nonsingular, then $\omega_Y = \mathbb{Q}_Y[2\dim_{\mathbb{C}} Y]$. Given $f : X \to Y$, we have $\omega_X = f^! \omega_Y$. Define a contravariant functor

$$D \; : \; \mathcal{D}_Y \longrightarrow \mathcal{D}_Y, \qquad K \longmapsto D(K) \, (= K^\vee) \; := \; R\mathcal{H}om(K, \omega_Y).$$

We have $D^2 = \mathrm{Id}$, $(K[i])^\vee = K^\vee[-i]$ and $\omega_Y = \mathbb{Q}_Y^\vee$. The complex $K^\vee$ is called the (Verdier) *dual* of $K$. Poincaré-Verdier Duality consists of the canonical isomorphism

$$H^i(Y, K^\vee) = H_c^{-i}(Y, K)^\vee$$

which is a formal consequence of the fact that $(f_!, f^!)$ are an adjoint pair. The usual Poincaré Duality for topological manifolds is the special case when $Y$ is smooth and orientable, for then a choice of orientation gives $\omega_Y = \mathbb{Q}_Y[\dim_{\mathbb{R}} Y]$.

We have the important relations

$$Df_! = f_* D, \qquad Df^! = f^* D.$$

A *t-category* is a triangulated category endowed with a $t$-structure ([8, 107]). The *standard t-structure* on $\mathcal{D}_Y$ is the prototype of a *t-structure* and is defined using the standard truncation functors

$$\tau_{\leq i} \; : \; \mathcal{D}_Y^{\leq i} \longrightarrow \mathcal{D}_Y, \qquad \tau_{\geq i} \; : \; \mathcal{D}_Y^{\geq i} \longrightarrow \mathcal{D}_Y,$$

where $\mathcal{D}_Y^{\leq i} \subseteq \mathcal{D}_Y$ is the full subcategory of complexes $K$ with $\mathcal{H}^j(K) = 0$ for $j \geq 0$, etc. The functor $\mathcal{H}^0$ is cohomological. The intersection $\mathcal{D}_Y^{\leq 0} \cap \mathcal{D}_Y^{\geq 0}$ is the abelian category of constructible sheaves on $Y$. We have $\mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y^{\leq 0}, \mathcal{D}_Y^{\geq 1}) = 0$ and given any $K \in \mathcal{D}_Y$ there is the truncation triangle $\tau_{\leq 0} K \to K \to \tau_{\geq 0} K \to \tau_{\leq 0} K[1]$. Another important $t$-structure is the (middle) *perverse* $t$-structure (§3.4)

There are the following notions of exactness. A functor of abelian categories is *exact* if it preserves exact sequences. There are the companion notions of *left and right exactness*. A functor of triangulated categories (i.e. additive an commuting with translations) is *exact* if it preserves triangles. A functor of $t$-categories $F : \mathcal{D} \to \mathcal{D}'$ is a functor of the underlying triangulated categories. It is *exact* if it preserves triangles. It is *left t-exact* if $F : \mathcal{D}^{\geq 0} \to \mathcal{D}'^{\geq 0}$. Similarly, for *right t-exact*. It is *t-exact* if it is both left and right $t$-exact, in which case it preserves the abelian hearts.

**Perverse $t$-exactness.**
If $\dim f^{-1}y \leq d$, then

$$f_!, f^* : {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq d}, \qquad f^!, f_* : {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq -d}.$$

If $f$ is quasi finite (= finite fibers), then $d = 0$ above.
If $f$ is affine, e.g. the embedding of the complement of a Cartier divisor, the embedding of an affine open subset, or the projection of the complement of a universal hyperplane section etc., then

$$f_* : {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0} \quad \text{(right } t\text{-exact)}, \qquad f_! : {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0} \quad \text{(left } t\text{-exact)}.$$

More generally, if locally over $Y$, $X$ is the union of $d+1$ affine open sets, then

$$f_* : {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq d}, \qquad f_! : {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq -d}.$$

If $f$ is quasi finite and affine, then $f_!$ and $f_*$ are $t$-exact.
If $f$ is finite (= proper and finite fibers), then $f_! = f_*$ are $t$-exact.
If $f$ is a closed embedding, then $f_! = f_*$ are $t$-exact and fully faithful. In this case it is customary to drop $f_*$ from the notation, e.g. $IC_X \in \mathcal{D}_Y$.
If $f$ is smooth of relative dimension $d$, then $f^![-d] = f^*[d]$ are $t$-exact.
In particular, if $f$ is étale, then $f^! = f^*$ are $t$-exact.
If $f$ is a normally nonsingular inclusion of codimension $d$ with respect to a stratification $\Sigma$ of $Y$, then $f^![d] = f^*[-d] : \mathcal{D}_Y^{\Sigma} \to \mathcal{D}_X$ are $t$-exact.

The following splitting criterion ([52, 54]) plays an important role in the proof of the Decomposition Theorem:

**Theorem 5.4** *Let $K \in \mathcal{D}_X$ and $\eta : K \to K[2]$ such that $\eta^l :{}^{\mathfrak{p}} \mathcal{H}^{-l}(K) \to{}^{\mathfrak{p}} \mathcal{H}^l(K)$ is an isomorphism for all $l$. Then there is an isomorphism:*

$$K \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(K)[-i].$$

### 5.2.1 Familiar objects from algebraic topology

Here is a brief list of some of the basic objects of algebraic topology and a short discussion of how they relate to the formalism in $\mathcal{D}_Y$.

**(Co)homology etc.:**
singular cohomology: $H^l(Y, \mathbb{Q}_Y)$;
singular cohomology with compact supports: $H_c^l(Y, \mathbb{Q}_Y)$;
singular homology $H_l(Y, \mathbb{Q}) = H_c^{-l}(Y, \omega_Y)$;
Borel-Moore homology: $H_l^{BM}(Y, \mathbb{Q}) = H^{-l}(Y, \omega_Y)$;
relative (co)homology: if $i : Z \to Y$ is a locally closed embedding and $j : (Y \setminus Z) \to Y$, then we have $H^l(Y, Z, \mathbb{Q}) = H^l(Y, i_! i^! \mathbb{Q})$ and $H_l(Y, Z, \mathbb{Q}) = H_c^{-l}(Y, j_* j^* \omega_Y)$.

**Intersection (co)homology**. The intersection homology groups $IH_j(Y)$ of an $n$-dimensional irreducible variety $Y$ are defined as the $j$-th homology groups of chain complexes of geometric chains with closed supports subject to certain admissibility conditions. Similarly, one defines intersection homology with compact supports. There are natural maps

$$IH_j(Y) \longrightarrow H_j^{BM}(Y), \qquad IH_{c,j}(Y) \longrightarrow H_j(Y).$$

Intersection cohomology: $IH^j(Y) := IH_{2n-j}(Y) = H^{-n+j}(Y, IC_Y)$.
Intersection cohomology with compact supports: $IH_c^j(Y) := IH_{c,2n-j}(Y) = H_c^{-n+j}(Y, IC_Y)$.

**Duality and pairings.** Verdier Duality implies we have canonical identifications

$$H_l(Y,\mathbb{Q})^\vee = H_c^{-l}(Y,\omega_Y)^\vee \simeq H^l(Y,\mathbb{Q}), \qquad H_l^{BM}(Y,\mathbb{Q})^\vee = H^{-l}(Y,\omega_Y)^\vee \simeq H_c^l(Y,\mathbb{Q}).$$

If $Y$ is nonsingular of dimension $n$, then we have *Poincaré Duality*:

$$H^{n+l}(Y,\mathbb{Q}) \simeq H_{n-l}^{BM}(Y,\mathbb{Q}), \qquad H_{n+l}(Y,\mathbb{Q}) \simeq H_c^{n-l}(Y,\mathbb{Q}).$$

There are two ways to express the classical nondegenerate Poincaré intersection pairing

$$H^{n+l}(Y,\mathbb{Q}) \times H_c^{n-l}(Y,\mathbb{Q}) \longrightarrow \mathbb{Q}, \qquad H_{n-l}^{BM}(Y,\mathbb{Q}) \times H_{n+l}(Y,\mathbb{Q}) \longrightarrow \mathbb{Q}.$$

While the former one is given by wedge product and integration, the latter can be described geometrically as the intersection form in $Y$ as follows. Given a Borel-Moore cycle and a usual, i.e. compact, cycle in complementary dimensions, one changes one of them, say the first one, to one homologous to it, but transverse to the other. Since the ordinary one has compact supports, the intersection set is finite and one gets a finite intersection index.
Let $Y$ be compact, $Z$ be a closed subvariety such that $Y \setminus Z$ is a smooth and of pure dimension $n$. We have *Lefschetz Duality*

$$H_q(Y,Z;\mathbb{Q}) = H_c^{-q}(Y, j_*j^*\omega_Y) = H^{-q}(Y, j_*j^*\omega_Y) = H^{-q}(Y\backslash Z, \mathbb{Q}_Y[2n]) = H^{2n-q}(Y\backslash Z, \mathbb{Q}).$$

*Goresky-MacPherson's Poincaré Duality*: since $IC_Y = IC_Y^\vee$, we have canonical isomorphsims

$$IH^{n+l}(Y,\mathbb{Q}) \simeq IH_c^{n-l}(Y,\mathbb{Q})^\vee.$$

**Functoriality.** The usual maps in (co)homology associated with a map $f: X \to Y$ arise from the adjunction maps

$$\mathbb{Q}_Y \longrightarrow f_*f^*\mathbb{Q}_Y = f_*\mathbb{Q}_X, \qquad f_!f^!\omega_Y = f_!\omega_X \longrightarrow \omega_Y.$$

by taking cohomology. In general, for an arbitrary map $f$, there are no maps associated with Borel-Moore and cohomology with compact supports. If $f$ is proper, then $f_* = f_!$ and one gets pull-back for proper maps in cohomology with compact supports and push-forward for proper maps in Borel-Moore homology. These maps are dual to each other.

If $f$ is an open immersion, then $f^* = f^!$ and one has the restriction to an open subset map for Borel-Moore homology and the push-foward for an open subset map for cohomology with compact supports. These maps are dual to each other.

**Cup and Cap products.** The natural identification $H^l(Y, \mathbb{Q}) = \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y, \mathbb{Q}_Y[l])$ and the canonical isomorphisms $\mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y, \mathbb{Q}_Y[l]) \simeq \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y[k], \mathbb{Q}_Y[k+l])$ identify the *cup product*

$$\cup : H^l(Y, \mathbb{Q}) \times H^k(Y, \mathbb{Q}) \to H^{k+l}(Y, \mathbb{Q})$$

with the composition

$$\mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y, \mathbb{Q}_Y[l]) \times \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y[l], \mathbb{Q}_Y[k+l]) \longrightarrow \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y, \mathbb{Q}_Y[k+l]).$$

Similarly, the *cap product*

$$\cap : H_k^{BM}(Y, \mathbb{Q}) \times H^l(Y, Y \setminus Z, \mathbb{Q}) \longrightarrow H_{k-l}^{BM}(Z, \mathbb{Q})$$

relative to a closed imbedding $i : Z \to Y$ is obtained as a composition of maps in the derived category as follows:

$$\begin{tikzcd}[column sep=small]
 & & H^l(Y, Y\setminus Z, \mathbb{Q}) = \mathrm{Hom}_{\mathcal{D}_Z}(\mathbb{Q}_Z, i^!\mathbb{Q}_Y[l]) \\
 & & \times \\
H_k^{BM}(Y, \mathbb{Q}) = \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y, \omega_Y[-k]) \arrow[r] & \mathrm{Hom}_{\mathcal{D}_Z}(i^!\mathbb{Q}_Y, i^!\omega_Y[-k]) = \mathrm{Hom}_{\mathcal{D}_Z}(i^!\mathbb{Q}_Y, \omega_Z[-k]) \arrow[d] & \\
 & H_{k-l}^{BM}(Z, \mathbb{Q}) = \mathrm{Hom}_{\mathcal{D}_Z}(\mathbb{Q}_Z, \omega_Z[l-k]). &
\end{tikzcd}$$

**Gysin Map.** Let $i : Z \to Y$ be the closed embedding of a codimension $d$ submanifold of the complex manifold $Y$. We have $i_* = i_!$ and $i^! = i^*[-2d]$, the adjunction map for $i_!$ yields

$$i_*\mathbb{Q}_Z = i_! i^* \mathbb{Q}_Y = i_! i^! \mathbb{Q}_Y[2d] \longrightarrow \mathbb{Q}_Y[2d]$$

and by taking cohomology we get the Gysin map

$$H^l(Z, \mathbb{Q}) \longrightarrow H^{l+2d}(Y, \mathbb{Q}).$$

Geometrically, this can be viewed as equivalent via Poincaré Duality to the proper push-forward map in Borel-Moore homology $H_j^{BM}(Z, \mathbb{Q}) \to H_j^{BM}(Y, \mathbb{Q})$.

**Fundamental Class.** Let $i : Z \to Y$ be the closed immersion of a $d$-dimensional subvariety of the manifold $Y$. The space $Z$ carries a fundamental class in $H_{2d}^{BM}(Z)$. The fundamental class of $Z$ is the image of this class in $H_{2d}^{BM}(Y) \simeq H^{2n-2d}(Y, \mathbb{Z})$.

**Mayer-Vietoris.** There is a whole host of Mayer-Vietoris sequences (cf. [107], 2.6.10), e.g.:

$$\ldots \longrightarrow H^{l-1}(U_1 \cap U_2, K) \longrightarrow H^l(U_1 \cup U_2, K) \longrightarrow H^l(U_1, K) \oplus H^l(U_2, K) \longrightarrow \ldots$$

**Relative (co)homology** Let $U \xrightarrow{j} Y \xleftarrow{i} Z$ be the inclusions of an open subset $U \subset Y$ and of the closed complement $Z := Y \backslash U$. There are the following "attaching" distinguished triangles:

$$i_! i^! C \longrightarrow C \longrightarrow j_* j^* C \xrightarrow{[1]}, \qquad j_! j^! C \longrightarrow C \longrightarrow i_* i^* C \xrightarrow{[1]}.$$

The long exact sequences of relative (co)homology (including the versions with compact supports) arise by taking the associated long exact sequences.

**Refined intersection forms** Let $i : Z \to Y$ be a closed immersion into a nonsingular variety $Y$ of dimension $n$. There are maps

$$i_! \omega_Z[-n] = i_! i^! \omega_Y[-n] \longrightarrow \omega_Y[-n] \simeq \mathbb{Q}_Y[n] \longrightarrow i_* i^* \mathbb{Q}_Y[n] = i_* \mathbb{Q}_Z[n].$$

Taking cohomology we get the *refined intersection form* on $Z \subseteq Y$ :

$$H^{BM}_{n-l}(Z) \longrightarrow H^{n+l}(Z), \quad \text{or} \qquad H^{BM}_{n-l}(Z) \times H_{n+l}(Z) \longrightarrow \mathbb{Q}.$$

It is called refined because we are intersecting cycles in the nonsingular $Y$ which are supported on $Z$. By using Lefschetz Duality, this pairing can be viewed as the cup product in relative cohomology. This is an essential ingredient in our proof of the Decomposition Theorem [44, 46].

### 5.2.2 A formulary

Throughout this section, $f : X \to Y$ and $g : Y' \to Y$ and $h : Y \to Z$ are maps of varieties, $C \in \mathcal{D}_X$ is a constructible complex on $X$ and $K, K', K_i \in \mathcal{D}_Y$ are constructible complexes on $Y$. An equality sign actually stands for the existence of a suitably canonical isomorphism. Since we use field coefficients, the tensor product is exact and it coincides with the associated left derived functor. Perversity means middle perversity on complex varieties. All operations preserve stratifications of varieties and of maps. We use the simplified notation $f_* := Rf_*$, $f_! := Rf_!$. Some standard references are [107, 84, 16, 103, 79, 8, 81].

**Cohomology via map to a point or space.**

$$H(X, C) = H(pt, f_*C), \qquad H_c(X, C) = H(pt, f_!C);$$

$$H(X, C) = H(Y, f_*C), \qquad H_c(X, C) = H_c(Y, f_!C).$$

**Translation functors.** Let $T := f^*, f_*, f_!$ or $f^!$:

$$T \circ [j] = [j] \circ T.$$

$${}^{\mathfrak{p}}\tau_{\leq i} \circ [j] = [j] \circ {}^{\mathfrak{p}}\tau_{\leq i+j}, \qquad {}^{\mathfrak{p}}\tau_{\geq i} \circ [j] = [j] \circ {}^{\mathfrak{p}}\tau_{\geq i+j}; \qquad \text{same for } \tau.$$

$$\mathcal{H}^i \circ [j] = \mathcal{H}^{i+j}, \qquad {}^{\mathfrak{p}}\mathcal{H}^i \circ [j] = {}^{\mathfrak{p}}\mathcal{H}^{i+j}.$$

$$\mathrm{RHom}(K,K')[j]=\mathrm{RHom}(K,K'[j])=\mathrm{RHom}(K[-j],K').$$
$$R\mathcal{H}om(K,K')[j]=R\mathcal{H}om(K,K'[j])=R\mathcal{H}om(K[-j],K').$$
$$(K\otimes K')[j]=K\otimes K'[j]=K[j]\otimes K'.$$

**Morphism in $\mathcal{D}_Y$.**

$$\mathrm{Ext}^i_{\mathcal{D}_Y}(K,K')=\mathrm{Hom}_{\mathcal{D}_Y}(K,K'[i])=H^0(\mathrm{RHom}(K,K'[i]))=H^0(Y,R\mathcal{H}om(K,K'[i])).$$

If $K\in {}^{\mathfrak{p}}\mathcal{D}^{\leq i}_Y$ and $K'\in {}^{\mathfrak{p}}\mathcal{D}^{\geq i}_Y$, then (same for the standard case)

$$\mathrm{Hom}_{\mathcal{D}_Y}(K,K')=\mathrm{Hom}_{\mathcal{P}_Y}({}^{\mathfrak{p}}\mathcal{H}^i(K),{}^{\mathfrak{p}}\mathcal{H}^i(K')).$$

For sheaves, $\mathrm{Ext}^{<0}(F,G)=0$ and, $\mathrm{Ext}^{i>0}(F,G)$ is the group of Yoneda $i$-extensions of $G$ by $F$. The group $\mathrm{Ext}^1(F,G)$ is the set of equivalence classes of short exact sequences $0\to F\to ?\to G\to 0$ with the Baer sum operation. For complexes, $\mathrm{Ext}^1(K,K')$ classifies, distinguished triangles $K\to ?\to K'\to K[1]$.

**Adjunction**

$$\mathrm{RHom}(f^*K,C)=\mathrm{RHom}(K,f_*C),\qquad \mathrm{RHom}(f_!C,K)=\mathrm{RHom}(C,f^!K),$$
$$\mathrm{RHom}(K_1\otimes K_2,K_3)=\mathrm{RHom}(K_1,R\mathcal{H}om(K_2,K_3));$$
$$f_*R\mathcal{H}om(f^*K,C)=R\mathcal{H}om(K,f_*C),\qquad R\mathcal{H}om(f_!C,K)=f_*R\mathcal{H}om(C,f^!K),$$
$$R\mathcal{H}om(K_1\otimes K_2,K_3)=R\mathcal{H}om(K_1,R\mathcal{H}om(K_2,K_3)).$$

If all $\mathcal{H}^j(K_3)$ are locally constant, then

$$R\mathcal{H}om(K_1,K_2\otimes K_3)=R\mathcal{H}om(K_1,K_2)\otimes K_3.$$

**Transitivity.**

$$(hf)_*=h_*f_*,\quad (hf)_!=h_!f_!,\quad (hf)^*=f^*h^*,\quad (hf)^!=f^!h^!,$$
$$f^*(K\otimes K')=f^*K\otimes f^*K',\qquad f^!R\mathcal{H}om(K,K')=R\mathcal{H}om(f^*K,f^!K').$$

**Change of coefficients.**

$$K\otimes f_!C\simeq f_!(f^*K\otimes C).$$

**Duality exchanges.**

$$DK:=K^\vee:=R\mathcal{H}om(K,\omega_Y),\qquad \omega_Y:=\gamma^!\mathbb{Q}_{pt},\ \ \gamma:Y\to pt.$$

$$D:{}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y\longrightarrow {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y,\qquad D:{}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y\longrightarrow {}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y,\qquad D:\mathcal{P}_Y\simeq \mathcal{P}^{opp}_Y.$$

If $F : \mathcal{D}_X \to \mathcal{D}_Y$ is left (right, resp.) $t$-exact, then $D \circ F \circ D$ is right (left, resp.) $t$-exact. Similarly, for $G : \mathcal{D}_Y \to \mathcal{D}_X$.

$$\omega_Y = \mathbb{Q}_Y^\vee;$$

$$D \circ [j] = [-j] \circ D;$$

$$D_Y \circ f_* = f_! \circ D_X, \qquad D_X \circ f^* = f^! \circ D_Y;$$

$$D \circ {}^{\mathfrak{p}}\tau_{\leq j} = {}^{\mathfrak{p}}\tau_{\geq -j} \circ D, \qquad D \circ {}^{\mathfrak{p}}\tau_{\geq j} = {}^{\mathfrak{p}}\tau_{\leq -j} \circ D, \qquad {}^{\mathfrak{p}}\mathcal{H}^j \circ D = D \circ {}^{\mathfrak{p}}\mathcal{H}^{-i};$$

$$D(K \otimes K') = R\mathcal{H}om(K, DK').$$

$$D^2 = Id \quad \text{(biduality)}.$$

**Poincaré-Verdier Duality.**

$$H^j(Y, DK) \simeq H_c^{-j}(Y, K)^\vee.$$

If $Y$ is smooth of pure complex dimension $n$ and is canonically oriented:

$$\omega_Y = \mathbb{Q}_Y[-2n].$$

**Support conditions for perverse sheaves.**
Support conditions: $K \in {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0}$ iff $\dim \mathrm{Supp}\mathcal{H}^i(K) \leq -i$, for every $i$.
Co-support conditions: $K \in {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0}$ iff $\dim \mathrm{Supp}\mathcal{H}^i(DK) \leq -i$, for every $i$.
A perverse sheaf is a complex subject to the support and co-support conditions.

**Base Change.** Consider the Cartesian square, where the ambiguity of the notation does not generate ambiguous statements:

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

Base change isomorphisms:

$$g^! f_* = f_* g^!, \qquad f_! g^* = g^* f_!.$$

For the immersion of a point $g : y \to Y$

$$H_c^l(f^{-1}(y), C) = (R^l f_! C)_y; \qquad H^l(f^{-1}(y), C) = (R^l f_* C)_y \quad (f \text{ proper}).$$

Base change maps:

$$g^* f_* \longrightarrow f_* g^*, \qquad f_! g^! \simeq g^! f_!.$$

Proper (Smooth, resp.) Base Change: if $f$ is proper ($g$ is smooth, resp.), then the base change maps are isomorphisms.

There are natural maps

$$g_! f_* \longrightarrow f_* g_!, \qquad f_! g_* \longrightarrow g_* f_!.$$

**Intermediate extension functor.** For $f$ a locally closed embedding

$$f_{!*} : \mathcal{P}_X \longrightarrow \mathcal{P}_Y, \quad P \longmapsto \mathrm{Im}\,\{\,{}^{\mathfrak{p}}\mathcal{H}^0(f_! P) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^0(f_* P)\}.$$

If $X = U_{l+1}$ is the union of strata of dimension $\geq l+1$, then $f_{!*}P$ is computable by iteration of the formula

$$j_{l!*}P = \tau_{\leq -l-1}\, j_{l*}\, P, \qquad j_l : U_{l+1} \to U_l.$$

For an open immersion, the intermediate extension is characterized as the extension with no subobjects and no quotients supported on the boundary (however, it may have such subquotients).

**Intersection Cohomology complexes.** Let $L$ be a local system on a nonsingular Zariski dense open subset $j : U \to Y$ of the irreducible $n$-dimensional $Y$.

$$IC_Y(L) := j_{!*}L[n] \;\in\; \mathcal{P}_Y.$$

If the smallest dimension of a stratum is $d$, then

$$\mathcal{H}^l(IC_Y(L) = 0, \quad \forall j \neq [-n, -d-1];$$

note that for a general perverse sheaf, the analogous range is $[-n, -d]$.
As to duality:

$$D(IC_Y(L)) = IC_Y(L^{\vee}).$$

The category $\mathcal{P}_Y$ is Artinian and Noetherian. The simple objects are the intersection cohomology complexes of simple local systems on irreducible subvarieties.

**Nearby and vanishing cycles.** With a regular function $f : Y \to \mathbb{C}$ are associated the two functors $\Psi_f, \Phi_f : \mathcal{D}_Y \to \mathcal{D}_{Y_0}$, where $Y_0 = f^{-1}(0)$. If $Y \setminus Y_0 \xrightarrow{j} Y \xleftarrow{i} Y_0$, there are exact triangles:

$$i^*K \longrightarrow \Psi_f(K) \xrightarrow{\mathrm{can}} \Phi_f(K)[1] \xrightarrow{[1]}, \qquad i^!K \longrightarrow \Phi_f(K) \xrightarrow{\mathrm{var}} \Psi_f(K)[-1] \xrightarrow{[1]}.$$

The functors $\Psi_f, \Phi_f$ are endowed with the monodromy automorphism $T$ and

$$\mathrm{can} \circ \mathrm{var} = T - I : \Phi_f(K) \to \Phi_f(K) \qquad \mathrm{var} \circ \mathrm{can} = T - I : \Psi_f(K) \to \Psi_f(K).$$

There is the distinguished triangle

$$i^*j_*j^*K \longrightarrow \Psi_f(K) \xrightarrow{T-I} \Psi_f(K) \xrightarrow{[1]}$$

Up to a shift, the functors $\Psi_f, \Phi_f$ commute with duality and are $t$-exact:

$$\Psi_f \circ D = D \circ \Psi_f \circ [2], \quad \Phi_f \circ D = D \circ \Phi_f \circ [2], \qquad \Psi_f[-1], \Phi_f[-1] : \mathcal{P}_Y \longrightarrow \mathcal{P}_{Y_0}.$$

For $K \in \mathcal{P}_{Y \setminus Y_0}$, the long exact sequence for the triangle above gives:

$$
\begin{aligned}
{}^{\mathfrak{p}}\mathcal{H}^{-1}(i^* j_* K) \; &= \; \mathrm{Ker}\{\, \Psi_f(K)[-1] \xrightarrow{T-I} \Psi_f(K)[-1] \,\}, \\
{}^{\mathfrak{p}}\mathcal{H}^{0}(i^* j_* K) \; &= \; \mathrm{Coker}\{\, \Psi_f(K)[-1] \xrightarrow{T-I} \Psi_f(K)[-1] \,\},
\end{aligned}
$$

$j_*K$ and $j_!K \in \mathcal{P}_Y$ and comparing the above equalities with the triangle:

$$i^* j_* K \xrightarrow{[1]} j_! K \longrightarrow j_* K \longrightarrow$$

yields

$$
\begin{aligned}
\mathrm{Ker}\,\{\, j_! K \to j_{!*} K \,\} \; &\simeq \; \mathrm{Ker}\,\{\, \Psi_f(K)[-1] \xrightarrow{T-I} \Psi_f(K)[-1] \,\}, \\
\mathrm{Coker}\,\{\, j_{!*} K \to j_* K \,\} \; &\simeq \; \mathrm{Coker}\,\{\, \Psi_f(K)[-1] \xrightarrow{T-I} \Psi_f(K)[-1] \,\}.
\end{aligned}
$$

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Authors' addresses:

Mark Andrea A. de Cataldo, Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA. e-mail: *mde@math.sunysb.edu*

Luca Migliorini, Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, ITALY. e-mail: *migliori@dm.unibo.it*