Intersection forms, topology of maps and motivic decomposition for resolutions of threefolds

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Contents

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1 Introduction

This paper has two aims.
The former is to give an introduction to our earlier work [8] and more generally to some of the main themes of the theory of perverse sheaves and to some of its geometric applications. Particular emphasis is put on the topological properties of algebraic maps.
The latter is to prove a motivic version of the decomposition theorem for the resolution of a threefold Y. This result allows to define a pure motive whose Betti realization is the intersection cohomology of Y.

We assume familiarity with Hodge theory and with the formalism of derived categories. On the other hand, we provide a few explicit computations of perverse truncations and intersection cohomology complexes which we could not find in the literature and which may be helpful to understand the machinery.

We discuss in detail the case of surfaces, threefolds and fourfolds. In the surface case, our "intersection forms" version of the decomposition theorem stems quite naturally from two well-known and widely used theorems on surfaces, the Grauert contractibility criterion for curves on a surface and the so called "Zariski Lemma," cf. [1].

The following assumptions are made throughout the paper

Assumption 1.0.1 *We work with varieties over the complex numbers. A* map *$f : X \to Y$ is a proper morphism of varieties. We assume that X is smooth. All (co)homology groups are with rational coefficients.*

These assumptions are placed for ease of exposition only, for the main results remain valid when X is singular if one replaces the cohomology of X with its intersection cohomology, or the constant sheaf $\mathbb{Q}_X$ with the intersection cohomology complex of X.

It is a pleasure to dedicate this work to J. Murre, with admiration and respect.

2 Intersection forms

2.1 Surfaces

Let $D = \cup D_k \subseteq X$ be a finite union of compact irreducible curves on a smooth complex surface. There is a sequence of maps

$$H_2(D) \xrightarrow{r_*} H_2^{BM}(X) \overset{PD}{\simeq} H^2(X) \xrightarrow{r^*} H^2(D). \tag{1}$$

The group $H_2(D)$ is freely generated by the fundamental classes $[D_k]$.
The group $H^2(D) \simeq H_2(D)^\vee$ and, via Mayer-Vietoris, it is freely generated by the classes associated with points $p_k \in D_k$. The map

$$H_2(D) \xrightarrow{cl} H^2(X), \qquad cl := PD \circ r_*$$

is called the *class map* and it assigns to the fundamental class $[D_k]$ the cohomology class $c_1(\mathcal{O}_X(D_k))$.
The restriction map r, or rather $r \circ PD$, assigns to a Borel Moore $2-$cycle meeting transversely all the D_k, the points of intersection with the appropriate multiplicities.
The composition $H_2(D) \longrightarrow H^2(D)$, gives rise to the so-called refined intersection form on $D \subseteq X$:

$$\iota : H_2(D) \times H_2(D) \longrightarrow \mathbb{Q} \tag{2}$$

with associated symmetric intersection matrix $||D_h \cdot D_k||$.
If X is replaced by the germ of a neighborhood of D, then X retracts to D so that all four spaces appearing in (1) have the same dimension $b_2(D)$ =numbers of curves in D.
In this case the restriction map r is an isomorphism: the Borel Moore classes of disks transversal to the D_k map to the point of intersection.
On the other hand, cl may fail to be injective, e.g. $(\mathbb{C} \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1)$.

The following are two classical results results concerning the properties of the intersection form ι, dealing respectively with resolutions of normal surface singularities and one dimensional families of curves. They are known as the Grauert's Criterion and the Zariski Lemma (cf. [1], p.90).

Theorem 2.1.1 *Let $f : X \to Y$ be the contraction of a divisor D to a normal surface singularity. Then the refined intersection form ι on $H_2(D)$ is negative definite. In particular, the class map cl is an isomorphism.*

Theorem 2.1.2 *Let $f : X \to Y$ be a surjective proper map of quasi-projective smooth varieties, X a surface, Y a curve. Let $D = f^{-1}(y)$ be any fiber. Then the rank of cl is $b_2(D)-1$. More precisely, let $F = \sum_k a_k D_k$, $a_k > 0$, be the cycle-theoretic fiber. $F \cdot F = 0$ and the induced bilinear form*

$$\frac{H_2(D)}{\langle [F] \rangle} \times \frac{H_2(D)}{\langle [F] \rangle} \longrightarrow \mathbb{Q}$$

is non degenerate and negative definite.

Remark 2.1.3 Theorem 2.1.1 can be interpreted in terms of the topology of the "link" $\mathcal{L}$ of the singularity. Let N be a small contractible neighborhood of a singular point y and $\mathcal{L}$ be its boundary. Choose analytic disks $\Delta_1, \cdots, \Delta_r$ cutting transversally the divisors $D_1, \cdots, D_r$ at regular points. The classes of these disks, generate the Borel-Moore homology $H_2^{BM}(f^{-1}(N)) \simeq H^2(f^{-1}(N))$. The statement 2.1.1 implies that each class Δ_i is homologous to a rational linear combination of exceptional curves. Equivalently, for every index i some multiple of the $1-$cycle $\Delta_i \cap \mathcal{L}$ bounds in the link $\mathcal{L}$ of y. This is precisely what fails in the aforementioned example $(\mathbb{C} \times \mathbb{P}^1, \{0\} \times \mathbb{P}^1)$. A similar interpretation is possible for the "Zariski lemma."

In view of the important role played by these theorems in the theory of complex surfaces it is natural to ask for generalization to higher dimension. We next define what is the analogue of the intersection form for a general map $f : X \to Y$ (cf. 1.0.1)

2.2 Intersection forms associated to a map

General theorems, due to J. Mather, R. Thom and others (cf. [16]) ensure that a projective map $f : X \to Y$ can be stratified, i.e. there is a decomposition $\mathfrak{Y} = \coprod S_l$ of Y with locally closed nonsingular subvarieties S_l, the strata, so that $f : f^{-1}(S_l) \to S_l$ is, for any l, a topologically locally trivial fibration. Such stratification allows us, when X is nonsingular, to define a sequence of intersection forms. Let L be the pullback of an ample bundle on Y. The idea is to use sections of L to construct transverse slices and reduce the strata to points, and to use a very ample line bundle η on X to fix the ranges:

Let $\dim S_l = l$, let s_l a generic point of the stratum S_l and Y_s a complete intersection of l hyperplane sections of Y passing through s_l, transverse to S_l;

As we did for surfaces, we consider the maps:

$$I_{l,0} : H_{n-l}(f^{-1}(s_l)) \times H_{n-l}(f^{-1}(s_l)) \longrightarrow \mathbb{Q}.$$

obtained intersecting cycles supported in $f^{-1}(s)$ in the smooth $(n-l)-$dimensional ambient variety $f^{-1}(Y_s)$:

$$H_{n-l}(f^{-1}(s_l)) \to H_{n-l}(f^{-1}(Y_s)) \simeq H^{n-l}(f^{-1}(Y_s)) \to H^{n-l}(f^{-1}(s_l)).$$

We can define other intersection forms, in different ranges, cutting the cycles in $f^{-1}(s_l)$ with generic sections of η.

The composition:

$$H_{n-l-k}(f^{-1}(s)) \to H_{n-l-k}(f^{-1}(Y_s)) \simeq H^{n-l+k}(f^{-1}(Y_s)) \to H^{n-l+k}(f^{-1}(s)).$$

gives maps

$$I_{l,k} : H_{n-l-k}(f^{-1}(s_l)) \times H_{n-l+k}(f^{-1}(s_l)) \longrightarrow \mathbb{Q}.$$

Let us denote by

$$\cap\eta^k : H_{n-l+k}(f^{-1}(s_l)) \to H_{n-l-k}(f^{-1}(s_l)),$$

the operation of cutting a cycle in $f^{-1}(s_l)$ with k generic sections of η.

Composing this map with $I_{l,k}$, we obtain the intersection forms we will consider:

$$I_{l,k}(\cap\eta^k\cdot, \cdot) : H_{n-l+k}(f^{-1}(s_l)) \times H_{n-l+k}(f^{-1}(s_l)) \longrightarrow \mathbb{Q}.$$

Remark 2.2.1 These intersection forms depend on η but not on the particular sections used to cut the dimension. They are independent of L. In fact we could define them using a local slice of the stratum S_l and its inverse image, without reference to sections of L.

Example 2.2.2 Let $f : X \to Y$ be a resolution of singularities of a threefold Y, with a stratification $Y_0 \coprod C \coprod y_0$, defined so that f is an isomorphism over Y_0, the fibers are one-dimensional over C, and there is a divisor $D = \cup D_i$ contracted to the point y_0. We have the following intersection forms:

– let c be a general point of C and $s \in H^0(Y, \mathcal{O}(1))$ be a generic section vanishing at c; there is the form $H_2(f^{-1}(c) \times H_2(f^{-1}(c)) \longrightarrow \mathbb{Q}$ which is nothing but the Grauert-type form on the surface $f^{-1}(\{s = 0\})$;

– similarly, over y_0, there is the form on $H_4(D)$ given by $\eta \cap [D_i] \cdot [D_j]$; it is a Grauert-type form, computed on a hyperplane section of X with respect to η;

– finally, we have the more interesting $H_3(D) \times H_3(D) \longrightarrow \mathbb{Q}$.

One of the dominant themes of this paper is that *Hodge theory affords non degeneracy results for these forms and that this non degeneration has strong cohomological consequences.*

To see why Hodge theory is relevant to the study of the intersection forms, let us sketch a proof of Theorem 2.1.1, in the hypothesis that X and Y are projective. The proof we give is certainly not the most natural or economic. Its interest lies in the fact that, while the original proof seems difficult to generalize to higher dimension, this one can be generalized. It is based on the observation that the classes $[D_i]$ of the exceptional curves are "primitive" with respect to the cup product with the first Chern class of any ample line bundle pulled back from Y. Even though such a line bundle is certainly not ample, some parts of the "Hodge package," namely the Hard

Lefschetz theorem and the Hodge Riemann bilinear relations, go through. To prove this, we introduce a technique, which we call *approximation of $L-$primitives,* which plays a decisive role in what follows.

Proof of 2.1.1 in the case X and Y are projective.
Let L be the pullback to X of an ample line bundle on Y. Since the map is dominant, $L^2 \neq 0$, and we get the Hodge-Lefschetz type decomposition:

$$H^2(X, \mathbb{R}) \,=\, \mathbb{R}\langle c_1(L) \rangle \oplus \mathrm{Ker}\ \{c_1(L)\wedge : H^2(X) \to H^4(X)\}.$$

Denote the kernel above by P^2. This decomposition is orthogonal with respect to the Poincaré duality pairing which, in turn, is non degenerate when restricted to the two summands. The decomposition holds with rational coefficients. However, real coefficients are more convenient in view of taking limits.
Consider a sequence of Chern classes of ample $\mathbb{Q}-$line bundles L_n, converging to the Chern class of L, e.g. $L_n = L + \frac{1}{n}\eta$, η ample on X. Define $P^2_{1/n} = \mathrm{Ker}\ \{c_1(L_n) : H^2(X) \to H^4(X)\}$. These are $(b_2-1)-$dimensional subspaces of $H^2(X)$. Any limit point of the sequence $P^2_{1/n}$ in $\mathbb{P}^{b_2}(\mathbb{R})$ gives a codimension one subspace $W \subseteq H^2(X)$, contained in $\mathrm{Ker}\ \{c_1(L) : H^2(X) \to H^4(X)\} = P^2$. Since $\dim W = b_2 - 1 = \dim P^2$, we must have $\lim_n P^2_{1/n} = P^2$.

The Hodge Riemann Bilinear Relations hold on $P^2_{1/n}$ by classical Hodge theory. The duality pairing on the limit P^2 is non degenerate. It follows that the Hodge Riemann Bilinear Relations hold on P^2 as well.

The classes of the exceptional curves D_i are in P^2, since we can choose a section of the very ample line bundle on Y not passing through the singular point and pull it back to X.

The fact that these classes are independent is known classically. Let us briefly mention here that if there is only one component D_i then $0 \neq [D_i] \in H^2(X)$ in the Kähler X. In general, one may also argue along the following lines (cf. [6], [5], §8): use the Leray spectral sequence over an affine neighborhood V of the singularity y to show that $H^2(f^{-1}(V)) \to H^2(f^{-1}(y))$ is surjective; use the basic properties of mixed Hodge structures to deduce that $H^2(X) \to H^2(f^{-1}(y))$ is also surjective; conclude by dualizing and by Poincaré Duality.

The classes $[D_i]$ are real of type $(1,1)$ and for such classes $\alpha \in P^2 \cap H^{1,1}$ the Hodge Riemann bilinear relations give

$$\int_X \alpha \wedge \alpha \,<\, 0$$

whence the statement of 2.1.1. $\square$

2.3 Resolutions of isolated singularities in dimension 3

In this section we study the intersection forms in the case of the resolution of three-dimensional isolated singularities. Many of the features and techniques used in the general case emerge already in this case. Besides motivating what follows, we believe that the statements and the techniques used here are of some independent interest.

We prove all the relevant Hodge-theoretic results about the intersection forms associated to the resolution of an isolated singular point on a threefold. This example will be reconsidered in the last section, where we give a motivic version of the Hodge theoretic decomposition proved here.

As is suggested in the proof of Theorem 2.1.1 sketched at the end of the previous section, in order to draw conclusions on the behaviour of the intersection forms, we must investigate the extent to which the Hard Lefschetz theorem and the Hodge Riemann Bilinear Relations hold when we consider the cup product with the Chern class of the pullback of an ample bundle by a projective map. In order to motivate what follows let us recall an inductive proof of the Hard Lefschetz theorem based on the Hodge Riemann relations:

Hard Lefschetz and Hodge-Riemann relations in dimension $(n-1)$ and Weak Lefschetz in dimension n imply Hard Lefschetz in dimension n.

Let X be projective nonsingular and X_H be a generic hyperplane section with respect to a very ample bundle η. Consider the map $c_1(\eta) : H^{n-1}(X) \to H^{n+1}(X)$. The Hard Lefschetz theorem states it is an isomorphism. By the Weak Lefschetz Theorem $i^* : H^{n-1}(X) \to H^{n-1}(X_H)$ is injective, and its dual $i_* : H^{n-1}(X_H) \to H^{n+1}(X)$, with respect to Poincaré duality on X and X_H, is surjective. The cup product with $c_1(\eta)$ is the composition $i_* \circ i^*$

$$\begin{array}{ccc} H^{n-1}(X) & \xrightarrow{\quad c_1(\eta)\quad} & H^{n+1}(X) \\ & {\scriptstyle i^*}\searrow \quad \nearrow{\scriptstyle i_*} & \\ & H^{n-1}(X_H) & \end{array}$$

and is therefore an isomorphism if and only if the bilinear form $\int_{X_H}$ remains non degenerate when restricted to the subspace $H^{n-1}(X) \subseteq H^{n-1}(X_H)$. This inclusion is a Hodge substructure. The Hodge Riemann relations on X_H imply that the Hodge structure $H^{n-1}(X_H)$ is a direct sum of Hodge structures polarized by the pairing $\int_{X_H}$. It follows that the restriction of the Poincaré form $\int_{X_H}$ to $H^{n-1}(X)$ is non degenerate, as wanted. The other cases of the Hard Lefschetz Theorem (i.e. $c_1(\eta)^k$ for $k \geq 2$) follow immediately from the weak Lefschetz theorem and the Hard Lefschetz theorem for X_H. $\square$

Assumption 2.3.1 *Y is projective with an isolated singular point y, dim$Y = 3$. X is a resolution and $f : X \to Y$ is an isomorphism when restricted to $f^{-1}(Y - y)$. Suppose $D = f^{-1}(y)$ is a divisor and let D_i be its irreducible components.*

As usual in this paper, we will denote by η a very ample line bundle on X, and by L the pullback to X of a very ample line bundle on Y. Of course L is not ample. We want to investigate whether the Hard Lefschetz theorem and the Hodge Riemann relations hold if we consider cup-product with $c_1(L)$ instead of with an ample line bundle.

Remark 2.3.2 Since $c_1(L)^3 \neq 0$ we have an isomorphism $c_1(L)^3 : H^0(X) \to H^6(X)$.

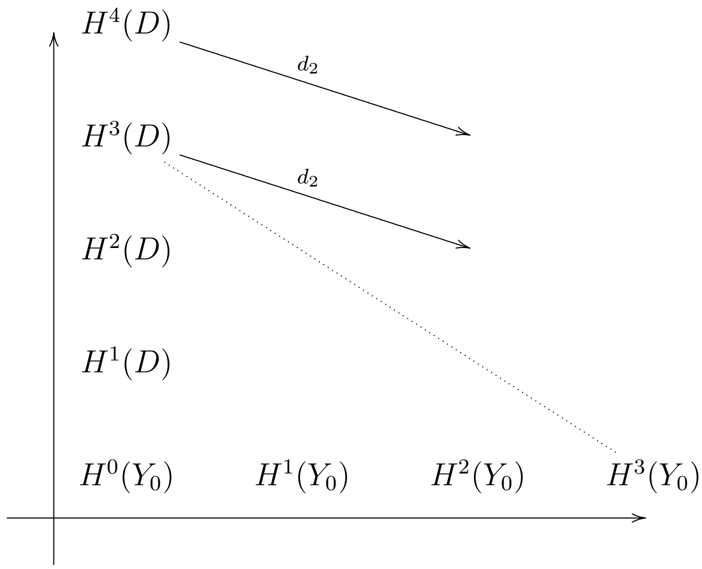
Remark 2.3.3 Clearly the classes $[D_i] \in H^2(X)$ are killed by the cup product with $c_1(L)$, since we can pick a generic section of $\mathcal{O}_Y(1)$ not passing through y and its inverse image in X will not meet the D_i. Since $[D_i] \neq 0$, it follows that $c_1(L) : H^2(X) \to H^4(X)$ is not an isomorphism.

We now prove that in fact the subspace Im $\{H_4(D) \to H^2(X)\}$ generated by the classes $[D_i]$ is precisely Ker $c_1(L) : H^2(X) \to H^4(X)$.

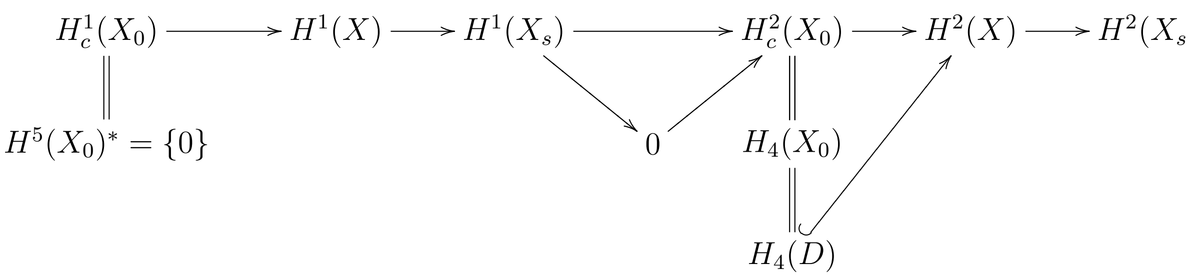
Theorem 2.3.4 *Let $s \in \Gamma(Y, \mathcal{O}_Y(1))$ be a generic section and $X_s = f^{-1}(\{s = 0\}) \xrightarrow{i} X$. Then:*

a. $i^* : H^1(X) \to H^1(X_s)$ is an isomorphism.
b. $i_* : H^3(X_s) \to H^5(X)$ is an isomorphism.
c. $i^* : H^2(X)/(\mathrm{Im}\ \{H_4(D) \to H^2(X)\}) \to H^2(X_s)$ is injective.
d. $i_* : H^2(X_s) \to \mathrm{Ker}\ \{H^4(X) \to H^4(D)\}$ is surjective.
e. The map $H_3(D) \to H^3(X)$ is injective.

Proof. Set $X_0 = X \setminus X_s$ e $Y_o = Y \setminus \{s = 0\}$ and let us consider the Leray spectral sequence for $f : X_0 \to Y_0$. Since Y_0 is affine, we have $H^k(Y_0) = 0$ for $k > 3$.



The sequence degenerates so that we have surjections $H^3(X_0) \to H^3(D)$ and $H^4(X_0) \to H^4(D)$. But from [13], Proposition 8.2.6, $H^3(X) \to H^3(D) \to 0$ and $H^4(X) \to H^4(D) \to 0$ are also surjective. We have the long exact sequence



The other statements are obtained applying duality. $\square$

Since on $H^2(X_s)$ the bilinear relation of Hodge-Riemann hold, the argument given at the beginning of this section shows that

$$c_1(L)^2 : H^1(X) \longrightarrow H^5(X) \text{ is an isomorphism}$$

and

$$c_1(L) \,:\, H^2(X)/H_4(D) \longrightarrow \text{Ker } \{H^4(X) \to H^4(D)\} \text{ is an isomorphism.}$$

The Hodge Riemann relations hold for $P^1 := H^1(X)$, $P^2 := \text{Ker } c_1(L)^2 : H^2(X)/H_4(D) \to H^6(X)$ since, by the weak Lefschetz Theorem, they follow from those for X_s.

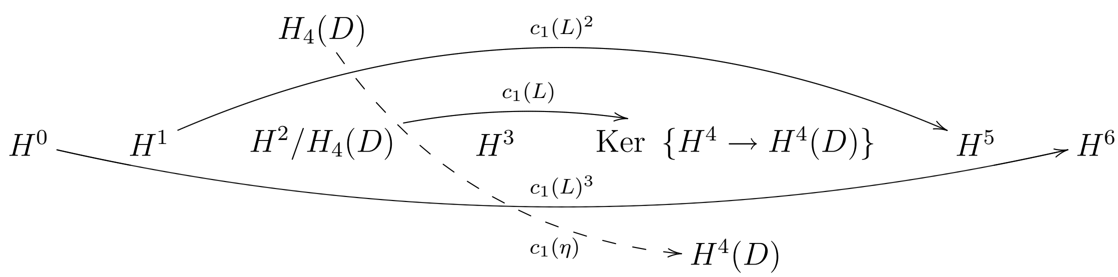
The Hodge Riemann relations for $P^3 = \text{Ker } \{c_1(L) : H^3(X) \to H^5(X)\}$, of which $H_3(D)$ is a subspace, must be considered separately: the main technique to be used here is the *approximation of primitives* introduced in the previous section to prove Theorem 2.1.1.

Theorem 2.3.5 *The Poincaré pairing $\int_X$ is a polarization of P^3*

Proof. Since $c_1(L)^2 : H^1(X) \to H^5(X)$ is an isomorphism, there is a decomposition, orthogonal with respect to the Poincarè pairing, $H^3(X) = P^3 \oplus c_1(L)H^1(X)$ and, in particular, $\dim P^3 = b_3 - b_1$, just as if L were ample. The Poincarè pairing remains nondegenerate when restricted to P^3. The classes $c_1(L) + \frac{1}{n}c_1(\eta)$ are Chern classes of ample line bundles, hence $P^3_{1/n} = \text{Ker } \{c_1(L) + \frac{1}{n}c_1(\eta) : H^3(X) \to H^5(X)\}$ are $b_3 - b_1-$dimensional subspaces of $H^3(X)$. As in the proof of 2.1.1 a limit point of the sequence $P^3_{1/n}$, considered as points in the real Grassmannian $Gr(b_3 - b_1, b_3)$, gives a subspace of $H^3(X)$, contained in Ker $\{c_1(L) : H^3(X) \to H^5(X)\} = P^3$ and, by equality of dimensions, $\lim P^3_{1/n} = P^3$. The Hodge Riemann relations must then hold on the limit P^3 as explained in the proof of Theorem 2.1.1. □

Finally, let us remark that the cup-product with η gives an isomorphism $c_1(\eta) : H_4(D) \to H^4(D)$ via the bilinear form $\int_X c_1(\eta) \wedge [D_i] \wedge [D_j]$ which is negative definite. As we remarked in 2.2.2 this form is just the intersection form on the exceptional curves of the restriction of f to a hyperplane section (with respect to η) of X.

Summarizing:



the groups in the central row behave, with respect to L, as the cohomology of a projective nonsingular variety on which L is ample.

We are now in position to prove the first nontrivial fact on intersection forms which generalizes 2.1.1:

Corollary 2.3.6 *$H^3(D)$ has Hodge structure which is pure of weight 3 and the Poincaré form is a polarization. In particular, the (skew-symmetric) intersection form $H_3(D) \times H_3(D) \to \mathbb{Q}$ is non degenerate.*

Proof. This follows because $H_3(D) \to H^3(X)$, is injective and identifies $H^3(D)$ with a Hodge substructure of the polarized Hodge structure $P^3 \subseteq H^3(X)$. □

This is a nontrivial criterion for a configuration of (singular) surfaces contained in a nonsingular threefold to be contractible to a point. See [8], Corollary 2.1.11 for a generalization to arbitrary dimension.
For example, the purity of the Hodge structure implies that $H^3(D) = \oplus H^3(D_i)$.

We will see that the non degeneracy statement of 2.3.6 also plays an important role in the motivic decomposition of X described in section 5.

Remark 2.3.7 The same analysis can be carried on with only notational changes for an arbitrary generically finite map from a nonsingular threefold X, e.g. assuming that there is also some divisor which is blown down to a curve etc. In this case the Hodge structure of X can be further decomposed, splitting of a piece corresponding to the contribution to cohomology of this divisor.

Remark 2.3.8 The classical argument of Ramanujam [20], [14], to derive the Akizuki-Kodaira-Nakano Vanishing Theorem from Hodge theory and Weak Lefschetz can be adapted to give the following sharp version: if L is a line bundle on a threefold X, with $L^3 \neq 0$, a multiple of which is globally generated, then

$$H^p(X, \Omega^q_X \otimes L^{-1}) = 0$$

for $p+q < 2$, and for $p+q = 2$ but $(p,q) \neq (1,1)$. More precisely $H^1(X, \Omega^1_X \otimes L^{-1}) \neq 0$ if and only if some divisor is contracted to a point.

2.4 Resolutions of isolated singularities in dimension 4

Let us quickly consider another similar example in dimension 4,

Assumption 2.4.1 $f : X \to Y$, *where* Y *still has a unique singular point* y *and* X *is a resolution. As before,* η *will denote a very ample bundle on* X*, and* L *the pull-back of a very ample bundle on* Y*. Set* $D = f^{-1}(y)$.

An argument completely analogous to the one used in the previous example shows that the sequence of spaces $H^0(X)$, $H^1(X)$, $H^2(X)/H_6(D)$, $H^3(X)/H_5(D)$, $H^4(X)$, Ker $\{H^5(X) \to H^5(D)\}$, Ker $\{H^6(X) \to H^6(D)\}$, $H^7(X)$, $H^8(X)$ satisfies the Hard Lefschetz Theorem with respect to the cup product with L. The corresponding primitive spaces P^1, P^2, P^3, are endowed with pairing satisfying the Hodge Riemann bilinear relations. The new fact that we have to face shows up when studying the Hodge Riemann bilinear relations on $H^4(X)$. The "approximation of primitives" technique here must be modified, since the dimension of

$P^4 = \text{Ker } \{c_1(L) : H^4(X) \to H^6(X)\}$ is greater than $b_4 - b_2$. Hence, if we introduce the primitive spaces $P^4_{1/n} = \text{Ker } \{c_1(L) + \frac{1}{n}c_1(\eta) : H^4(X) \to H^6(X)\}$ with respect to the ample classes $c_1(L) + \frac{1}{n}c_1(\eta)$, their limit is a proper subspace, of dimension $b_4 - b_2$, of P^4. We can determine the exact dimension of P^4:

Lemma 2.4.2 $\dim \text{Ker } \{c_1(L) : H^4(X) \to H^6(X)\} \;=\; b_4 - b_2 + \dim H_6(D)$.

Proof. Since $c_1(L)^2 : H^2(X)/H_6(D) \stackrel{c_1(L)}{\to} H^4(X) \stackrel{c_1(L)}{\to} \text{Ker } \{H^6(X) \to H^6(D)\}$ is an isomorphism, we have an orthogonal decomposition

$$H^4(X) = P^4 \oplus \text{Im } \{c_1(L) : H^2(X) \to H^4(X)\}.$$

The statement follows from: $\text{Ker } \{c_1(L) : H^2(X) \to H^4(X)\} = H_6(D)$. □

The "excess" dimension of P_4 is thus $\dim H_6(D)$. On the other hand P^4 contains an obvious subspace of this dimension, namely $c_1(\eta)H_6(D)$, the subspace generated by the classes obtained intersecting the irreducible components of the exceptional divisor with a generic hyperplane section.

Remark 2.4.3 The intersection form $\int_X c_1(\eta)^2 \wedge [D_i] \wedge [D_j]$ is negative definite, as it is just the intersection form on the exceptional curves of a double hyperplane section of X.

This last remark implies the following orthogonal decomposition

$$H^4(X) = c_1(\eta)H_6(D) \oplus (c_1(\eta)H_6(D))^{\perp} \cap P^4 \oplus \text{Im } \{c_1(L) : H^2(X) \to H^4(X)\}.$$

and $(c_1(\eta)H_6(D))^{\perp} \cap P^4$ has dimension $b_4 - b_2$. This subspace turns out to be the subspace of "approximable $L-$primitives" we are looking for, as shown in the following

Theorem 2.4.4

$$\lim_{n\to\infty} \text{Ker } \{c_1(L) + \frac{1}{n}c_1(\eta) : H^4(X) \to H^6(X)\} \;=\; (c_1(\eta)\, H_6(D))^{\perp} \cap P^4.$$

Proof. The two subspaces have the same dimension, so it is enough to prove that

$$\text{Ker } \{c_1(L) + \frac{1}{n}c_1(\eta) : \{H^4(X) \to H^6(X)\}\} \subseteq (c_1(\eta)H_6(D))^{\perp}.$$

If $(c_1(L) + \frac{1}{n}c_1(\eta)(\alpha)) = 0$, then, using $c_1(L)[D_i] = 0:$

$$\int_X c_1(\eta) \wedge [D_i] \wedge \alpha \;=\; -n \int_X c_1(L) \wedge [D_i] \wedge \alpha = 0.$$

□

Corollary 2.4.5 *The Poincaré pairing is a polarization of the weight* 4 *pure Hodge structure* $(c_1(\eta)H_6(D))^\perp \cap P^4$.

Let us spell out the consequences of this analysis for the intersection form $H_4(D) \times H_4(D) \to \mathbb{Q}$. First notice that the same argument used in the proof of 2.3.4.e shows that the map $H_4(D) \to H^4(X)$ is injective. It follows that $H_4(D)$ has a pure Hodge structure which is the direct sum of two substructures polarized (with opposite signs) by the Poincaré pairing.

The next result shows that in fact $H_4(D)$ is the direct sum of two substructures, polarized (with opposite signs) by the Poincaré pairing. This gives a clear indication of what happens in general:

Corollary 2.4.6 *The intersection form* $H_4(D) \times H_4(D) \to \mathbb{Q}$ *is non degenerate. There is a direct sum decomposition:*

$$H_4(D) = c_1(\eta)H_6(D) \oplus (c_1(\eta)H_6(D))^\perp$$

orthogonal with respect to the intersection form, which is negative definite on the first summand and positive on the second.

3 Intersection forms and Decomposition in the Derived Category

We now show how the results we quoted at the beginning of the first section can be translated in statements about the decomposition in the derived category of sheaves of the direct image of the constant sheaf. We will freely use the language of derived categories. In particular we will use the notion of a constructible sheaf and the functors $Rf_*, Rf_!, f^*, f^!$.

In section 3.3 we briefly review the classical E_2-degeneration criterion of Deligne [10], [11] in order to motivate the construction of the perverse cohomology complexes. These complexes are a natural generalization of the higher direct image local systems for a smooth map. The construction of perverse cohomology is carried out in section 4.

We denote by $S(Y)$ the abelian category of sheaves of $\mathbb{Q}-$vector spaces on Y, and by $D^b(Y)$ the corresponding derived category of bounded complexes.

We shall make use of the following splitting criterion in the derived category. We state it in the form we need it in this paper. For a more general statement and a proof the reader is referred to [6] and [8].

Let (U, y) be a germ of an isolated $n-$dimensional singularity with the obvious stratification $= V \coprod y$, $j : V \to U \leftarrow y : i$ be the obvious maps, P be a self-dual complex on U with $P_{|V} = \mathcal{L}[n]$, $\mathcal{L}$ a local system on V, and $P \simeq \tau_{\leq 0} P$. We wish to compare P, $IC_U(\mathcal{L}) := \tau_{\leq -1} Rj_* \mathcal{L}[n]$ and the stalk $\mathcal{H}^0(P)_y$.

Lemma 3.0.7 *The following are equivalent:*
1) there is a canonical isomorphism in the derived category

$$P \simeq IC_U(\mathcal{L}) \oplus \mathcal{H}^0(P)_y[0];$$

2) the natural map $\mathcal{H}^0(P) \longrightarrow \mathcal{H}^0(Rj_*j^*P) = R^n j_* \mathcal{L}$ *is zero.*

3.1 Resolution of surface singularities

For a normal surface Y, let $j : Y_{reg} \to Y$ be the open embedding of its regular points. The intersection cohomology complex, which we will consider in much more detail in the next section, is $IC_Y = \tau_{\leq -1} Rj_* \mathbb{Q}_{Y_{reg}}[2]$. The following, which we will prove as a consequence of 2.1.1, is the first case of the Decomposition theorem which needs to be stated in the derived category and not just in the category of sheaves.

Theorem 3.1.1 *Let $f : X \to Y$ be a proper birational map of quasi-projective surfaces, X smooth, Y normal. There is a canonical isomorphisms*

$$Rf_* \mathbb{Q}_X[2] \xrightarrow{\simeq} IC_Y \oplus R^2 f_* \mathbb{Q}_X[0].$$

Proof. We work locally on Y. Let (Y, p) be the germ of an analytic normal surface singularity, $f : (X, D) \to (Y, p)$ be a resolution. The fiber $D = f^{-1}(p)$ is a connected union of finitely many irreducible compact curves D_k. Note that $j^* Rf_* \mathbb{Q} \simeq \mathbb{Q}_{Y \setminus p}$. Consider the following diagram

$$\begin{array}{cccc}
Rf_*\mathbb{Q}_X[2] & \xrightarrow{r} & Rj_*j^*Rf_*\mathbb{Q}_X[2] & = Rj_*\mathbb{Q}_{Y\setminus p}[2] \\
 & \searrow^{l_0} & \uparrow & \\
 & & \tau_{\leq 0} Rj_*j^*Rf_*\mathbb{Q}_X[2] & \\
 & \searrow^{l_{-1}} & \uparrow & \\
 & & \tau_{\leq -1} Rj_*j^*Rf_*\mathbb{Q}_X[2] & =: IC_Y \\
 & \searrow^{l_{-2}} & \uparrow & \\
 & & \tau_{\leq -2} Rj_*j^*Rf_*\mathbb{Q}_X[2] & = \mathbb{Q}_Y[2]
\end{array}$$

We are looking for the obstructions to the existence of the lifts l_0, l_{-1} and l_{-2}.

Since $R^k f_* \mathbb{Q}_X = 0$, $k \geq 3$, we have that $\tau_{\leq 0} R f_* \mathbb{Q}_X[2] \simeq R f_* \mathbb{Q}_X[2]$. In particular, l_0 exists and it is unique.
From the exact triangle

$$\to \tau_{\leq -1} R j_* j^* R f_* \mathbb{Q}_X[2] \to \tau_{\leq 0} R j_* j^* R f_* \mathbb{Q}_X[2] \to \mathcal{H}^0(R j_* j^* R f_* \mathbb{Q}_X[2]) \simeq R^2 j_* \mathbb{Q}_{Y \setminus p} \stackrel{+1}{\to}$$

l_{-1} exists iff the natural map

$$\rho : R^2 f_* \mathbb{Q}_X \to R^2 j_* \mathbb{Q}_{Y \setminus p}$$

is trivial. Using the isomorphisms $(R^2 f_* \mathbb{Q}_X)_p \simeq H^2(X)$ and $(R^2 j_* \mathbb{Q}_{Y \setminus p})_p \simeq H^2(X \setminus D)$, the map ρ can be identified with the restriction map ρ appearing in the long exact sequence of the pair (X, D) :

$$\ldots \longrightarrow H_2(D) \stackrel{cl}{\longrightarrow} H^2(X) \simeq H^2(D) \stackrel{\rho}{\longrightarrow} H^2(X \setminus D) \longrightarrow \ldots$$

where have identified $H_2(D) \simeq H^2(X, X \setminus D)$ via Lefschetz Duality.
By Theorem 2.1.1, cl is an isomorphism and ρ is trivial.
This lift l_{-1} is unique and splits by Lemma 3.0.7. □

Remark 3.1.2 It can be shown easily that a lift $R f_* \mathbb{Q}_X[2] \longrightarrow \tau_{\leq -2} R j_* \mathbb{Q}_U[2] = \mathbb{Q}_Y[2]$ exists iff $H^1(f^{-1}(p), \mathbb{Q}) = \{0\}$, i.e. iff $IC_Y \simeq \mathbb{Q}_Y[2]$, iff (Y, p) is a rational homology manifold.
It follows that, in general, the natural map $\mathbb{Q}_Y \to R f_* \mathbb{Q}_X$ does *not* split and $R f_* \mathbb{Q}_X$ does *not* decompose as a direct sum of its shifted cohomology sheaves as in (6).

Example 3.1.3 Let $f : X = \mathbb{C} \times \mathbb{P}^1 \to Y$ be the real algebraic map contracting precisely $D := \{0\} \times \mathbb{P}^1$ to a point $p \in Y$. One has a *non* split exact sequence in the category $P(Y)$ of perverse sheaves on Y :

$$0 \longrightarrow IC_Y \longrightarrow R f_* \mathbb{Q}_X[2] \longrightarrow H^2(\mathbb{P}^1)_p[0] \longrightarrow 0.$$

It is remarkable that while the lift l_{-2} does not exist in general, the lift l_{-1} always exists. While looking for a nontrivial map $R f_* \mathbb{Q}_X[2] \to \mathbb{Q}_Y[2]$, one ends up finding another *more interesting* map to IC_Y.

Recall that the dualizing sheaf $\omega_X \simeq \mathbb{Q}_X[2n]$. Dualizing the canonical isomorphism of Theorem 3.1.1 and, keeping in mind that IC_Y and $H_2(D)_p[0]$ are simple objects in $P(Y)$ (cf. section 4.2), we get

Corollary 3.1.4 *There are canonical isomorphisms*

$$H_2(D)_p[0] \oplus IC_Y^* \stackrel{\simeq}{\longrightarrow} Rf_*\omega_X[-2] \stackrel{PD}{\simeq} Rf_*\mathbb{Q}_X[2] \stackrel{\simeq}{\longrightarrow} IC_Y \oplus H^2(D)_p[0],$$

such that the composition is a direct sum map and induces the intersection form ι on $H_2(D)$ and the Poincaré-Verdier pairing on the self-dual IC_Y.
In particular, if X is compact, then the induced splitting injection

$$IH^\bullet(Y) \subseteq H^\bullet(X)$$

exhibits the lhs as the pure Hodge substructure of the rhs orthogonal to the space $cl(H_2(D)) \subseteq H^2(X)$ with respect to the Poincaré pairing on X.

3.2 Fibrations over curves

Let $f : X \to Y$ be a map of from a smooth surface onto a smooth curve. Denote by $\hat{f} : \hat{X} \to \hat{Y}$ the smooth part of the map f, by $j : \hat{Y} \to Y$ the open immersion, by $T^i := R^i\hat{f}_*\mathbb{Q}_{\hat{X}} = R^i f_*\mathbb{Q}_{X|\hat{Y}}$. For ease of exposition we assume that f has connected fibers.
Fix an ample line bundle η on X. The isomorphism stated in the next proposition will depend on η.

Proposition 3.2.1 *There is an isomorphism*

$$Rf_*\mathbb{Q}_X[2] \simeq j_*T^0[2] \oplus P \oplus j_*T^2[0],$$

with P a suitable self-dual (with respect to the Verdier duality functor) object of $D^b(Y)$.

Proof. We work around one critical value $p \in Y$ and replace Y by a small disk centered at p, X by the preimage of this disk, etc.
Since the fibers are connected, $\mathbb{Q}_Y \simeq f_*\mathbb{Q}_X \simeq j_*T^0 \simeq j_*T^2$.
Since η is $f-$ample, $\eta : T^0 \simeq T^2$, which in this case implies that $\eta : j_*T^0 \simeq j_*T^2$.
There are the natural truncation maps $f_*\mathbb{Q}_X \to Rf_*\mathbb{Q}_X \to R^2f_*\mathbb{Q}_X[-2]$.
There is the natural adjunction map $R^2f_*\mathbb{Q}_X \to j_*T^2$. It is splitting/surjective in view of the presence of η.
Putting together, there is a sequence of maps

$$j_*T^0[2] \stackrel{c}{\to} Rf_*\mathbb{Q}_X[2] \stackrel{\eta}{\to} Rf_*\mathbb{Q}_X[4] \stackrel{\pi}{\to} j_*T^2[2] \stackrel{\sigma}{\to} j_*T^0[2] \stackrel{c}{\to} \ldots \qquad (3)$$

where the composition $\eta\pi c$ is the isomorphism mentioned above $\eta : j_*T^0 \simeq j_*T^2$ and $\sigma := (\eta\pi c)^{-1}$.

The reader can verify that the composition

$$j_*T^0[2] \oplus j_*T^2[0] \stackrel{\gamma:=(c+\eta c\sigma)[-2]}{\longrightarrow} Rf_*\mathbb{Q}_X[2] \stackrel{\sigma\pi\eta\oplus\pi[-2]}{\longrightarrow} j_*T^0[2] \oplus j_*T^2[0]$$

is the identity, i.e. γ splits.
Let $P := \mathrm{Cone}(\gamma)$. There is a direct sum decomposition

$$Rf_*\mathbb{Q}_X[2] \simeq j_*T^0[2] \oplus P \oplus j_*T^2[0]. \tag{4}$$

The self-duality of P follows from the self-duality of $Rf_*\mathbb{Q}_X[2]$ and of $j_*T^0[2] \oplus j_*T^2[0]$. □

The object P introduced in the previous proposition has a simple structure:

Proposition 3.2.2 *Assumptions as in 3.2.1. The object P splits in $D^b(Y)$ as*

$$P = V \oplus j_*T^1[1],$$

where V = Ker $R^2f_*\mathbb{Q}_X \to j_*T^2$ is a skyscraper sheaf supported at $Y \setminus \hat{Y}$. This decomposition is canonical and compatible with Verdier duality.

Proof. By inspecting cohomology sheaves we see that $\mathcal{H}^i(P) = 0$ for $i \neq 0, 1$, that $\mathcal{H}^{-1}(P) = R^1f_*\mathbb{Q}_X$ and that $\mathcal{H}^0(P) = V$.
In view of Lemma 3.0.7, we need to show that

$$r' : \mathcal{H}^0(P) \to R^1j_*T^1 \tag{5}$$

is the zero map.
We now show that this is equivalent to the Zariski Lemma.
By applying adjunction to (4), we obtain the a commutative diagram

$$\begin{array}{ccc} Rf_*\mathbb{Q}_X[2] & \longrightarrow & Rj_*j^*Rf_*\mathbb{Q}_X[2] \\ \downarrow\simeq & & \downarrow\simeq \\ j_*T^0[2] \oplus P \oplus j_*T^2[0] & \longrightarrow & Rj_*T^0[2] \oplus Rj_*P \oplus Rj_*T^2[0] \end{array}$$

The associated map of spectral sequences $\mathbb{H}^p(Y, \mathcal{H}^q(-)) \Longrightarrow \mathbb{H}^{p+q}(Y, -)$ gives a commutative diagram

$$\begin{array}{ccccc} H_2(D) & \stackrel{cl}{\longrightarrow} & H^2(X) & \stackrel{r}{\longrightarrow} & H^2(X \setminus D) \\ & & \downarrow\simeq & & \downarrow\simeq \\ & & \mathcal{H}^0(P) \oplus (j_*T^2)_p & \stackrel{(r',id)}{\longrightarrow} & \mathcal{H}^1(Rj_*T^1) \oplus (j_*T^2)_p. \end{array}$$

It follows that, using the identifications above, $\mathrm{Im}(cl) = \mathrm{Ker}\ r = \mathrm{Ker}\ r' \subseteq \mathcal{H}^0(P)$. In particular, $r' = 0$, iff $\dim \mathrm{Ker}\ r = \dim \mathcal{H}^0(P)$. Note that $\dim \mathcal{H}^0(P) = b_2(f^{-1}(p)) - 1$.
It follows that $r' = 0$ iff $\dim \mathrm{Im}(cl) = b_2(f^{-1}(p)) - 1$. The latter is implied by Theorem 2.1.2.
We conclude by Lemma 3.0.7. □

Finally, we have:

Theorem 3.2.3 *There is an isomorphism*

$$Rf_*\mathbb{Q}_X[2] \simeq j_*T^0[2] \oplus j_*T^1[1] \oplus V[0] \oplus j_*T^2[0].$$

Remark 3.2.4 From 3.2.3 follows that $R^1 f_*\mathbb{Q}_X \simeq j_* R^1 \hat{f}_* \mathbb{Q}_{\hat{X}}$. Note that this implies the Local Invariant Cycle Theorem.
Since $R^2 f_*\mathbb{Q}_X = j_* R^2 \hat{f}_* \mathbb{Q}_{\hat{X}} \oplus V$, we have the coarser decomposition

$$Rf_*\mathbb{Q}_X[2] \simeq R^0 f_*\mathbb{Q}_X[2] \oplus R^1 f_*\mathbb{Q}_X[1] \oplus R^2 f_*\mathbb{Q}_X[0].$$

In particular, the Leray spectral sequence degenerates at E_2. It is easy to see that the Leray filtration on the cohomology of X is by Hodge substructures:

$$L^2 = f^*H(Y), \qquad L^1 = \mathrm{Ker}\ \{f_* : H(X) \to H(Y)\}.$$

3.3 Smooth maps

Even in the case of a smooth fibration $f : X \to Y$ of a surface over a curve, the study of the complex $Rf_*\mathbb{Q}_X$ is nontrivial, without any projectivity assumptions.

Example 3.3.1 Let X be a Hopf surface. There is a natural holomorphic smooth fibration $f : X \to \mathbb{P}^1$ with fibers elliptic curves. Since $b_1(X) = 1$, one sees easily that the Leray Spectral Sequence for f is not E_2-degenerate. In particular, $Rf_*\mathbb{Q}_X$ is *not* isomorphic to $\oplus_i R^i f_*\mathbb{Q}_X[-i]$.

Let us briefly list *some* of the important properties of a smooth projective map $f : X \to Y$ of smooth varieties.

The sheaves $R^i f_*\mathbb{Q}_X$ are locally constant over Y, i.e. they are local systems. In fact, f is differentiably locally trivial over Y in view of Ehresmann Lemma.

The model for a general decomposition theorem for $Rf_*\mathbb{Q}_X$ is the following

Theorem 3.3.2 *Let $f : X^n \to Y^m$ be a smooth projective map of smooth quasi-projective varieties of the indicated dimensions and η be an $f-$ample line bundle on X.*

$$Rf_*\mathbb{Q}_X \simeq_{D(Y)} \oplus R^i f_*\mathbb{Q}_X[-i]. \tag{6}$$

$$\eta^i : R^{n-m-i} f_*\mathbb{Q}_X \simeq R^{n-m+i} f_*\mathbb{Q}_X, \quad \forall\, i \geq 0. \tag{7}$$

$$\text{The local systems } R^j f_*\mathbb{Q}_X \text{ on } Y \text{ are semisimple.} \tag{8}$$

The first Chern class of the line bundle $\eta \in H^2(X,\mathbb{Q}) = Hom_{D(X)}(\mathbb{Q}_X, \mathbb{Q}[2])$, defines maps

$$\eta : \mathbb{Q}_X \longrightarrow \mathbb{Q}_X[2], \quad \eta : Rf_*\mathbb{Q}_X \longrightarrow Rf_*\mathbb{Q}_X[2], \quad \eta^r : Rf_*\mathbb{Q}_X \longrightarrow Rf_*\mathbb{Q}_X[2r]$$

and finally

$$\eta^r : R^i f_*\mathbb{Q}_X \longrightarrow R^{i+2r} f_*\mathbb{Q}_X.$$

Theorem 3.3.2.7 is then just a re-formulation of the Hard Lefschetz Theorem for the fibers of f and can be named the Relative Hard Lefschetz Theorem for smooth maps.

We remind the reader that a functor $\mathcal{T} : D(Y) \to A$, A an abelian category, is said to be cohomological (cf. [25] II, 1.1.5.) if, setting $\mathcal{T}^i(K) = \mathcal{T}^0(K[i])$, to a distinguished triangle

$$K \to L \to M \stackrel{+1}{\to}$$

corresponds a long exact sequence in A

$$\to \mathcal{T}^i(K) \to \mathcal{T}^i(L) \to \mathcal{T}^i(M) \to \mathcal{T}^{i+1}(K) \to ...$$

The cohomology sheaf functor $\mathcal{H}^0 : D(Y) \to S(Y)$ is cohomological. Noting that $\mathcal{H}^i(Rf_*) = R^i f_*$, Theorem 3.3.2.6 can be re-phrased by saying that $Rf_*\mathbb{Q}_X$ is decomposable with respect to the functor $\mathcal{H}^0$.
It is important to note that (7) implies (6) by the Deligne-Lefschetz Criterion.
Theorem 3.3.2.8 states that every local subsystem $\mathcal{L} \subseteq R^j f_*\mathbb{Q}_X$ admits a complement, i.e. a local system $\mathcal{L}'$ such that $\mathcal{L} \oplus \mathcal{L}' = R^j f_*\mathbb{Q}_X$.

Let us note *some* of the important consequences of Theorem 3.3.2.
The $\mathcal{H}^0-$decomposability (6) of $Rf_*\mathbb{Q}_X$ implies immediately the E_2-degeneration of the Leray spectral sequence, i.e. of the spectral sequence associated with the cohomological functor $\mathcal{H}^0$:

$$H^p(Y, R^q f_*\mathbb{Q}_X) \Longrightarrow H^{p+q}(X,\mathbb{Q}) \text{ is } E_2-\text{degenerate.} \tag{9}$$

This degeneration implies the surjection

$$H^k(X,\mathbb{Q}) \longrightarrow H^0(Y, R^k f_* \mathbb{Q}_X) = H^k(X_y,\mathbb{Q})^{\pi_1(Y,y)}, \tag{10}$$

i.e. the so-called Global Invariant Cycle Theorem.
The Theory of MHS allows to show, using a smooth compactification of X, that in fact the monodromy invariants are a Hodge substructure of $H^k(X_y,\mathbb{Q})$, which as a PHS is independent of $y \in Y$ (Theorem of the Fixed Part). In fact, (8) is a consequence of this fact.

In general, if f is not smooth, Theorem 3.3.2 fails completely.
The Relative Hard Lefschetz Theorem (7) fails due to the presence of singular fibers, i.e. fibers along which the differential of f drops rank.
The sheaves $R^j f_* \mathbb{Q}_X$ are no longer locally constant. Moreover, they are not semisimple in the category of constructible sheaves: e.g. $j_! \mathbb{Q}_{\mathbb{C}^*} \to \mathbb{Q}_{\mathbb{C}}$.
The following examples shows that the $\mathcal{H}^0-$decomposability (6) fails in general and so does the E_2-degeneration of the Leray Spectral Sequence (9).

Example 3.3.3 Let X be the blowing up of $\mathbb{CP}^2$, along ten points lying on an irreducible cubic C' and C be the strict transform of C' on X. Since $C^2 = -1$ the curve contracts to a point under a birational map $f : X \to Y$. We leave to the reader the task to verify that 1) the Leray Spectral Sequences for $H^2(X,\mathbb{Q})$ and for $IH^2(Y,\mathbb{Q})$ are not E_2-degenerate and that, though the Leray spectral sequence always degenerates over suitably small Euclidean neighborhoods on Y, 2) the complex IC_Y does not split as a direct sum of its shifted cohomology sheaves.

The following more general class of examples shows that the failure of the E_2-degeneration is very frequent.

Example 3.3.4 Let $f : X \to Y$ be a projective resolution of the singularities of a projective and normal variety Y such that there is at least one index i such that the natural MHS on $H^i(Y,\mathbb{Q})$ is not pure (e.g. $i = 2$ in3.3.3). Then the Leray Spectral Sequence for f is not E_2-degenerate. If it were, then the edge sequence would give an injection of MHS $f^* : H^j(Y,\mathbb{Q}) \to H^j(X,\mathbb{Q})$, forcing the MHS of such a Y to be pure.

However, not everything is lost.

4 Perverse sheaves and the Decomposition Theorem

One of the main ideas leading to the theory of perverse sheaves is that Theorem 3.3.2, which holds for smooth maps, can be made to hold for arbitrary proper algebraic maps provided that it is re-formulated using the perverse cohomology functor ${}^p\mathcal{H}^0$ in place of the cohomology sheaf functor $\mathcal{H}^0$. Just as this latter is $\tau_{\leq 0}\tau_{\geq 0}$, with τ the standard truncation functors of a complex, the perverse cohomology functor will be expressed as ${}^p\mathcal{H}^0 = {}^p\tau_{\leq 0}{}^p\tau_{\geq 0}$, where ${}^p\tau$ is the so called perverse truncation functor. Roughly speaking, the perverse truncation functor (with respect to middle perversity, which is the only case we will consider) is defined by gluing standard truncations on the strata, shifted by a term which depends on the dimension of the stratum. The choice of the shifting is dictated by the behavior of the standard truncation with respect to duality, as we suggest in 4.1.1. In this context and keeping this in mind, perverse truncation becomes quite natural. We believe it can be useful to give a few details of its construction and an example of computation, related to the examples given in section 2.2. In analogy with the cohomology sheaf functor $\mathcal{H}^0$, the perverse cohomology functor ${}^p\mathcal{H}^0$ will be a cohomological functor which takes values in an abelian subcategory of $D^b(Y)$, whose object are the so called perverse sheaves. For a general proper map these objects play the role played by local system for smooth maps.

4.1 Truncation and Perverse sheaves

Let $D^b(Y)$ be the bounded derived category of the category $S(Y)$ of sheaves of rational vector spaces on Y. We are interested in the full subcategory $D(Y)$ of those complexes whose cohomology sheaves are constructible. This means that, given an object F of $D(Y)$, there is an algebraic Whitney stratification $Y = \coprod S_l$, depending on F, such that $\mathcal{H}^j(F)_{|S_l}$ is a finite rank local system. By the Thom Isotopy Lemmata, $Rf_*\mathbb{Q}_X$, and in fact any other complex appearing in this paper, is an object of $D(Y)$. One is interested in direct sum decompositions of this complex, in the geometric meaning of the summands and in the consequences, both theoretical and practical, of such splittings.

We now define the t-structure on $D(Y)$ associated with the middle perversity. Instead of insisting on its axiomatic characterization (cf. [2]), we give the explicit construction of the *perverse truncations* ${}^p\tau_{\leq m} : D(Y) \longrightarrow D(Y)$, and ${}^p\tau_{\geq m} : D(Y) \longrightarrow D(Y)$. These come with natural morphisms ${}^p\tau_{\leq m}F \longrightarrow F$ and $F \longrightarrow {}^p\tau_{\geq m}F$.
We start with the following:

Lemma 4.1.1 *Let Z be nonsingular of complex dimension r, and $F \in D(Z)$ with locally constant cohomology sheaves. Then there are natural isomorphisms:*

$$\tau_{\leq k}\mathcal{D}F \simeq \mathcal{D}\tau_{\geq -k-2r}F \qquad \tau_{\geq k}\mathcal{D}F \simeq \mathcal{D}\tau_{\leq -k-2r}F$$

Proof. Since the dualizing complex is in this case isomorphic to $\mathbb{Q}_Z[2r]$, it is enough to prove that there are natural isomorphisms

$$\tau_{\leq k}Rhom(F,\mathbb{Q}_Z) \simeq Rhom(\tau_{\geq -k}F,\mathbb{Q}_Z) \qquad \tau_{\geq k}Rhom(F,\mathbb{Q}_Z) \simeq Rhom(\tau_{\leq -k}F,\mathbb{Q}_Z).$$

We prove the first statement. The proof of the second is analogous. Applying $Rhom$ and $\tau_{\leq k}$ to the map $F \to \tau_{\geq -k}F$, we get:

$$\begin{array}{ccc} Rhom(\tau_{\geq -k}F,\mathbb{Q}_Z) & \longrightarrow & Rhom(F,\mathbb{Q}_Z) \\ \uparrow & & \uparrow \\ \tau_{\leq k}Rhom(\tau_{\geq -k}F,\mathbb{Q}_Z) & \longrightarrow & \tau_{\leq k}Rhom(F,\mathbb{Q}_Z). \end{array}$$

To prove the statement it is enough to show that the three complexes $Rhom(\tau_{\geq -k}F,\mathbb{Q}_Z)$, $\tau_{\leq k}Rhom(\tau_{\geq -k}F,\mathbb{Q}_Z)$ and $\tau_{\leq k}Rhom(F,\mathbb{Q}_Z)$ have the same cohomology sheaves. Since F and $\mathbb{Q}_Z$ have locally constant cohomology sheaves, there are natural isomorphisms of complexes of vector spaces $Rhom(F,\mathbb{Q}_Z)_y \simeq Rhom(F_y,\mathbb{Q}_y) \simeq \oplus_i Hom(\mathcal{H}^{-i}F_y,\mathbb{Q}_y)[-i]$. The cohomology sheaves of the three complexes, are, therefore, equal to $Hom(\mathcal{H}^{-i}F_y,\mathbb{Q}_y)$ for $i \leq k$ and vanish otherwise. $\square$

The construction of the perverse truncation is done by induction on the strata of Y starting from the shifted standard truncation on the open stratum U_d. In the sequel we will indicate by U_l the union of strata of dimension bigger than or equal to l. With a slight abuse of notation, we will write $U_{l+1} = U_l \coprod S_l$, with S_l now denoting the union of strata of dimension l. Let $F \in Ob(D(Y))$ be $\mathfrak{Y}-$constructible for some stratification $\mathfrak{Y} = \coprod S_l$. All the constructions below will lead to $\mathfrak{Y}-$constructible complexes.

We define ${}^p\tau^{U_d}_{\leq 0} = \tau_{\leq -\dim Y}$ and ${}^p\tau^{U_d}_{\geq 0} = \tau_{\geq -\dim Y}$.
Suppose that ${}^p\tau^{U_{l+1}}_{\leq 0} : D(U_{l+1}) \longrightarrow D(U_{l+1})$ and ${}^p\tau^{U_{l+1}}_{\geq 0} : D(U_{l+1}) \longrightarrow D(U_{l+1})$ have been defined.
We proceed to define ${}^p\tau^{U_l}_{\leq 0}$ and ${}^p\tau^{U_l}_{\geq 0}$ on $U_l = U_{l+1}\coprod S_l$. Let $i : S_l \to U_l \longleftarrow U_{l+1} : j$ be the inclusions: the exact triangles

$$\tau'_{\leq 0}F \to F \to Rj_*\,{}^p\tau^{U_{l+1}}_{>0}j^*F \xrightarrow{[1]} \qquad \tau''_{\leq 0}F \to F \to i_*\tau_{>-dimS}i^*F \xrightarrow{[1]}$$

and

$$Rj_!\,{}^p\tau^{U_{l+1}}_{<0}j^!F \longrightarrow F \longrightarrow \tau'_{\geq 0}F \xrightarrow{[1]} \qquad i_!\tau_{<-\dim S}i^!F \longrightarrow F \longrightarrow \tau''_{\geq 0}F \xrightarrow{[1]}$$

define four functors (cf. [2], 1.1.10, 1.3.3 and 1.4.10), i.e. the four objects $\tau'_{\geq 0}F$, $\tau'_{\leq 0}F$, $\tau''_{\geq 0}F$ and $\tau''_{\leq 0}F$ which make the corresponding triangles exact, are determined up to unique isomorphism. Define

$$ {}^{p}\tau^{U_l}_{\leq 0} := \tau''_{\leq 0}\tau'_{\leq 0}, \qquad {}^{p}\tau^{U_l}_{\geq 0} := \tau''_{\geq 0}\tau'_{\geq 0}. $$

Define:

$$ {}^{p}\tau_{\leq 0} := {}^{p}\tau^{U_0}_{\leq 0}, \qquad {}^{p}\tau_{\geq 0} := {}^{p}\tau^{U_0}_{\geq 0}. $$

We have the following compatibilities with respect to shifts.

$$ {}^{p}\tau_{\leq m}(F[l]) \simeq {}^{p}\tau_{\leq m+l}(F)[l], \qquad {}^{p}\tau_{\geq m}(F[l]) \simeq {}^{p}\tau_{\geq m+l}(F)[l]. $$

These formulas hold for the ordinary truncation functors as well and we symbolically summarize them as follows

$$ (\tau_m([l]))[-l] \, = \, \tau_{m+l}. $$

The perverse truncations so defined have the following properties:

- By the construction above, if F is $\mathfrak{Y}-$cc, then so are ${}^{p}\tau_{\leq m}F$ and ${}^{p}\tau_{\geq m}F$.
- Let $P(Y)$ be the full subcategory of complexes Q such that

  $$ \dim\ \mathrm{Supp}\ (\mathcal{H}^{-i}(Q) \leq i \text{ for every } i \in \mathbb{Z} $$

  and the same holds for $\mathcal{D}(Q)$, the Verdier dual of Q. $P(Y)$ is an abelian category. The functor

  $$ {}^{p}\!\mathcal{H}^0(-) : D(Y) \longrightarrow P(Y), \qquad {}^{p}\!\mathcal{H}^0(F) := {}^{p}\tau_{\leq 0}\, {}^{p}\tau_{\geq 0}F \simeq {}^{p}\tau_{\geq 0}\, {}^{p}\tau_{\leq 0}F, $$

  is cohomological. Define

  $$ {}^{p}\!\mathcal{H}^m(F) := {}^{p}\!\mathcal{H}^0(F[m]). $$

  These functors are called *the perverse cohomology functors*. Any distinguished triangle $F \longrightarrow G \longrightarrow H \xrightarrow{[1]}$ in $D(Y)$ gives rise to a long exact sequence in $P(Y)$:

  $$ \ldots \longrightarrow {}^{p}\!\mathcal{H}^i(F) \longrightarrow {}^{p}\!\mathcal{H}^i(G) \longrightarrow {}^{p}\!\mathcal{H}^i(H) \longrightarrow {}^{p}\!\mathcal{H}^{i+1}(F) \longrightarrow \ldots. $$

  If F is $\mathfrak{Y}-$cc, then so are ${}^{p}\!\mathcal{H}^m(F)$, $\forall m \in \mathbb{Z}$.

- Poincaré- Verdier Duality induces functorial isomorphisms for $F \in Ob(D(Y))$

  $${}^p\tau_{\leq 0}\mathcal{D}F \simeq \mathcal{D}\,{}^p\tau_{\geq 0}F, \qquad {}^p\tau_{\geq 0}\mathcal{D}F \simeq \mathcal{D}\,{}^p\tau_{\leq 0}F \qquad \mathcal{D}({}^p\mathcal{H}^j(F)) \simeq {}^p\mathcal{H}^{-j}(\mathcal{D}(F)).$$

  This can be seen from the construction above. In fact, by Lemma 4.1.1, the isomorphisms hold for $U = U_d$, since ${}^p\tau^{U_d}_{\leq 0} = \tau_{\leq -\dim Y}$ and ${}^p\tau^{U_d}_{\geq 0} = \tau_{\geq -\dim Y}$. Suppose that ${}^p\tau^U_{\leq 0}\mathcal{D} \simeq \mathcal{D}\,{}^p\tau^U_{\geq 0}$ and ${}^p\tau^U_{\geq 0}\mathcal{D} \simeq \mathcal{D}\,{}^p\tau^U_{\leq 0}$ for $U = U_{l+1}$. It then follows that the same isomorphisms hold for $U = U_l$. In fact, applying the functor $\mathcal{D}$ to the triangle defining $\tau'_{\leq 0}\mathcal{D}F$, and the inductive hypothesis ${}^p\tau^U_{\leq 0}\mathcal{D} \simeq \mathcal{D}\,{}^p\tau^U_{\geq 0}$, we get the triangle defining $\tau'_{\geq 0}F$, so that $\mathcal{D}\tau'_{\leq 0}\mathcal{D}F \simeq \tau'_{\geq 0}F$. The argument for $\tau''_{\leq 0}$ is identical. We get $\mathcal{D}\tau''_{\leq 0}\mathcal{D}F \simeq \tau''_{\geq 0}F$. It follows that $\mathcal{D}\tau''_{\leq 0}\tau'_{\leq 0} \simeq \tau''_{\geq 0}\mathcal{D}\tau'_{\leq 0} \simeq \tau''_{\geq 0}\tau'_{\geq 0}\mathcal{D}$ and the first wanted isomorphism follows. The second is equivalent to the first one. The third one follows formally: $\mathcal{D}({}^p\mathcal{H}^m(F)) \simeq \mathcal{D}\,{}^p\tau_{\leq 0}\,{}^p\tau_{\geq 0}(F[-m]) \simeq {}^p\tau_{\geq 0}\,{}^p\tau_{\leq 0}(\mathcal{D}(F)[m]) \simeq {}^p\mathcal{H}^{-m}(\mathcal{D}F)$.

- For every F and m one constructs, functorially, a distinguished triangle

  $${}^p\tau_{\leq m}F \longrightarrow F \longrightarrow {}^p\tau_{\geq m+1}F \xrightarrow{[1]}.$$

The objects of the abelian category $P(Y)$ are called *perverse sheaves.* An object F of $D(Y)$ is perverse if and only if the two natural maps ${}^p\tau_{\leq 0}F \longrightarrow F$ and $F \longrightarrow {}^p\tau_{\geq 0}F$ are isomorphisms.

Example 4.1.2 Let $f : X \longrightarrow Y$ a surjective proper map of surfaces, X smooth. The direct image $Rf_*\mathbb{Q}[2]$ is a perverse sheaf.

Example 4.1.3 To give an example of how the truncation functors can be computed from the construction given above, let us examine the example of section 2.3. The assumptions in 2.3.1 are in force and we use the same notation. We show that:

$${}^p\tau_{\leq 0}Rf_*\mathbb{Q}_X[3] \simeq \tau_{\leq 0}Rf_*\mathbb{Q}_X[3], \qquad {}^p\tau_{\leq -1}Rf_*\mathbb{Q}_X[3] = H_4(D)_y[1].$$

Since ${}^p\tau^{Y-y}_{>0} = \tau_{>-3}$ and $j^*Rf_*\mathbb{Q}_X[3] = \mathbb{Q}_{Y\setminus y}[3]$, we have $\tau'_{\leq 0}Rf_*\mathbb{Q}_X[3] = Rf_*\mathbb{Q}_X[3]$. The perverse truncation ${}^p\tau_{\leq 0}Rf_*\mathbb{Q}_X[3] = \tau''_{\leq 0}\tau'_{\leq 0}Rf_*\mathbb{Q}_X[3] = \tau''_{\leq 0}Rf_*\mathbb{Q}_X[3]$ is computed by the triangle

$$\tau''_{\leq 0}Rf_*\mathbb{Q}_X[3] \longrightarrow Rf_*\mathbb{Q}_X[3] \longrightarrow i_*\,\tau_{>0}\,i^*Rf_*\mathbb{Q}_X[3] \xrightarrow{+1}.$$

Since $i^*Rf_*\mathbb{Q}_X[3] = \oplus_j H^{3-j}(D)_y[j]$, we have $i_*\tau_{>0}i^*Rf_*\mathbb{Q}_X[3] = H^4(D)_y[-1]$, so that

$${}^p\tau_{\leq 0}Rf_*\mathbb{Q}_X[3] \simeq \text{Cone}\,\{Rf_*\mathbb{Q}_X[3] \to H^4(D)_y[-1]\} \simeq \tau_{\leq 0}Rf_*\mathbb{Q}_X[3].$$

Keeping in mind the truncation rules, we have the triangle

$$\tau'_{\leq -1} Rf_* \mathbb{Q}_X[3] \longrightarrow Rf_* \mathbb{Q}_X[3] \longrightarrow Rj_* \, {}^p\tau^{Y-y}_{>-1} j^* \mathbb{Q}_Y[3] = Rj_* j^* \mathbb{Q}_Y[3] \stackrel{+1}{\longrightarrow}$$

from which we deduce that

$$\tau'_{\leq -1} Rf_* \mathbb{Q}_X[3] \simeq i_! i^! Rf_* \mathbb{Q}_X[3] \simeq \oplus_j H_j(D)_y[j-3].$$

The truncation ${}^p\tau_{\leq -1} Rf_* \mathbb{Q}_X[3] = \tau''_{\leq -1} \tau'_{\leq -1} Rf_* \mathbb{Q}_X[3] = \tau''_{\leq -1}(\oplus_j H_j(D)_y[j-3])$ is computed by the triangle

$$\tau''_{\leq -1}(\oplus_j H_j(D)_y[j-3]) \longrightarrow \oplus_j H_j(D)_y[j-3] \to i_* \tau_{>-1}(\oplus_j H_j(D)_y[j-3]) \stackrel{+1}{\longrightarrow}$$

from which the conclusion follows.

It is remarkable that the category of Perverse sheaves is Artinian and Noetherian, that its simple object can be completely characterized and have an important geometric meaning: they are the intersection cohomology complexes.

4.2 The simple objects of $P(Y)$

Goresky and MacPherson introduced the intersection cohomology groups of Y for an arbitrary perversity. Here we deal with the case of middle perversity. These groups were first defined as the homology of a chain sub-complex of the complex of geometric chains with twisted coefficients on Y. Later, following a suggestion by Deligne, they realized these groups as the hypercohomology of what they called the intersection cohomology complexes with twisted coefficients of Y.
These complexes are the building blocks of $P(Y)$. They are special examples of perverse sheaves and every perverse sheaf can be exhibited as a finite series of non trivial extensions of objects of this kind supported on closed subvarieties of Y.

Let $Z \subseteq Y$ be a closed subvariety, $Z^o \subseteq Z_{reg} \subseteq Z$ be an inclusion of Zariski-dense open subsets and L be a local system on Z^o.

Goresky-MacPherson associate with this data the intersection cohomology complex $IC_Z(L)$ in $P(Z)$.
Up to isomorphism, this complex is independent of the choice of Z^o : if L and L' are local systems on Z^o and $Z^{o\prime}$ respectively and $L_{|Z^o \cap Z^{o\prime}} \simeq L'_{|Z^o \cap Z^{o\prime}}$, then the associated intersection cohomology complexes on Z are canonically isomorphic.
The complex $IC_Z(L)$, when viewed as a complex on Y, is perverse on Y.
The intersection cohomology complex of Y is defined to be $IC_Y := IC_Y(\mathbb{Q}_{Y_{reg}})$. If Y is smooth, or a rational homology manifold, then $IC_Y \simeq \mathbb{Q}_Y[\dim Y]$.
If Z is smooth and L is a local system on Z, then $IC_Z(L) \simeq L[\dim Z]$.

Proposition 4.2.1 *The simple objects in $P(Y)$ are precisely the ones of the form $IC_Z(L)$, L simple on Z^o. In particular, if L is simple, then $IC_Z(L)$ does not decompose into non-trivial direct summands in $D(Y)$.*
The semisimple objects of $P(Y)$ are finite direct sums of such intersection cohomology complexes on possibly differing subvarieties.

Every perverse sheaf $Q \in P(Y)$ is supported on a finite union of closed subvarieties of Y. Let Z be any one of them. There is a Zariski-dense open subset $Z^o \subseteq Z_{reg}$, such that $Q_{|Z^o} \simeq L[\dim Z]$, where L is a local system on Z^o. The object Q admits a finite filtration where one of the quotients is $IC_Z(L)$ and all the others are all supported on $Supp(Q) \setminus Z^o$. It follows that Q admits a finite filtration where the quotients are intersection cohomology complexes supported on closed subvarieties of Y.

An intersection cohomology complex $IC_Z(L)$ is characterized by its not admitting subquotients supported on smaller dimensional subspaces of Z. Its eventual splitting is entirely due to a corresponding splitting of L.

Let us define the intersection cohomology complexes. Assume $\mathfrak{Y}$ is a stratification and L is a local system on the open stratum U_d. We start by defining $IC_{U_d}(L) := L[dimY]$. Now suppose inductively that $IC_{U_{l+1}}(L)$ has been defined on U_{l+1} and we define it on U_l by

$$IC_{U_l}(L) := \tau_{\leq -l-1} Rj_* IC_{U_{l+1}}(L).$$

Let us give formulae for $IC_Y(L)$ when Y and L have isolated singularities. It suffices to work in the Euclidean topology.
Let (Y, p) be a germ of an isolated singularity, $j : U := Y \setminus p \to Y$ be the open embedding and L be a local system on U. We have

$$IC_Y(L) = \tau_{\leq -1}(Rj_* L[\dim Y]). \tag{11}$$

If $\dim Y = 1$, then $IC_Y(L) = j_* L[1]$. The stalk at p are the invariants of L i.e. $H^0(U, L)$.
In general, when $\dim Y \geq 2$, then $IC_Y(L)$ is a complex, not a sheaf. If L is simple, then $IC_Y(L)$ is simple and does not split non-trivially in $D(Y)$. The cohomology sheaves $\mathcal{H}^j(IC_Y(L))$ are non trivial only for $j \in [-\dim Y, -1]$ and we have

$$\mathcal{H}^{-\dim Y}(IC_Y(L)) = j_* L, \qquad \mathcal{H}^{-\dim Y + l}(IC_Y(L)) = H^l(U, L)_p, \ 1 \leq l \leq \dim Y - 1, \tag{12}$$

where V_p denotes a skyscraper sheaf at $p \in Y$ with stalk V.

In order to familiarize ourselves with these complexes, we compute two important examples:

Example 4.2.2 We consider a threefold Y with an ordinary double point y and with associated link $\mathcal{L}$. Let $j : Y \setminus y \to Y$ be the open embedding, so that (11) gives

$$IC_Y = \tau_{\leq -1} Rj_* \mathbb{Q}[3].$$

The cohomology sheaves at y are

$$\mathcal{H}^k(IC_Y) = H^{k+3}(\mathcal{L})_p, \text{ for } k \leq -1, \qquad \mathcal{H}^k(IC_Y) = 0 \text{ otherwise.}$$

The singularity is analytically equivalent to a cone over a smooth quadric in projective space, hence its link is homeomorphic to the S^1-bundle over $S^2 \times S^2$ with Chern class $(1,1)$. The long exact sequence for this S^1-fibration gives

$$H^k(\mathcal{L}) = \mathbb{Q} \text{ for } k = 0,2,3,5 \qquad H^k(\mathcal{L}) = 0 \text{ otherwise,}$$

which in turn implies that

$$\mathcal{H}^k(IC_Y) = \mathbb{Q} \text{ for } k = -3,-1, \qquad \mathcal{H}^k(IC_Y) = 0 \text{ otherwise.}$$

We have a triangle in $D(Y)$, (not of perverse sheaves)

$$\mathbb{Q}_Y[3] \to IC_Y \to \mathbb{Q}_y[1] \stackrel{+1}{\to}$$

The fact that $\mathcal{H}^{-1}(IC_Y) = \mathbb{Q}_y$ should be compared with the existence of a small resolution with fiber a projective line over the singular point, and the statement of the Decomposition Theorem 4.4.1.

Example 4.2.3 Let $Y = \mathbb{C}^2$, and L be a local system on $\mathbb{C}^2 \setminus (x_1x_2 = 0)$ defined by the two monodromies T_1 and T_2 acting on the vector space $V = L_p$, the stalk of L at $p = (1,1) \in \mathbb{C}^2$. We first determine the intersection cohomology complex over $\mathbb{C}^2 \setminus \{0\}$. Denoting by $j : \mathbb{C}^2 \setminus \{x_1x_2 = 0\} \to \mathbb{C}^2 \setminus \{0\}$ the natural map, we have $IC_{\mathbb{C}^2 \setminus \{0\}}(L) = \tau_{\leq -2} Rj_* L[2] = (j_*L)[2]$. Denoting by $j' : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2$ the natural map, we have

$$IC_{\mathbb{C}^2}(L) \ = \ \tau_{\leq -1} Rj'_* IC_{\mathbb{C}^2 \setminus \{0\}(L)} \ = \ \tau_{\leq -1} Rj'_*(j_*L[2]).$$

In order to determine the cohomology sheaves of $IC_{\mathbb{C}^2}(L)$, we compute $H^i(\mathbb{C}^2 \setminus \{0\}, j_*L)$ for $i = 0,1$. More precisely, we should determine these groups for a fundamental system of neighborhoods of the origin; however the cohomology groups are in fact constant. Set $N_1 = T_1 - Id$, $N_2 = T_2 - Id$.
We have $H^0(\mathbb{C}^2 \setminus \{0\}, j_*L) = H^0(\mathbb{C}^2 \setminus \{x_1x_2 = 0\}, L) = \text{Ker } N_1 \cap \text{Ker } N_2 = V^{\pi_1}$. Since $j_*L = \tau_{\leq 0} Rj_*L$, and fundamental deleted neighborhoods around the axes are homotopic to circles, so that

$$\mathcal{H}^i(Rj_*L) = 0 \text{ for } i \geq 2,$$

we have the following exact triangle in $D(\mathbb{C}^2 \setminus \{0\})$

$$j_*L = \tau_{\leq 0} Rj_*L \longrightarrow Rj_*L \longrightarrow \mathcal{H}^1(Rj_*L)[-1] \xrightarrow{+1} .$$

The sheaf $\mathcal{H}^1(Rj_*L)$ is the local system on $(x_1x_2 = 0) \setminus \{(0,0)\} = D_1 \coprod D_2$, with fiber Coker N_1 and monodromy T_2 on D_1, and fiber Coker N_2 and monodromy T_1 on D_2. Since $\mathbb{C}^2 \setminus \{x_1x_2 = 0\}$ retracts to a torus T^2, the cohomology of L is isomorphic to the group cohomology of $\mathbb{Z}^2$ with values in V as a $\mathbb{Z}^2$-module via the monodromies T_1, T_2, which can be computed by the Koszul complex (see for instance [26])

$$0 \longrightarrow V \xrightarrow{\phi} V \oplus V \xrightarrow{\psi} V \longrightarrow 0,$$

with

$$\phi(v) = (N_1(v), N_2(v)) \qquad \psi(v_1, v_2) = N_2(v_1) - N_1(v_2).$$

The long exact sequence associated to the exact triangle above gives

$$\mathcal{H}^{-1}(IC_Y(L))_0 \simeq H^1(\mathbb{C}^2 \backslash \{0\}, j_*L) = \frac{(N_1(v_1), N_2(v_2)) \text{ such that } N_1N_2(v_1 - v_2) = 0}{(N_1(v), N_2(v))}.$$

More generally, a similar recipe holds for the cohomology sheaves of the intersection cohomology complex of a local system defined on the complement of a normal crossing, see [4].

4.3 Decomposability, E_2−degenerations and filtrations

Definition 4.3.1 Let $\mathcal{H} = \mathcal{H}^0$ be the sheaf cohomology functor. We say that F in $D(Y)$ is $\mathcal{H}$*−decomposable* if

$$F \simeq_{D(Y)} \bigoplus_i \mathcal{H}^i(F)[-i]$$

We say that F in $D(Y)$ is ${}^p\mathcal{H}$*−decomposable* if

$$F \simeq_{D(Y)} \bigoplus_i {}^p\mathcal{H}^i(F)[-i].$$

If F is $\mathcal{H}$−decomposable, then the spectral sequence

$$H^p(Y, \mathcal{H}^q(F)) \Longrightarrow \mathbb{H}^{p+q}(Y, F)$$

is E_2−degenerate. This spectral sequence is the Leray Spectral Sequence when $F = Rf_*(G)$. In this case the corresponding filtration is called the Leray filtration.

The analogous statements holds for ${}^p\mathcal{H}-$decomposability. The corresponding spectral sequence is called the Perverse Leray Spectral Sequence:

$$\mathbb{H}^p(Y,{}^p\mathcal{H}^q(Rf_*G)) \Longrightarrow \mathbb{H}^{p+q}(Y,G)$$

and the corresponding filtration is called the perverse filtration.

Definition 4.3.2 Let $f : X \to Y$ be a map, $n = \dim X$. The perverse filtration $H^{n+j}_{\leq b}(X) \subseteq H^{n+j}(X)$, $b, j \in \mathbb{Z}$ is defined to be the perverse filtration on $\mathbb{H}^j(Y, Rf_*\mathbb{Q}_X[n])$

It coincides, up to a shift, with the Leray filtration, when f is smooth.

If these decomposing isomorphisms exist, they are seldom unique. We now give the statement (not in the most general form) of one of the more general criteria for decomposability, see [10] and [11]:

Theorem 4.3.3 (Deligne degeneration criterion.) *Let K be an object of $D^b(Y)$, and let $\eta \in H^2(X)$. Suppose that $\eta^l : {}^p\mathcal{H}^{-l}(K) \to {}^p\mathcal{H}^l(K)$ is an isomorphism for all l. Then K is ${}^p\mathcal{H}-$decomposable. The same statement holds if we consider the functor $\mathcal{H}$.*

Example 4.3.4 By the computation done in 4.1.3, we have the following description of the perverse filtration for the resolution of a threefold:

$$H^i_{\leq -2}(X) = \{0\}, \qquad H^2_{\leq -1}(X) = \text{Im } \{H_4(D) \to H^2(X)\}, \qquad H^i_{\leq -1}(X) = 0 \text{ otherwise},$$

$$H^4_{\leq 0}(X) = \text{Ker } \{H^4(X) \to H^4(D)\}, \qquad H^i_{\leq 0}(X) = H^i(X) \text{ otherwise },$$

$$H^i(X)_{\leq 1} = H^i(X) \text{ for all } i.$$

The condition 4.3.3, that

$$\eta : {}^p\mathcal{H}^{-1}(Rf_*\mathbb{Q}_X[3]) = H_4(D)_y \longrightarrow H^4(D)_y = {}^p\mathcal{H}^1(Rf_*\mathbb{Q}_X[3])$$

be an isomorphism, is just the non degeneracy of the intersection form $\eta \cap [D_i] \cdot [D_j]$. Note that in this case, the explicit description makes it clear that the perverse filtration is given by Hodge substructures.

4.4 The Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber

We can now state the generalization of Deligne's Theorem 3.3.2 to the case of arbitrary proper maps. Recall that if X is smooth, then $IC_Y = \mathbb{Q}_X[n]$.

Theorem 4.4.1 *Let $f : X \to Y$ be a proper map of algebraic varieties. Then*

$$\text{the complex } Rf_*IC_X \simeq \bigoplus_i {}^p\mathcal{H}^i(Rf_*IC_X)[-i] \text{ is } {}^p\mathcal{H}-\text{decomposable} \tag{13}$$

The complexes ${}^p\mathcal{H}^j(Rf_*IC_X)$ are semisimple i.e. there is a canonical isomorphism

$$ {}^p\mathcal{H}^j(Rf_*IC_X) \simeq_{P(Y)} \oplus IC_{Z_a}(L_a) \tag{14}$$

for some finite collection, depending on j, of semisimple local systems L_a on smooth distinct varieties $Z_a^o \subseteq Z \subseteq Y$.
Let η be an $f-$ample line bundle on X. Then

$$\eta^r \, : \, {}^p\mathcal{H}^{-r}(Rf_*IC_X) \simeq {}^p\mathcal{H}^r(Rf_*IC_X). \tag{15}$$

The Verdier Duality functor is an autoequivalence $\mathcal{D} : D(Y) \to D(Y)$ which preserves $P(Y)$ and for which one has

$$\mathcal{D} \circ {}^p\mathcal{H}^{-j} \simeq {}^p\mathcal{H}^j \circ \mathcal{D}.$$

This fact implies that the summands appearing in the semisimplicity statement for j are pairwise isomorphic to the ones appearing for $-j$ and that the local systems L are self-dual.

Theorem 4.4.1 is the deepest known fact concerning the homology of algebraic maps.
The original proof uses algebraic geometry in positive characteristic. in an essential way.
M. Saito has given a transcendental proof of a more general statement concerning his mixed Hodge modules in the series of papers [22], [23], [22].
We give a proof for the push-forward of intersection cohomology (with constant coefficients) first in the case of semismall maps (cf. [6]) and then for arbitrary maps in [8]. Though at present our methods do not afford results concerning the push-forward with more general coefficients, they give new and precise results on the perverse filtration and on the refined intersection forms.
C. Sabbah [24] has recently proved a decomposition theorem for push-forwards of semisimple local systems.

Remark 4.4.2 It is now evident that the computations in 3.1 and 3.2 establish the Decomposition Theorem for maps from a smooth surface. In the case of the proper birational map $f : X \to Y$ of 3.1, in fact, the complex $Rf_*\mathbb{Q}_X[2]$ is perverse, as observed in 4.1.2, and 3.1.1 states that it splits into IC_Y and $R^2 f_*\mathbb{Q}_X[0]$. In the case of the family of curves treated in 3.2 we have that $j_*T^0[2] = {}^p\mathcal{H}^{-1}(Rf_*\mathbb{Q}_X[2])[1]$, and $j_*T^2[0] = {}^p\mathcal{H}^1(Rf_*\mathbb{Q}_X[2])[-1]$, and we showed in 3.2.1 that $\eta : j_*T^0 \to j_*T^2$ is an isomorphism. The perverse sheaf P splits, see 3.2.2, in $j_*T^1[1] = IC_Y(T^1)$, and V, concentrated on points.

Remark 4.4.3 For the case of the resolution of a threefold with isolated singularities, whose Hodge theory has been treated in 2.3, we have, as seen in 4.1.3,

$$ {}^p\mathcal{H}^{-1}(Rf_*\mathbb{Q}_X[3]) \simeq H_4(D)_y, \qquad {}^p\mathcal{H}^1(Rf_*\mathbb{Q}_X[3]) \simeq H^4(D)_y \simeq \eta \wedge H_4(D)_y, $$

and we have the splitting

$$ {}^p\mathcal{H}^0(Rf_*\mathbb{Q}_X[3]) \simeq IC_Y \oplus H_3(D)_y. $$

Similarly, for the 4-fold with isolated singularities, see 2.4,

$$ {}^p\mathcal{H}^{-2}(Rf_*\mathbb{Q}_X[4]) \simeq H_6(D)_y, \qquad {}^p\mathcal{H}^{-1}(Rf_*\mathbb{Q}_X[4]) \simeq H_5(D)_y, $$

$$ {}^p\mathcal{H}^2(Rf_*\mathbb{Q}_X[4]) \simeq H^6(D)_y \simeq \eta^2 \wedge H_6(D)_y, \qquad {}^p\mathcal{H}^1(Rf_*\mathbb{Q}_X[4]) \simeq H^5(D)_y \simeq \eta \wedge H_5(D)_y, $$

and we have the splitting

$$ {}^p\mathcal{H}^0(Rf_*\mathbb{Q}_X[4]) \simeq IC_Y \oplus H_4(D)_y. $$

4.5 Results on intersection forms

In this section we list some of the results of [8] which are related to the theme of this paper. For simplicity, we state them in the special case when $f : X \to Y$ is a map of projective varieties, X smooth. Let η and A be ample line bundles on X and Y respectively, $L := f^*A$.

Theorem 4.5.1 *For $l \geq 0$ and $b \in \mathbb{Z}$, the subspaces given by the perverse filtration (cf. 4.3)*

$$ H^l_{\leq b}(X) \subseteq H^l(X) $$

are pure Hodge sub-structures. The quotient spaces

$$ H^l_b(X) = H^l_{\leq b}(X)/H^l_{\leq b-1}(X) $$

inherit a pure Hodge structure of weight l.

The cup product with η verifies $\eta\, H^l_{\leq a}(X) \subseteq H^{l+2}_{\leq a+2}(X)$ and induces maps, still denoted $\eta : H^l_a(X) \to H^{l+2}_{a+2}(X)$. The cup product with L is compatible with the Decomposition Theorem 4.4.1 and induces maps $L : H^l_a(X) \to H^{l+2}_a(X)$.
These maps satisfy graded Hard Lefschetz Theorems (cf. [8], Theorem 2.1.4).
Define $P^{-j}_{-i} := \text{Ker}\ \ \eta^{i+1} \cap \text{Ker}\ \ L^{j+1} \subseteq H^{n-i-j}_{-i}(X)$, $i,\, j \geq 0$ and $P^{-j}_{-i} := 0$ otherwise.
In the same way in which the classical Hard Lefschetz implies the Primitive Lefschetz Decomposition for the cohomology of X, the graded Hard Lefschetz Theorems imply the double direct sum decomposition of

Theorem 4.5.2 *Let $i,\, j \in \mathbb{Z}$. There is a Lefschetz-type direct sum decomposition into pure Hodge sub-structures of weight $(n-i-j)$, called the $(\eta, L)-$decomposition:*

$$H^{n-i-j}_{-i}(X) = \bigoplus_{l,\, m \in \mathbb{Z}} \eta^{-i+l}\, L^{-j+m}\, P^{j-2m}_{i-2l}.$$

One can define bilinear forms $S^{\eta L}_{ij}$ on $H^{n-i-j}_{-j}(X)$ by modifying the Poincaré pairing

$$S^{\eta L}_{ij}([\alpha],[\beta]) \;:=\; \int_X \eta^i \wedge L^j \wedge \alpha \wedge \beta$$

and descending it to the graded groups. These forms are non degenerate. In fact their signature can be determined in the following generalization of the Hodge Riemann relations.

Theorem 4.5.3 *The 4.5.2 is orthogonal with respect to $S^{\eta L}_{ij}$. The forms $S^{\eta L}_{ij}$ induce polarizations of each $(\eta, L)-$direct summand.*

The homology groups $H^{BM}_*(f^{-1}(y)) = H_*(f^{-1}(y))$, $y \in Y$, are filtered by virtue of the decomposition theorem (one may call this the perverse filtration). The natural cycle class map $cl : H^{BM}_{n-*}(f^{-1}(y)) \to H^{n+*}(X)$ is filtered strict.

The following generalizes the Grauert Contractibility Criterion.

Theorem 4.5.4 *Let $b \in \mathbb{Z}$, $y \in Y$. The natural class maps*

$$cl_b : H^{BM}_{n-b,b}(f^{-1}(y)) \longrightarrow H^{n+b}_b(X)$$

is injective and identifies $H^{BM}_{n-b,b}(f^{-1}(y)) \subseteq \text{Ker}\ \ L \subseteq H^{n+b}_b(X)$ with a pure Hodge substructure, compatibly with the $(\eta, L)-$decomposition. Each $(\eta, L)-$direct summand of $H^{BM}_{n-b,b}(f^{-1}(y))$ is polarized up to sign by $S^{\eta L}_{-b,0}$.
In particular, the restriction of $S^{\eta L}_{-b,0}$ to $H^{BM}_{n-b,b}(f^{-1}(y))$ is non degenerate.

By intersecting in X cycles supported on $f^{-1}(y)$, we get the refined intersection form (see section 2.2) $H^{BM}_{n-*}(f^{-1}(y)) \to H^{n+*}(f^{-1}(y))$ which is filtered strict as well.

Theorem 4.5.5 *(***The Refined Intersection Form Theorem***) Let $b \in \mathbb{Z}$, $y \in Y$. The graded refined intersection form*

$$H^{BM}_{n-b,a}(f^{-1}(y)) \longrightarrow H^{n+b}_{a}(f^{-1}(y))$$

is zero if $a \neq b$ and it is an isomorphism if $a = b$.

We have seen in earlier sections how these results can be made explicit in the case of surfaces, threefolds and fourfolds. For more applications in any dimension see [8].

In fact, the method of proof of the results stated in this section is inspired by the low dimensional examples of surfaces, threefolds and fourfolds.

4.6 The decomposition mechanism

It is quite hard to describe what kind of geometric phenomena are expressed by the Decomposition Theorem. The complex $Rf_*\mathbb{Q}_X$ essentially describes $H^*(f^{-1}(U))$ for any neighborhood U of a point y. We gain some geometric insight if we represent, via Poincaré duality, the cohomology classes with Borel-Moore cycles in $f^{-1}(U)$. Let S be the stratum containing y. By the exact sequence

$$H^{BM}_*(f^{-1}(U \cap S)) \xrightarrow{i_*} H^{BM}_*(f^{-1}(U)) \xrightarrow{j^*} H^{BM}_*(f^{-1}(U \setminus S)),$$

the Borel Moore cycles in $f^{-1}(U)$ are of two kind: those in Im i_*, which are homologous to cycles supported on the inverse image of the stratum S, and those whose restriction to $f^{-1}(U \setminus S)$ is not trivial. Neither i_* is injective nor j^* is surjective: there are non trivial cycles in $f^{-1}(U \cap S)$ which become homologous to zero in $f^{-1}(U)$, and there are cycles in $f^{-1}(U \setminus S)$ which cannot be closed to cycles in $f^{-1}(U)$.

The Decomposition Theorem gives strong information on both types. The first deep aspect of the Theorem is that the subspace Im i_*, has a uniform behavior for all projective maps, related to the non degeneracy of the intersection forms. For instance, we already noticed in 2.1.3, how the Grauert Theorem 2.1.1 implies that the classes of disks transverse to exceptional curves are homologous to linear combinations of the classes of these curves. Such non degeneracy results, see 4.5.4, 4.5.5, are peculiar to algebraic maps and stem from "weight" considerations, either in characteristic 0 (Hodge Theory). or in positive characteristic (weights of Frobenius, cf. [2]).

The Decomposition Theorem, though, contains other deep information. Since $Rf_*\mathbb{Q}_X$ splits as a direct sum of terms associated with the strata, we have a splitting map, which can be made canonical after an ample line bundle η on X has been chosen, from the subspace Im $j^* \subseteq H_*^{BM}(f^{-1}(U \setminus S)$ to $H_*^{BM}(f^{-1}(U)$: i.e. the following is split exact

$$0 \longrightarrow \text{Im } i_* \longrightarrow H_*^{BM}(f^{-1}(U) \longrightarrow \text{Im } j^* \longrightarrow 0.$$

The image of this map defines a subspace of $H_*^{BM}(f^{-1}(U)$ which is complementary to Im i_* and consists of classes which are closures of some Borel Moore cycles in $f^{-1}(U \setminus S)$. The deep fact here, is that these cycles are governed by the intersection cohomology complex construction on Y; each stratum having S in its closure contributes to $H_*^{BM}(f^{-1}(U)$ via the intersection cohomology of a local system on the stratum.

5 Grothendieck motive decomposition for maps of threefolds

We assume we are in the situation 2.3.1. Again, this is for ease of exposition only. See Remark 5.0.6.

We have already shown that

$$H^2_{-1}(X) \;=\; \text{Im } \; H_4(D) \to H^2(X) \;=: \text{Im } \; i_*,$$
$$H^4_0(X) = \text{Ker } \; \{H^4(X) \to H^4(D)\} \;=: \text{Ker } \; i^*.$$

The choice of an ample line bundle η allows to split the perverse filtration:

$$H^2(X) = \text{Im } \; i_* \oplus (c_1(\eta) \wedge \text{Im } \; i_*)^\perp.$$
$$H^4(X) = \text{Ker } \; i^* \oplus (\text{Im } \; i_*)^\perp.$$

We have, canonically, that

$$H^3(X) = \text{Im } \; H_3(D) \oplus (\text{Im } \; H_3(D))^\perp.$$

so that

$$IH^i(Y) \;=\; H^i(X) \text{ for } i = 0,\, 1,\, 5,\, 6, \qquad IH^2(Y) \;=\; (c_1(\eta) \wedge \text{Im } \; H_4(D),)^\perp$$
$$IH^3(Y) \;=\; (\text{Im } \; H_3(D))^\perp, \qquad IH^4(Y) \;=\; (\text{Im } \; H_4(D))^\perp;$$

here we are using the convention for intersection cohomology compatible with singular cohomology: $IH^i(Y) := \mathbb{H}^{i-n}(Y, IC_Y))$.

We want to realize these splittings by algebraic cycles on $X \times X$, in order to find a Grothendieck motive for the intersection cohomology of Y. These cycles will be supported on $D \times D$.

We start with the following simple lemma.

Lemma 5.0.1 *Let X be a projective $n-$fold, and $Y \subseteq X$ be a subvariety. Let $W \subseteq \text{Im } \{H_s(Y) \to H^{2n-s}(X)\} \subseteq H^{2n-s}(X)$ be a vector subspace on which the restriction of the Poincarè pairing remains non degenerate, i.e. $H(X) = W \oplus W^{\perp}$. Then the projection $P_W \in End(H(X)) \simeq H(X \times X)$ on W relative to the above splitting can be represented by a cycle supported on $Y \times Y$.*

Proof. Let $\{e_1\}$ be a basis for $H(X)$ such that $e_1, \cdots, e_k \in W$ and $e_{k+1}, \cdots, e_N \in W^{\perp}$. For $i = 1, \cdots, k$, we can represent e_i by a cycle γ_i contained in Y. In force of the hypothesis, the dual basis $\{\check{e_i}\}$ is of the form

$$\check{e_i} = \sum_{j=1}^{k} a_{ij} e_j \qquad \text{for } 1 \leq i \leq k \qquad \check{e_i} = \sum_{j=k+1}^{N} a_{ij} e_j \qquad \text{for } k+1 \leq i \leq N.$$

In particular $\check{e_1}, \cdots, \check{e_k}$ are represented by the cycles $\check{\gamma_i} = \sum_{j=1}^{k} a_{ij}\gamma_j$ supported on Y. The projector $P_W = \sum_{i=1}^{k} e_i \otimes \check{e_i}$ is thus represented by the cycle $\sum_{i,j=1}^{k} a_{ij}\gamma_i \times \gamma_j$, which is supported on Y. □

The following is a standard but very useful application of "strictness" in Hodge Theory

Lemma 5.0.2 *Let $Y \subseteq X$ be a codimension d subvariety of an $n - fold$ and let $\pi : \tilde{Y} \to Y$ a resolution of singularities. Suppose $\beta \in \text{Im } \{H_{2k}(Y) \to H^{2(n-k)}(X)\} \cap H^{n-k,n-k}(X)$. Then there is $\tilde{\beta} \in H^{n-k-d,n-k-d}(\tilde{Y})$ such that $(i \circ \pi)_*(\tilde{\beta}) = \beta$.*

Proof. We consider the weights of the homology groups as given by their being dual of the cohomology groups. Thus $H_{2k}(Y)$ has weights $\geq -2k$. The map $H_{2k}(Y) \to H^{2(n-k)}(X)$ is of type (n,n). Since the Hodge structure on $H^{2(n-k)}(X)$ is pure, the strictness of maps of Hodge structures implies that

$$\text{Im } \{H_{2k}(Y) \to H^{2(n-k)}(X)\} \;=\; \text{Im } \{W_{-2k}H_{2k}(Y) \to H^{2(n-k)}(X)\}.$$

It follows that $\beta = i_*\beta'$ for some $\beta' \in W_{-2k}H_{2k}(Y)$. On the other hand this group coincides with $\text{Im } \{\pi_* : H_{2k}(\tilde{Y}) \to H_{2k}(Y)\}$ for any resolution, whence the statement.

□

Theorem 5.0.3 *Let $f : X \to Y$, D as before. Then there exist algebraic 3-dimensional cycles Z_{-1}, Z_0, Z_1, supported on $D \times D$ such that:*
Z_1 defines the projection of $H(X)$ onto $H^4_1(X) = c_1(\eta) \wedge \text{Im } \{H_4(D) \to H^2(X)\} \subseteq H^4(X)$;
Z_{-1} defines the projection of $H(X)$ onto $H^2_{-1}(X) = \text{Im } \{H_4(D) \to H^2(X)\} \subseteq H^2(X)$;
Z_0 defines the projection of $H(X)$ on $\text{Im } \{H_3(D) \to H^3(X)\} \subseteq H^3(X)$.

Proof. Let Λ be the inverse of the negative-definite intersection matrix $I_{ij} = \int_X c_1(\eta) \wedge [D_i] \wedge [D_j]$. We denote by $\eta \cap D_i$ the curve obtained intersecting the divisor D_i with a general section of η. Set:

$$Z_{-1} = \sum \lambda_{ij}[(\eta \cap D_i) \times D_j] \qquad Z_1 = \sum \lambda_{ij}[D_i \times (\eta \cap D_j)].$$

It is immediate to verify that Z_{-1} and Z_1 define the sought-for projectors.
The construction of Z_0 is not so direct: Since, by 2.3.5, the Poincaré paring is non degenerate on Im $\{H_3(D) \to H^3(X)\}$, by 5.0.1 we can represent the projection on $H_3(D)$ by a cycle supported on $D \times D$. Furthermore, the projection is a map of Hodge structures, hence its representative cycle $P_3 \in H^6(X \times X)$ has type $(3, 3)$. By 5.0.2 we have $P_3 = i_* \pi_* \beta$ for some $\beta \in H^{1,1}(\widetilde{D \times D})$, where $\widetilde{D \times D}$ is any resolution of $D \times D$. By the Lefschetz Theorem on $(1, 1)$, classes there is an algebraic cycle $\tilde{Z}$ such that $\beta = [\tilde{Z}]$. It is clear that $Z_0 = i_* \pi_* Z$ does the job. □

The following follows immediately.

Corollary 5.0.4 *The Grothendieck motive,* $(X, \Delta_X - Z_0 - Z_1 - Z_{-1})$ *is a Betti realization of the Intersection cohomology of* Y.

We can be more specific: the projector $\Delta_X - Z_0 - Z_1 - Z_{-1}$ is supported on the fiber product $X \times_Y X$, therefore defines a relative motive over Y in the sense of [9] (see also [7]). By [9], Lemma 2.23, the isomorphism of algebras

$$End(Rf_*\mathbb{Q}_X[3]) = H_6^{BM}(X \times_Y X)$$

ensures that the Betti realization of this relative motive is the projector associated with the splitting for $Rf_*\mathbb{Q}_X[3]$ we have used in this section.

Remark 5.0.5 From the construction of the cycles it is evident that Z_{-1} and Z_1 define in fact Chow motives, not only Grothendieck motives. Under some hypothesis it is possible to construct a Chow projector for Z_0 as well. For instance, if D is smooth irreducible, and its conormal bundle $\mathcal{I}_D/\mathcal{I}_D^2$ is ample. In this case, let Z_0 the cycle in $D \times D$ representing the Hodge Λ operator with respect to the polarization given by the conormal bundle. It is immediate to verify that Z_0 defines the Chow motive we need. In general some knowledge of the nature of the resolution may allow one to find a Chow motive whose Betti realization is intersection cohomology.

Remark 5.0.6 It is not difficult to modify the proofs to produce a Grothendieck motive for the intersection cohomology of Y for an arbitrary three-dimensional variety Y, (e.g. with non-isolated singularities). If, for example, some divisor D' is blown down to a curve C, then one needs to construct a further projector, represented by a cycle which is a linear combination of the components of $D' \times_C D'$. This projector splits off the contribution of D' to the cohomology of X. We leave this task to the reader.

Remark 5.0.7 If Y is a fourfold with isolated singularities, then the computations in 2.4 express its intersection cohomology as a Hodge substructure of the cohomology of a resolution X. The method developed in this section does not apply in general since we do not know whether the classes of the projectors, which are pushforward of classes of type (p, p) on a resolution of the product of the exceptional divisor with itself, are represented by algebraic cycles. On the other hand, this can be achieved in the presence of supplementary information on the singularities of Y or on the exceptional divisor. For example: if the singularities are locally isomorphic to toric singularities. This allows to define a motive for the intersection cohomology in several interesting cases.

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