LECTURES ON p-ADIC HODGE THEORY BY BHARGAV BHATT

Abstract. These are notes poorly taken by Mark Andrea de Cataldo.

Contents

lectureone

1. Lecture I

l1s1

1.1. Goal of p-adic Hodge theory.

Classical Hodge theory has two parts: 1) the special properties of Hodge structures coming from geometry; 2) the abstract properties of Hodge structures. One can say the same about p-adic Hodge theory.

Fix a prime p. Consider a finite extension $K/\mathbb{Q}_p$ with fixed algebraic closure $\overline{K}$ and with absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$; this is a pro-finite group. Let X/K be an algebraic variety.

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In this situation, a possible analogue of the Betti (=singular) cohomology of a complex variety is the étale cohomology $H^i_{et}(\overline{X}, \mathbb{Z}_p)$. This is a (continuous) G_K-module: the variety is defined over K and the constructions of étale cohomology yield such a structure automatically, by considerations of all finite extensions of K inside $\overline{K}$.

In these lectures, the goal of p-adic Hodge theory is to understand this action in terms of the geometry of X/K.

abvar **Example 1.1.1.** *Let A/K be an abelian variety of dimension g. Then $H_1(X_{\overline{K}}, \mathbb{Z}_p) = T_p(A)$, the p-Tate module of A, i.e. the inverse limit of the groups $A[p^n](\overline{K})$. As a group this is abstractly isomorphic to $\mathbb{Z}_p^{2g}$. However, it is also a G_K-module, and as such it is not isomorphic to $\mathbb{Z}_p^{2g}$ (with the trivial G_K-structure): the Galois group does not act trivially on torsion points which are not K-rational.*

l1s2

1.2. The Hodge-Tate decomposition.

The key result is the Hodge-Tate decomposition; proved by J. Tate in the late 1960's; it jump-started the subject.

htd **Theorem 1.2.1. (Usual Hodge decomposition)** *$X/\mathbb{C}$ smooth and proper. Then*

$$H^n(X, \mathbb{C}) = \bigoplus_{i+j=n} H^{i,j}(X) := \bigoplus_{i+j=n} H^j(X, \Omega^i_X) \qquad (1)$$ hd

A p-adic analogue of such a decomposition necessitates the notion of Tate twist.

tt **Definition 1.2.2. (Tate twist)**

(1) *$\mathbb{Z}_p(1) := T_p(\mathbb{G}_{m,K})$ (inverse limit of p^n-th roots of unity in $\overline{K}$) (it is a G_K-module; it is a $\mathbb{Z}_p$-module (abstractly isomorphic to $\mathbb{Z}_p$ as a $\mathbb{Z}_p$-module via a choice of a primitive p-th root of unity; but not isomorphic to it as G_K-module). We have the G_K-modules $\mathbb{Z}_p(i) := \mathbb{Z}_p(1)^{\otimes i}$, $i \in \mathbb{Z}$ (tensor product over $\mathbb{Z}_p$; negative powers defined by taking duals).*

(2) *Let V be a G_K-module. Define $V(i) := V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$.*

Remark 1.2.3.

(1) *$\mathbb{Z}_p(1)$ is a geometric Galois representation, i.e. it arises as the homology/cohomology of a variety. In fact, it is naturally isomorphic to*

$$\pi_1^{et}(\mathbb{G}_m, \overline{K})\hat{}_p;$$

this is because $p^n : \mathbb{G}_m \to \mathbb{G}_m$ is a μ_{p^n}-torsor.[1]

(2) *As mentioned above, $\mathbb{Z}_p(1)$ is not the trivial G_K representation. To show this you need to observe (exercise) that no finite extension of K can contain all p^n-roots of unity, for every n, so that not all of the elements of $\mathbb{Z}_p(1)$ are G_K-invariants. (Hint: prove the analogue statement for algebraic number fields and use that our K is the completion of such a number field ...). In fact, more is true: there are no non-trivial G_K-invariants, namely*

$$H^0(G_K, \mathbb{Z}_p(1))^{G_K} = \{0\}. \qquad (2)$$ noinv

[1] It was asked why not simply point-out that $\mathbb{Z}_p(-1) = H^2(\mathbb{P}^1, \mathbb{Z}_p)$. This is true, but the fundamental group as G_K-module seems like a "more direct" example.

(3) *Tate has showed that, if we take the completion $\mathbb{C}_p$ of $\overline{K}$ (which is algebraically closed, hence non-canonically isomorphic to $\mathbb{C}$), then*

$$H^1(G_k, \mathbb{C}_p(j)) = \begin{cases} 0, & \text{if } j \neq 0 \\ K, & \text{if } j = 0. \end{cases} \quad (3)$$

nointer

We may express the above by saying that "different Tate twists do not interact".

The following has been conjectured by Tate in 1967, proved by him for Abelian varieties, first proved by Faltings ...

tmhtd **Theorem 1.2.4. (Hodge-Tate decomposition)** *Let X/K be smooth and proper. Then there is a canonical isomorphism*

$$\rho_{HT} : H^n_{et}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{\cong} \bigoplus_{i+j=n} H^j(X, \Omega^i_X) \otimes_K \mathbb{C}_p(-i). \quad (4)$$

htd

N.B.: On the lhs, we have the tensor product of two G_K-modules: the first is what we are trying to understand; the second is "algebraic" determined by $\overline{K}/K$. On the rhs we have three factors, where the first has the trivial G_K-action (it is a K-vectorspace) and the other two, which are thus completely responsible for the G_K-action, are also "algebraic", determined by $\overline{K}/K$. Can this be interpreted as some kind of "triviality"?

Remark 1.2.5.

(1) *There is a variant for X/K open/singular.*[2]
(2) *The Hodge-Tate decomposition exists for every smooth, proper, rigid analytic space $/K$ (this is due to P. Scholze).*
(3) *By combining the Hodge-Tate decomposition (4) with the principle (3) that distinct Tate twists do not interact, we deduce that*

$$H^j(X, \Omega^i_X) = \left(H^n(X_{\overline{K}}, \mathbb{C}_p)(i)\right)^{G_K}, \quad (5)$$

hodgen

or, in English, the Galois action on étale cohomology knows the Hodge numbers!

N.B.: Ito (Hodge numbers)-Batyrev (Betti numbers): two birational Calabi-Yau have the same Hodge numbers. This could have been observed by a use of the above Hodge-Tate decomposition theorem (how?). But if this had been done, maybe Kontsevich would not have introduced the notion of motivic integration that connects to such an identification of Hodge numbers! (say better ...).

(4) *All existing proofs of the Hodge-Tate decomposition theorem (4) give a Hodge-type filtration on $H^*(X_{\overline{K}}, \mathbb{C}_p)$ with graded pieces $H^j(X, \Omega^i_X) \otimes_K \mathbb{C}_p$. Once you have this filtration, you use the machinery of Galois cohomology to get a canonical splitting of it, i.e. the Hodge-Tate decomposition (4).* [3]

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[2] Somebody asked whether we should expect a weight filtration. This should be no problem, since such a filtration already exists by Deligne on étale cohomology (probably for $\mathbb{Q}_p$-coefficients, but we are tensoring with $\mathbb{C}_p$ ..., so it should be OK ...).

[3] Note that for $X/\mathbb{C}$ smooth proper, the Hodge filtration on $H^*_{dR}(X/\mathbb{C})$ does not split canonically: roughly, if it did, it would do so in a family, giving rise to a holomorphic splitting into (p, q)-parts, and this is not the case, such splittings are typically only C^∞)

However, the natural maps go "backwards" with respect to the classical Hodge filtration, where we have $\mathbb{C} \to \mathcal{O}_X$*, giving*

$$H^*(X, \mathbb{C}) \to H^*(Y, \mathcal{O}_X);$$

instead in the p-adic context, we have

$$H^*(X, \mathcal{O}_X) \otimes_K \mathbb{C}_p \longrightarrow H^*_{et}(X_{\overline{K}}, \mathbb{C}_p).$$

(To stress this point, it was said that if we start with X/K*, then we get the canonical Hodge-Tate decomposition, so a natural presentation of the filtration by assembling direct summands. In this case, if we do not consider the map above, we could still be on the fence as to the "direction" of this Hodge-Tate filtration; but if we have a variety defined over a transcendental extension of* $\mathbb{Q}_p$*, then such a canonical splitting does not exist; what we (presumably) have is the Hodge-Tate filtration, which, indeed, goes "backwards.")*

(5) *The Hodge-Tate decomposition (4) yields a functor*

$$D_{HT} : \{\textit{geom Galois reps}\} \longrightarrow \{\textit{graded } K\textit{-vect}\}, \quad V \longmapsto \oplus_{i \in \mathbb{Z}} (V \otimes_{\mathbb{Z}_p} \mathbb{C}_p(i))^{G_K}. \qquad (6)$$ dht

This D_{HT} *is NOT fully faithful: take two elliptic curves, one with* K*-rational* p*-torsion, the other one without. Then the two rhs of (4) are isomorphic, but the two etale cohomology groups are not isomorphic as* G_K*-modules.*[4]

It was asked to explain why we do not take $H_{et}(X_{\overline{K}}, \mathbb{Z}_\ell)$ etc, instead of $\mathbb{Z}_p$-coefficients. Consider X/K and its "enlargement" over $Spec\, \mathcal{O}_K$, with special fiber defined over a finite field of order $q = p^?$, which has its own geometric Frobenius generating (topologically) the absolute Galois group of the finite field (abstractly isomorphic to $\hat{\mathbb{Z}}$). In this case, the Galois representation $H^n_{et}(X_{\overline{K}}, \mathbb{Z}_\ell)$ factors through $\hat{\mathbb{Z}}$ (why?), so one expects to be able to extract less information from ℓ-adic cohomology, then from p-adic cohomology: morally, G_K is "larger" than $\hat{\mathbb{Z}}$, so if we are factoring through this smaller group, we should expect to be able to observe less structure ....

drcomp

1.3. The p-adic-to-de Rham comparison theorem.

drtm **Theorem 1.3.1. (Betti-de Rham comparison Theorem)** *Let* $X/\mathbb{C}$ *be smooth proper. Then*

$$H^*_B(X, \mathbb{C}) \cong H^*_{dR}(X/\mathbb{C}) := H^*(X, \Omega^\bullet_X).$$

Here is a naive étale-to-p-adic analogue:

$$H^*_{et}(X_{\overline{K}}, \mathbb{C}_p) \;???\; \cong_{G_K} \;???\; H^*_{dR}(X/K) \otimes_K \mathbb{C}_p. \qquad (7)$$ naiveet2d

This is maximally false!

maxflz **Example 1.3.2.** $X = \mathbb{P}^1$. $H^2_{et}(\mathbb{P}^1_{\overline{K}}, \mathbb{C}_p) = H^1_{et}(\mathbb{G}_{m,\overline{K}}, \mathbb{C}_p) = \mathbb{C}_p(-1)$*, which has no* G_K*-invariants! (see (2)). On the other hand,* $H^2_{dR}(\mathbb{P}^1_K) \otimes_K \mathbb{C}_p$ *contains an isomorphic copy of* K *as* G_K*-invariants.*

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[4] This is saying that D_{HT} is not conservative (i.e. $D_{HT}(f)$ iso IFF f iso). But fully faithful implies conservative (the converse is false: take the groupoids associated with groups and a functor modeled on a group homomorphism: that is conservative, but, in general, neither injective, nor surjective on arrows). So D_{HT} is not fully faithfu).

In order to formulate a correct etale-to-p-adic analogue of the Betti-de Rham comparison theorem, we first re-write the Hodge-Tate decomposition. To do so, we define the G_K-module

$$B_{HT} := \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i), \tag{8}$$ bht

which we may call the big Hodge-Tate module (but probably B stands for Barsotti). We can then rewrite the Hodge-Tate decomposition (4) as follows (exercise)

$$H^n_{et}(X_{\overline{K}}, \mathbb{Z}_p) \otimes B_{HT} \cong \left(H^n_{Hodge} := \oplus_{i+j=n} H^j(X, \Omega^i_X)\right) \otimes_k B_{HT}. \tag{9}$$ htd2

In order to get the etale-to-de Rham comparison theorem, we first deform B_{HT}.

bplus **Fact 1.3.3.** *There is a complete DVR $B^+_{dR}/\overline{K}$ –to be defined in Lecture III– such that:*

1. *the residue field is $\mathbb{C}_p$;*
2. *G_K acts on everything in sight, compatibly with the structures in sight;*
3. *the maximal ideal $\mathfrak{m}$ is automatically G_K-invariant and $\mathfrak{m}/\mathfrak{m}^2 \cong_{G_K} \mathbb{C}_p(1)$.*
   The above three properties are also enjoyed by B_{HT}; the following sets B^+_{dR} apart
4. *the map $B^+_{dR} \to \mathbb{C}_p$ does NOT have a G_K-equivariant section (every Noetherian local ring as a section). (In particular, $B^+_{dR} \cong \mathbb{C}_p[[t]]$, but not canonically).*

Define B_{dR} to be the fraction field of B^+_{dR}. It is naturally filtered by the $\mathfrak{m}^k$.
We can now state

p2drtm **Theorem 1.3.4.** **(p-adic-to-de Rham comparison theorem)** *Let X/K be smooth and proper. There is a canonical isomorphism*

$$\rho_{dR} : H^n_{et}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{dR} \xrightarrow{\cong} H^n_{dR}(X/K) \otimes_K B_{dR}, \tag{10}$$ p2dr

which is compatible with the G_K-actions and with the Hodge-type filtrations discussed above.

N.B.: $H^n_{dR}(X/K)$ has the trivial G_K-action (X/K) and its own Hodge filtration; B_{dR} has its own "stupid" G_K-action and an interesting (but independent of X/K!) Hodge filtration; $H^n_{et}(X, \mathbb{Z}_p)$ is a G_K-module (remember the goal is to understand it!) and has the trivial Hodge filtration.

recfilet **Remark 1.3.5.**

1. *We can recover the Hodge filtration on de Rham cohomology from (10) as follows:*
   $$F^i H^n_{dR}(X/K) = \left(H^n_{et}(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{dR}\{i\}\right)^{G_K}, \tag{11}$$ recfil
   where $\{i\}$ denotes the shift of filtration by i ($\pm i$ ...) (exercise).
2. *We get a functor D_{dR} that lifts D_{HT} (6):*
   $$D_{dR} : \{\textit{geom Galois reps}\} \longrightarrow \{\textit{filtered } K\textit{-vect}\}, \quad V \longmapsto \left(V \otimes_{\mathbb{Z}_p} B_{dR}\{i\}\right)^{G_K} \tag{12}$$ ddr
   Again, this is not fully faithful: take two elliptic curves over $\mathbb{Q}$ X, Y; X with ordinary reduction, i.e. $X(\mathbb{Q}_p^{unram})[p] \neq 0$; Y with supersingular reduction (negate the above condition). Then $D_{dR}(H^1_{et}(X_{\overline{K}}, \mathbb{Z}_p)) \cong D_{dR}(H^1_{et}(Y_{\overline{K}}, \mathbb{Z}_p))$, but $H^1_{et}(X_{\overline{K}}, \mathbb{Z}_p) \ncong H^1_{et}(Y_{\overline{K}}, \mathbb{Z}_p)$, as G_K-modules, so that D_{dR} is not conservative, hence not fully faithful.

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2. Lecture II

Recall that our main goal is, given a variety X over a finite extension $K/\mathbb{Q}_p$, to understand $H^*_{et}(X_{\overline{K}}, \mathbb{Z}_p)$. as a G_K-module (Galois representation).

We ended Lecture I with the non-fully-faitful functor D_{dR} (12), from geometric Galois representations into filtered K-vectorials. The goal today is to identify an improvement of this functor, one that is fully faithful.

l2s1

2.1. **Crystals.**

The set-up is again X/K, and we take $K = \mathbb{Q}_p$ for simplicity (presumably because then $\mathcal{O}_K = \mathbb{Z}_p$, with residue field $\mathbb{F}_p$?).

We can then view X/K as the general fiber $\mathcal{X}_K/K$ of a morphism $\mathcal{X}/\mathcal{O}_K$ with special fiber $\mathcal{X}_0/\mathbb{F}_p$

$$\begin{array}{ccccc} \mathcal{X}_K & \longrightarrow & \mathcal{X} & \longleftrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow & & \downarrow \\ SpecK & \longrightarrow & Spec\mathcal{O}_K & \longleftarrow & SpecF_p \end{array}$$

Then, $\mathcal{X}_0/\mathbb{F}_p$ yields $H^*_{crys}(\mathcal{X}_0, \mathbb{Z}_p)$ which is an object in the category Dieudonne module over $\mathbb{Z}_p$, whose objects are

$$(M, \phi), \text{ with } M \text{ a f.g. } \mathbb{Z}_p\text{-mod, } \phi : M \to M, \text{ iso after inverting } p,$$

where, usually, ϕ arises via Frobenius.

Barghav is what follows ok? Should it appear before the previous sentnece? We assume good reduction, i.e. $\mathcal{X}_0/\mathbb{F}_p$ smooth as well. We consider this set-up to be a bit analogous to a smooth proper map $/\mathbb{C}$, where Ehresmann lemma tells us that we have a locally trivial fiber bundle.

fcrysdr

Fact 2.1.1. *There is a natural isomorphism*

$$H^*_{crys}(\mathcal{X}_0, \mathbb{Z}_p) \cong H^*_{dR}(\mathcal{X}/\mathbb{Z}_p), \tag{13}$$

crysdr

so that, having $\mathcal{X}/\mathbb{Z}_p$ produces Frobenius ϕ on $H^*_{dR}(\mathcal{X}/\mathbb{Z}_p)$. More generally, if $Y/\mathbb{F}_q$ is smooth, then you define (Bhargav: is below a def, or a thm????)

$$H^*_{crys}(Y, W(\mathbb{F}_q)) := H^*_{dR}(\widetilde{Y}/W(\mathbb{F}_q))$$

where $\widetilde{Y}$ is a lift of Y to the ring $W(\mathbb{F}_q)$ of Witt vectors of $\mathbb{F}_q$, assuming it exists.

The aim of crystal theory is to understand $H^*_{dR}(X/K)$ which is a filtered vector space (Hodge filtration) together with an automorphism (ϕ, Frobenius).

bcr

Fact 2.1.2. *There is a subring (which like all the B-type rings we have met so far have been introduced by Fontaine) $B_{crys} \subseteq B_{dR}$ s.t.*

1. *it is G_K stable;*
2. *it inherits the "Hodge" filtration from B_dR;*
3. *it has Frobenius ϕ.*

sctdr

2.2. The crystalline-to-de Rham comparison theorem.

Recall that we have already met ρ_{HT} (4) and ρ_{dR} (10). We now meet (14 via the following

tctdr **Theorem 2.2.1.** *Let $\mathcal{X}/\mathbb{Z}_p$ be proper smooth. There exists a canonical isomorphism (cfr. (13)*

$$\rho_{crys} : H^*_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{crys} \xrightarrow{\cong} H^*_{dR}(\mathcal{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{crys} \qquad (14)$$ ctdr

which is compatible with the G_K-actions, the Hodge filtrations and the ϕ actions.

As before, we get a functor:

$$D_{crys} : \{\text{geom. } G_K\text{-reps. w/good reduction}\} \longrightarrow \{\text{filtered } K\text{-vectorials w/ automorphisms}\} \qquad (15)$$ dcrys

where, $Filtr^0$ is given by Bhargav: $\otimes_{?????}$

$$V \longmapsto (V \otimes_{???} B_{crys})^{G_k}.$$

and for $Filtr^i$ use the shift of filtration $\{i\}$, as done earlier in (11) and ??? replace dR with crys???.

The big difference is that now D_{crys} is fully faithful!

fo **Theorem 2.2.2. (Fontaine)** *D_{crys} is fully faithful.*

ttec **Example 2.2.3.** *Recall the remark following (12), to the effect that D_{dR} (12) is not fully faithful. Take the same example. In this case, we see that D_{crys} is able to distinguish between the two non isomorphic Galois representations by mean of the eigenvalues of Frobenius: 1 and p in the first case and $\sqrt{p}$ twice in the second (Bhargav: are the weights off in the first case?)*

What happens to p-torsion? There are examples of surfaces $S/\mathbb{Z}_2$ (in fact, there are examples for every p) s.t. $H^2(S_{\overline{K}}, \mathbb{Z}_2)$ is torsion free, whereas $H^2_{crys}(S_0, \mathbb{Z}_2)_{tors} \neq 0$. This says we cannot expect an identity when it comes to an "integral"-type of statement, as we would in the context of smooth proper maps $/\mathbb{C}$ (Ehresmann lemma).

Nevertheless, we have the following

tmz **Theorem 2.2.4.** *Let $\mathcal{X}/\mathcal{O}_K$ be proper and smooth. Then*

$$length\, H^i_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_p)_{tors} \leq length\, H^i_{dR}(\mathcal{X}/\mathbb{Z}_p)_{tors}.$$

sthtd

2.3. The Hodge-Tate decomposition.

Bhargav: is the above title accurate: there is no HT decomposition ...

Recall that, without assuming good reduction, we have the following

5t **Theorem 2.3.1.** *X/K smooth proper. There is an E_2 (sic!) spectral sequence*

$$E_2^{ij} = H^i(X, \Omega^j_X) \otimes_K \mathbb{C}_p \Longrightarrow H^{i+j}_{et}(X_{\overline{K}}, \mathbb{C}_p),$$

Hence a resulting Hodge filtration on the abutment.

Bhargav: this is not the usual spectral sequence for $H^*(X, \Omega^\bullet_X)$, right? That one starts at E_1 with i and j swtiched ...

This has an analogue in characteristic p given by the following cart **Theorem 2.3.2.** *Let k be a field of characteristic p an let Y/k be a smooth variety (also proper?). Then there is an E_2 spectral sequence*

$$H^i(Y^{(1)}, \Omega_{Y^{(1)}}) \Longrightarrow H^{i+j}_{dR}(Y/K).$$

(in what above $Y^{(1)}$ is the k-variety obtained by base change via the Frobenius for k (I think)).

Bhargav: you mentioned the conjugate spectral sequence at this point ... You also said: they are related in the last lecture. I do not know what "they" refers to.

Proof. The differential of $Frob_* \Omega^\bullet_{Y/K}$ is $\mathcal{O}_{Y^{(1)}}$-linear! Cartier shows that

$$\mathcal{H}^i(Frob_* \Omega^\bullet_{Y/K}) = \Omega^i_{\mathcal{O}_{Y^{(1)}}}!$$

Note that Deligne-Illusie show that we have E_2-degeneration if Y lifts to second ring W_2 of Witt vectors of the field.

htabsc

2.4. Hodge-Tate for abelian schemes.

Set-up $A/\mathbb{Z}_p$ an abelian scheme, $K = \mathbb{Q}_p$ (again, for simplicity only, I assume).

Goal: construct the Hodge-Tate filtration and show why it splits.

There are two steps: one uses p-adic geometry; the other uses arithmetic:

(1) We use p-adic geometry to get an injection $\nu_A : H^1(A, \mathcal{O}_A \otimes_K \mathbb{C}_p \longrightarrow H^1_{et}(A_{\overline{K}}, \mathbb{C}_p)$. This gives the sought-after H-T filtration.

(2) For the splitting, we need a map in the opposite direction

$$\mu_A : H^0(A, \Omega^1_A) \otimes_{\mathbb{Z}_p} \mathbb{C}_p(-1) \longrightarrow H^1_{et}(A_{\overline{K}}, \mathbb{C}_p).$$

We dualize A, adjusts the shifts, and replace A^* with A to get μ_A.

Of course, one needs to check that $\mu_A \circ \nu_A = id$ (omitted), after which, we have the desired splitting.

Let us now discuss the geometric part (1) a bit more. Set $\mathcal{X} := A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}$ (the last symbol is the ring of integers of $\mathbb{C}_p$ (it is a non-Noetherian, $\mathbb{Q}$-valued valuation ring). To obtain the map ν_A is it enough to construct $H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to H^1_{et}(\mathcal{X}_{\mathbb{C}_p}, \mathbb{C}_p)$. The key idea is to take the inverse limit scheme $\mathcal{X}_\infty$ with transition maps the finite maps $[p] : \mathcal{X} \to \mathcal{X}$ given by multiplication by p. Let us make two observations.

(a) The group $\mathcal{X}[p](\mathcal{O}_{\mathbb{C}_p}) = \mathcal{X}_{\mathbb{C}_p}(\mathbb{C}_p)[p]$ acts on the transition maps above (translation on the domain; trivial action on the target). By taking the inverse limit, the Tate module $T_p(\mathcal{X}_{\mathbb{C}_p})(\cong \mathbb{Z}_p^{2g})$ acts on $\mathcal{X}_\infty \to \mathcal{X}$. This yields arrows

$$R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \longrightarrow R\Gamma_{cont}(T_p(\mathcal{X}), R\Gamma(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty})) \to \text{its completion}.$$

(b) We have that $[p]^*$ acts on $H^i(\mathcal{X}, \mathcal{O}_\mathcal{X})$ via multiplication by p^i (because we are dealing with an abelian scheme). This implies that $H^i(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty}) = H^i(\mathcal{X}, \mathcal{O}_\mathcal{X})[1/p]$, so that $H^i(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty})\hat{} = 0$ for every $i > 0$. It follows that the completion $(R\Gamma(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty})))\hat{} = (H^0)\hat{} = \mathbb{C}_p$.

The upshot of these two observations is that we get arrows

$$R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) \longrightarrow R\Gamma_{cont}(T_p(\mathcal{X}), R\Gamma(\mathcal{X}_\infty, \mathcal{O}_{\mathcal{X}_\infty}))\hat{} = R\Gamma_{cont}(T_p(\mathcal{X}_{\mathbb{C}_p}), \mathcal{O}_{\mathbb{C}_p}) =$$
$$= R\Gamma_{et}(\mathcal{X}_{\mathbb{C}_p}, \mathcal{O}_{\mathbb{C}_p})$$

(where we have used that the cohomology of an abelian variety with lisse coefficients (hence π_1-rep) is the same as the group cohomology $H^*(\pi_1, rep)$).

The composition of these arrows is the promised map! This ends our discussion of the geometric item (1) above.

ps **Remark 2.4.1.** *$\mathcal{X}_\infty$ is a concrete example of a perfectoid space.*

Before we discuss the arithmetic item (2) above, let us make

infmz **Remark 2.4.2.** *We have shown that there is the functor from abelian varieties to filtered vector spaces sending $A/\mathbb{C}_p$ to the H-T filtration given by $H^1(A, \mathcal{O}_A) \subseteq H^1_{et}(A, \mathbb{C}_p)$. This works in families and we have the following theorem of Scholze's: 1) there is a perfectoid space $A_g[p^\infty]$ such that $A_g[p^\infty](\mathbb{C}_p)$ is given by the pairs (A, ϕ), with $A/\mathbb{C}_p$ a PPAV and ϕ a symplectic iso $H^1(A, \mathbb{Z}_p) \cong \mathbb{Z}_p^{2g}$; 2) there is a H-T period map $\pi_{HT} : A_g[p^\infty] \to Grass(g, \mathbb{C}_p^{2g})$, sending (A, ϕ) to the filtration above; 3) π_{HT} is equivariant for the Hecke operators.*

OK, now let us discuss the arithmetic point (2). The goal is to construct, for an abelian scheme $A/\mathbb{Z}_p$, a map $H^0(A, \Omega^1_A) \otimes_{\mathbb{Z}_p} \mathbb{C}_p(-1) \to H^1_{et}(A_{\overline{K}}, \mathbb{C}_p)$. This is the same as a map into $Hom_{G_K}(T_p(A_{\overline{K}}), \mathbb{C}_p(1))$ (since T_p is like a π_1, this is reminiscent of the pairing of 1-forms and loops). We have the following

tmfo **Theorem 2.4.3. (Fontaine)** *$dlog : \mu_{p^\infty} \to \Omega^1_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p}$, $\alpha \to d\alpha/\alpha$ induces, by taking T_p and inverting p an iso*

$$\mathbb{C}_p(1) \cong T_p(\Omega^1_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p})[1/p]$$

(so that we can view Tate twists as differential forms).

Now, it suffices to construct $H^0(A, \Omega^1_A) \to Hom(A[p^\infty](\overline{K}), \Omega^1_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p})$. Apply $T_p[1/p]$, get $Hom(T_p(A_{\overline{K}}, \mathbb{C}_p(1))$. So our map looks like this: $\Omega \in H^0(A, \Omega^1_A)$ goes to the arrow sending $(x \in Spec\overline{\mathbb{Z}_p} \to A)$ to $x^*\omega \in \Omega^1_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p})$. Then you check this is the inverse to the previous construction.

le3

3. Lecture III

The goal in the last two lectures is to explain the de Rham comparison theorem. It is originally due to Faltings. We follow Beilinson's 2012 ca. approach.

Today we introduce the period ring which appears in the comparison theorem

3.1. **The de Rham comparison theorem over $\mathbb{C}$.**

Let us first review the classical situation over $\mathbb{C}$, in a way that is more suited to the p-adic context.

The complex Ω gives rise to a contravariant functor (presheaf) $\{\text{complex manifolds}\}^{op} \to D(Vect)$, $X \mapsto R\Gamma(X, \Omega_X)$.

shdr **Theorem 3.1.1.** *The natural morphism $\mathbb{C}_X \to \Omega_X$ becomes an iso after sheafification.*

(Evaluate at balls, use the Poincaré Lemma).

Today we discuss something similar, but over $\mathbb{Q}_p$! In order to do so, we need about twenty minutes of commutative algebra.

derco

3.2. Derived completions.

Given a group, we can form its p-adic completion, which involves taking (inverse) limits.

defderco

Definition 3.2.1. *Let $K \in D(Ab)$. Set*

$$\hat{K} := \textit{derived product completion} := R\lim_n \left(K \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^n\right) \in D(\mathbb{Z}_p).$$

excvb

Example 3.2.2.

(1) *$K = M[0]$ (single entry in degree zero). Then $\hat{K} = \hat{M}[0] = \lim_n M/p^n$. (He made a statement about the torsion being bounded so that it does not matter when taking $R\lim_n \ldots$)*

(2) *If each $H^i(K)$ is uniquely p-divisible (i.e. a $\mathbb{Z}[1/p]$-module), then the transition function $p^n : M \to M$ are iso and $\hat{K} = 0$ (reminiscent of the vanishing observed for abelian schemes in Lecture II).*

(3) *If $K = M[0]$ with M p-divisible (there may be p-torsion, but everything is divisible by p, e.g. $\mathbb{Q}_p\mathbb{Z}_p$), then $M/P^n = 0$, so that $M \otimes_{\mathbb{Z}}^L \mathbb{Z}/p^n =$ (since the transition functions are $p^n : M \to M$) $= (M[p^n])[1]$ (N.B.: $M[p^n]$ is the kernel, i.e. the p^n-torsion). Then $\hat{K} = \lim_n (M[p^n])[1]$ (transition functions are $\cdot p$, they are surjective, so there is no higher derived limits).*

As a special case, $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} = \mathbb{Z}_p[1]$.

(4) *$M := \mu_{p^\infty} \subseteq \overline{K}^*$. This is isomorphic to the first example:*

$$\hat{M} = \mathbb{Z}_p(1)[1] = T_p(\mu_{p^\infty})[1].$$

(5) *We saw Fontaine's theorem in Lecture II. The set-up was the (vertical) tower of extensions*

$$\begin{array}{ccc} \mathbb{C}_p = \overline{K} & \supseteq & \mathcal{O}_{\mathbb{C}_p} \\ \overline{K} & \supseteq & \mathcal{O}_{\overline{K}} = \overline{\mathbb{Z}_p} \\ K & \supseteq & \mathcal{O}_K \\ \mathbb{Q}_p & \supseteq & \mathbb{Z}_p, \end{array}$$

where there is no ramification in $\overline{K}/\mathbb{Q}_p$, but there is plenty of ramification integrally in $\overline{\mathbb{Z}}_p/\mathbb{Z}_p$. Fontaine's is

$$T_p(\Omega^1_{\overline{\mathbb{Z}}_p/\mathbb{Z}_p}) = \mathcal{O}_{\mathbb{C}_p}(1)$$

(the argument on the lhs is p-divisible).

It was asked how the p-adic ramification compares with $\overline{\mathbb{Z}}/\mathbb{Z}$: the answer is that

$$(\Omega^1_{\overline{\mathbb{Z}}/\mathbb{Z}})_p = \widehat{\Omega^1_{\overline{\mathbb{Z}}_p/\mathbb{Z}_p}}$$

We have

$$\widehat{\Omega^1_{\overline{\mathbb{Z}}_p/\mathbb{Z}_p}} \cong_{\textit{small torsion}} \mathcal{O}_{\mathbb{C}}(1)[1] \tag{16}$$ ty6

(not sure the exact meaning of "small torsion":)

cotco

3.3. The cotangent complex.

The construction of the cotangent complex is due to Illusie after some ideas of Quillen. It is rather involved, but the properties are easy to work with.

Let $f : X \to S$ be a map of schemes. Then the cotangent complex $L_{X/S}$ is a complex in $D^{\leq 0}_{qc}((X)$ satisfying

(1) It lifts $\Omega^1_{X/S}$, i.e. $H^0(L_{X/S}) = \Omega^1_{X/S}$.
(2) if f is smooth, then $L_{X/S} = \Omega^1_{X/S}$.
(3) Important computational tool: if f factors $X \to P \to S$, with $P \to S$ smooth and $X \to P$ a regular embedding (lci), then (2-step complex of vector bundles!)

$$L_{X/S} = \left(I_X/I_X^2 \xrightarrow{d} \Omega^1_{P/S_{|X}} \right)$$

oq **Remark 3.3.1.** *There is a partial converse. Theorem (Avramov): $S = Speck$ and X/S of finite type. Then $L_{X/k}$ is bounded iff X is lci.*

Example 3.3.2. *In our case, not so hard to show that $\mathcal{O}_K = \mathbb{Z}_p/f(x)$, so $\mathbb{Z}_p \to \mathcal{O}_K$ is an lci. Then we have*

$$L_{\mathcal{O}_K/\mathbb{Z}_p} \cong (f(x)/f(x)^2 \to \mathcal{O}_K dx) \cong (\mathcal{O}_K \to \mathcal{O}_K) \;\; 1 \mapsto f'(x).$$

Note that, by base change, $L_{\mathcal{O}_K/\mathbb{Z}_p}[1/p] = L_{K/rat_p} = 0$ (unramified field extension, char zero). Then $H^{-1}(L_{\mathcal{O}_K/\mathbb{Z}_p}) = 0$, so that $L_{\mathcal{O}_K/\mathbb{Z}_p} = \Omega^1_{\mathcal{O}_K/\mathbb{Z}_p}$. Pass to the limit ($\overline{\mathbb{Z}}_p$ is a union of finite extensions and the filtered colimit is exact) to get first lhs iso below

$$L_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p} \cong \Omega^1_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p} \cong \mathcal{O}_{\mathbb{C}_p}(1)[1]$$

where we have seen earlier the rhs iso above.

(4) The reason for $L_{X/S}$ is to study the deformation theory of $f : X \to S$: the ext-2 of it are the obstructions to extend to thickenings of S; the obstruction vanishing, there is an ext-1 worth of such extensions.

If $L_{X/S} = 0$, then a def of S gives a unique def of X.

prft **Example 3.3.3.** *Let $R/\mathbb{F}_p$ be perfect ($x \to x^p$ iso iff Frob auto iff exist unique p-th roots). Then $L_{X/S} = 0$. (Sketch of proof: for any R, frobenius induces the zero map $L_{R/\mathbb{F}_p} \to L_{R/\mathbb{F}_p}$; for R perfect this is an iso). This implies that every def of the finite field yields a unique def of R. In particular, take the second Witt vectors ...*

!!! At this point he mentioned that perfectoid spaces are modeled on affinoids, which are objects whore reduction mod p is perfect !!!

derdrco

3.4. **Derived de Rham cohomology.**

Let $f : X \to S$ be a map of schemes. Set

$$dR^H_{X/S} : Tot^{\Pi}(\mathcal{O} \to L_{X/S} \to \bigwedge^2 L_{X/S} \to \ldots) \in DF(X, f^{-1}\mathcal{O}_S).$$

(H is for Hodge; Tot^{Π} is for the product totalization of the complex –direct product instead of direct sum; there is a version w/out H and it the usual Tot; this is ok in char zero –only case we need today–; in pos. char need Dolde-Puppe resolutions; the filtration comes from the stupid filtration; the double complex is a second quadrant complex).

dfre **Remark 3.4.1.** (1) *By using* Π*-totalization, we get a spectral sequence*

$$E_1^{pq} = H^q(X, \bigwedge^p L_{X/S}) \Longrightarrow H^{p+q})X, dR^H_{X/S})$$

(w/ direct sum totalization total, there is no convergence).

(2) X/S *smooth implies* $dR^H_{X/S} = \Omega^{\bullet}_{X/S}$ *and the resulting filtration is the Hodge one (stupid).*

(3) *In char. zero, we can do better: take* X/K *a variety, then*

$$R\Gamma(X, dR^H) = R\Gamma(X^{an}, \mathbb{C}),$$

hence a filtration (finer than Deligne's Hodge filtration (cfr, one of Bhatt's papers).

(4) *There is a log-smooth variant of* $L_{X/S}$ *and* $dR^H_{X/S}$ *due to Gabber and to Olsson.*

pr

3.5. **The period rings.**

We have

$$A_{dR} : \widehat{dR^H_{\mathbb{Z}_p/\mathbb{Z}_p}} \in DF(\mathbb{Z}_p); \tag{17}$$ pr1

$$B^+_{dR} : A_{dR}[1/p] \in DF(\mathbb{Q}_p); \tag{18}$$ pr2

(the $[1/p]$ is in the derived sense ????; also, I wrote: A_{dR} is a commutative ring ????).

We have (derived completion and wedge commute) (also, nice exercise: $\wedge^i(V[i]) = Sym^i(V)[\ldots]$) ($* =^*$ refers to "up to small torsion", whatever that means) Γ^i are the divided symmetric powers) (the last terms is all in degree zero!)

$$Gr^H_i(A_{dR}) = \bigwedge^i \widehat{L_{\mathbb{Z}_p/\mathbb{Z}_p}}[-i]^* =^* \bigwedge^i \left(\mathcal{O}_{\mathbb{C}_p}(1)[1]\right)[-i] = \left(\Gamma^i\left(\mathcal{O}_{\mathbb{C}_p}(1)\right)[1]\right)[-i]^* =^* \mathcal{O}_{\mathbb{C}_p}(i)[0].$$

Then (B_HT is a direct sum over all $\mathbb{Z}$ and has p-inverted)

$$gr^H_* A_{dR} = \bigoplus_{i \geq 0} \mathcal{O}_{\mathbb{C}_p}(i) \text{``}=\text{''} B_{HT},$$

so that

$$gr^H_* B^+_{dR} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i).$$

The ring B^+_{dR} is a complete DVR w/residue field $\mathbb{C}_p$ and $m/m^2 \cong \mathbb{C}_p(1)$.

Define

$$B_{dR} := Frac(B^+_{dR}). \tag{19}$$ pr3

This is the most basic period ring. It is a filtered field whose gr is $B_{HT} = \oplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$.

Any DVR has a canonical log structure (the monoid is obtained by consideration of the invertible functions on open set obtained by removing the special point/fiber; the trivial log structure is instead when you take the invertible functions on the whole thing):

$$M: \quad \overline{\mathbb{Z}_p} \setminus \{0\} \xrightarrow{\alpha} \overline{\mathbb{Z}_p}.$$

N.B.: $M/\overline{\mathbb{Z}_p}^*$ is uniquely p-divisible, hence $\widehat{(-)^{grp}} = 0$, so that $\widehat{(M/\overline{\mathbb{Z}_p}^*)^{grp}} = 0$, so that: do not understand what I wrote:

$$\widehat{L_{\overline{\mathbb{Z}_p}/\mathbb{Z}_p}}, \quad \widehat{L_{\overline{\mathbb{Z}_p},M}/\mathbb{Z}_p}$$

and the same for dR cohomology.

le4

4. Lecture IV