Lectures on perverse sheaves and decomposition theorem

October 28, 2011

Abstract

- This a preliminary version. It is certainly full of typos and mistakes. I am not sure if/when there will be a substantial revision. The lectures will cover a strictly smaller subset of these notes, which can be used to start digging deeper into the material. There are several exercises. They are meant to be tackled by using the material that preceds them.
- The goal of the lectures is to introduce non experts to perverse sheaves and to a crowning achievement of this theory: the decomposition theorem concerning the homology of proper maps of complex algebraic varieties. Since derived categories are a necessary tool, some effort is made to introduce the reader to them by introducing the various concepts and tools in a concrete settings. I have chosen two such setting to be the the Leray-Hirsch theorem and the contraction of curves on surfaces. The former is an illustration of the notion of splitting in the derived cateogry (that is what the DT is), the latter yields very naturally to perverse sheaves and to intersection complexes (the building blocks of perverse sheaves).
- Here is list of the principal items to be covered (non linearly):
  1. Leray-Hirsch, contracting curves on surfaces.
  2. Review of derived categories and associated functors.
  3. Deligne thoerem for smooth projective maps.
  4. Pervers sheaves.
  5. Decomposition and relative hard Lefschetz theorem.
  6. Example of application of DT and RHL.

Contents

1 Contracting curves on surfaces and Leray-Hirsch

The two topics are un-related, except for the fact that we view them as an exercise in trying to extract topological information on the domain of a map in terms of the topology of the base and of some features of the map.
In this section we carry out this study by only using basic algebraic topology, but being careful to lay some grounds for a more sheaf-theoretic approach.
Our goal is to explain the statement of the decomposition theorem. which is best understood as a statement in the derived category.
This theorem can be viewed as a generalization to singular complex algebraic maps of a theroem of Deligne concerning smooth projective maps of complex quasi projective varieties. This result is also best understood in the derived category.
We shall introduce the language and some of the properties of derived categories by revisiting and re-proving the Leray-Hirsch theorem as well as the examples of contractions of curves on surfaces.
Once that is done, we discuss and prove Deligne theorem by introducing more properties of the derived category.

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Unless otherwise stated, we use rational cohomology.

1.0.1 Contractions of curves on complex surfaces

1.0.2 Four examples to keep in mind

Let $f : (X, E) \to (Y, v)$ be the contraction of a configuration of compact complex curves $E = \cup_j E_j$ on a complex algebraic surface X,
i.e. $Y = X/E$, f is the quotient map and $v = f(E)$.

The map $f : (X, E) \to (Y, v)$ is a proper map of $\mathbb{R}$-varieties (of which we are only considering the real points) of real dimension 4 (Exercise 1.0.2.1).

The map f is holomorphic IFF $||E_j \cdot E_k|| < 0$ (Grauert).

We have the following 4 key examples in mind.
$(|L|, C)$: $|L|$ the total space of a holomorphic line bundle L on a complex nonsingular projective curve C identified with the zero section. Let $o \in C$ be a point.

1. $(X, E) = (|L|, C)$, with $L =$ trivial.
2. $(X, E) = (|L|, C)$, with $L < 0$
3. (X, E) the blow-up of $(|L|, C)$, $L =$ trivial, $E = \widehat{E} \cup \mathcal{E}$,
   ($\widehat{E}$ the strict transform of C and $\mathcal{E}$ the exceptional divisor of the blow up).

4. (X, E) as above, but $L < 0$.

In Examples 1 and 3, the space Y is not a complex surface.
In Examples 2 and 4, f is a complex algebraic map.

Exercise 1.0.2.1 (**Contractions in real algebraic geometry**) A textbook reference is [1], Proposition 3.5.6.

1. Prove that $\mathbb{P}^n_{\mathbb{R}}$ is affine. Hint:

$$(x_0 : \ldots : x_n) \longmapsto \left(\frac{x_j x_k}{||x||^2}\right) \in M_{(n+1)\times(n+1)}(\mathbb{R}) = \mathbb{R}^{(n+1)^2}.$$

   Deduce tht real projective varieties are affine.

2. Find a closed embedding of real algebraic varieties

$$\mathbb{P}^n_{\mathbb{C}} \to \mathbb{R}^?.$$

   Hint: See hint above and modify it by changing x_k to $\overline{x_k}$ and landing in $\mathbb{R}^{2(n+1)^2}$.

3. Let X be real affine algebraic (e.g. projective!), let $\emptyset \neq Y \subseteq X$ be a closed algebraic subset. There is a map $\Phi : X \to Z$ of real algebraic varieties where Z is affine, Φ contracts Y to a point $z \in Z$ and it is otherwise a biregular isomorphism.

   The exercise is to fill-in the details of the following sketch of proof.

   We may assume that $Y \subseteq X \subseteq \mathbb{R}^n$.

   Let $P(x)$ be a real polynomial vanishing precisely at Y (sum of squares of generators of the ideal of X).

   Then $X = \coprod_{t\in\mathbb{R}} X_t$ (X_t the level “hypersurfaces” for P).

   Set $X' = \{(x,t) \mid tP(x) = 1\} \subseteq X \times \mathbb{R}$ (with $\pi : X' \to \mathbb{R}$ the projection).

   We have X_t“ = ”$X'_{1/t} := \pi^{-1}(1/t)$, for all $t \neq 0$.

   We lost $Y = X_0$ which has been sent to infinity in the sense that the X_t“ = ”$X'_{1/t} \subseteq \mathbb{R}^n \times \{1/t\}$ are getting further and further away from the origin of $\mathbb{R}^{n+1}$ as $t \to 0$.

   Take the inversion involution $\imath : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$, $u \mapsto u/||u||^2$.

   This way, the $X'_{1/t}$ are getting closer and closer to the origin as $t \to 0$.

The set $Z := \mathfrak{i}(X') \coprod \{0\} \subseteq \mathbb{R}^{n+1}$ is algebraic and closed ($\mathfrak{i}$ is an algebraic iso, so $\mathfrak{i}(X')$ is algebraic and Z is its Zariski closure) (BTW: this is an instance of algebraic Kolmogorov one point compactification).

Set

$$\Phi(x) := \left(\frac{P^2(x)}{||x||^2 P^2(x) + 1}, \frac{P(x)}{||x||^2 P^2(x) + 1} \right).$$

This does the job. □

1.0.3 The long exact sequence of relative cohomology

Let us study the topology of the four examples in §1.0.1.

We set $U = X \setminus E =_f Y \setminus v$ and we have the open/closed embeddings:

$$U \xrightarrow{j} X \xleftarrow{i} E.$$

By Lefschetz duality on the oriented 4-fold X, we have that:

$$H^i X, U = H_{4-i} E.$$

Exercise 1.0.3.1 Lefschetz duality can also be expressed as $H_i X, X^* = H^{4-i} E$. In example §1.0.1.1, take $i = 2$ and view a disk $\Delta := D \times \{p\}$ as a cycle in $H_2 X, X^*$. Then view the same in $H^2 E = (H_2 E)^*$ as the map sending the fundamental class of E to the intersection number, here $+1$, of E with Δ.
Find the analogus picture for the remaining 3 examples.

In all 4 examples, X retracts onto E, so that

$$i^* : H^* X \cong H^* E.$$

There is the cycle class map $cl_E : H_2 E \to H^2 X$.

We have the following commuative diagram, where the horizontal string is the long exact sequence (les) of relative cohomology (modulo the Lefschetz duality identifications) and the map ι is defined by the commutativity of the diagram:

$$\begin{tikzcd}
 & & & H^2 X, U \arrow[d, "LD"', "="] & & H^3 X, U \arrow[d, "LD"', "="] & & \\
0 \arrow[r] & H^1 X \arrow[r, "j_1^*"] & H^1 U \arrow[r, "b_1"] & H_2 E \arrow[r, "cl_E"] \arrow[dr, "\iota_E"'] & H^2 X \arrow[r, "j_2^*"] \arrow[d, "i^*"', "\cong"] & H^2 U \arrow[r, "b_2"] & H_1 E \arrow[r] & 0 \\
 & & & & H^2 E. & & &
\end{tikzcd}$$

By exactness, we have
$$\operatorname{Coker} b_1 = \operatorname{Im} cl_E = \operatorname{Ker} j_2^*.$$
By splicing, we get the two es:

$$\begin{tikzcd}
0 \arrow[r] & H^1X \arrow[r, "j_1^*"] & H^1U \arrow[r, "b_1"] & H_2E \arrow[r, "epi", "cl_E"'] \arrow[dr, "\iota_E"'] & \operatorname{Im} cl_E \arrow[r] \arrow[d, "mono"] & 0 \\
 & & & & H^2E, &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im} cl_E \arrow[r, "\subseteq"] & H^2X \arrow[r, "j_2^*"] \arrow[d, "i^*"', "\cong"] & H^2U \arrow[r, "b_2"] & H_1E \arrow[r] & 0 \\
 & & H^2E. & & &
\end{tikzcd}$$

The following key remark puts the map cl_E in the center of the stage and sets apart, among the 4 examples, the complex algebraic examples as the ones for which cl_E is an iso.

Remark 1.0.3.2
What follows is clear in view of exactness and of $\dim H_2E = \dim H^2X = \dim H^2E$.

1. $cl_E =$ iso IFF $\iota_E =$ iso (obvious since ι is defined as above).
2. $cl_E =$ iso IFF $cl_E =$ mono IFF $cl_E =$ epi.
3. $cl_E =$ iso IFF $b_1 = 0$ IFF $j_1^* =$ iso IFF $j_s^* = 0$ IFF $b_2 =$ iso.
4. $cl_E = 0$ IFF $\operatorname{Im} cl_E = 0$ IFF $\iota_E = 0$ (in which case we get two obvious ses).

Fact 1.0.3.3 (**Borel-Moore homology and Poincaré duality** [2]) Recall that singular homology is the homology of the chain complex made of finite chains and that Borel-Moore homology is the homology of chains which are locally finite. There is a natural map $H_* \to H_*^{BM}$. There is a natural iso $H_i^{BM} = H_c^{i\vee}$. On an oriented manifold M, there is the natural intersection pairing

$$I_M \ : \ H_i \times H^{BM}_{\dim -i} \longrightarrow \mathbb{Q}$$

and Poincaré duality takes the following forms

$$H^i = (H_c^{\dim -i})^\vee, \qquad H^i \otimes H_c^{\dim -i} \xrightarrow{\int} \mathbb{Q} \qquad \text{non degenerate;}$$

$$H_i^\vee = H^i = H^{BM}_{\dim -i}, \qquad H_i \times H^{BM}_{\dim -i} \xrightarrow{I_M} \mathbb{Q} \qquad \text{non degenerate.}$$

Exercise 1.0.3.4 For the following oriented manifolds X first calculate all groups by finding explicit geometric cycles as generators in the homology-type groups and closed differential forms in the cohomology-type one and then use these generators to describe the intersection forms I_X:

1. $X = \mathbb{R}$;
2. $X = \mathbb{R}^2$;
3. $X = \mathbb{R}^2 \setminus \{(0,0)\}$;
4. $X = C \times \mathbb{C}$, the trival line bundle on the curve C;
5. $X = \widetilde{\mathbb{C}^2}$, the blowing-up of $\mathbb{C}^2$ at the origin. Let L be the strict transform of a line in $\mathbb{C}^2$ and let E be the exceptional divisor. We must have $L \sim aE$; determine a. Describe I_X by using E and L. Do the same by using only E. Describe cl_E by using E. Describe ι_E using E. Finally, view ι_E as a bilinear map $H_2E \times H_2E \to \mathbb{Q}$ by using E. How many ways do you know to interpret and prove the statement that $E^2 = -1$?
6. Do the same as above for all 4 examples in §1.0.1.

   (The example above is a special case of one of the four. Which one?)

Exercise 1.0.3.5

1. In the complex algebraic Examples 1.0.1.2 and 4, we have cl =iso.
2. In the non-holomorphic Examples 1.0.1.1 and 3, we have $cl \neq$ iso.

   In fact, we have that the rank is 0 and 1, respectively.

1.0.4 The topology of the two complex algebraic contractions

In this case, we know that cl = iso.
We thus get the following information

$$H_2E \xrightarrow[\cong]{\iota} H^2E; \qquad H^1X \xrightarrow[\cong]{j^*} H^1U; \qquad H_2E \xrightarrow[\cong]{cl} H^2X \xrightarrow{j^*=0} H^2U.$$

Preview:
as we shall see, this information can be written as as iso of graded vector spaces

$$HX = IHY \oplus H_2E[-2],$$

where IHY is the intersection cohomology module of Y.

1.0.5 The topology of the other two contractions

In this case, all we can do is simply re-write the two exact sequences in §1.0.3 as follows:

$$0 \longrightarrow H^1X \xrightarrow{j^*} H^1U \xrightarrow{b} \mathrm{Ker}\, cl \longrightarrow 0.$$

$$0 \longrightarrow \mathrm{Im}\, cl \xrightarrow{\subseteq} H^2X \xrightarrow{j^*} H^2U \xrightarrow{b} H_1E \longrightarrow 0.$$

This makes it clear that, since the rank of cl depends on the situation (Exercise 1.0.3.5.2), we will meet hurdles (i.e. non trivial extensions see **?????**) when trying to describe HX in terms of something on Y, e.g. IHY.

1.0.6 The refined intersection form

The map $cl_E : H_2E \to H^2X$ is the class map sending the fundamental class of E to the associated cohomology class, i.e. the Poincaré dual to the same fundamental cycle of E, but viewed on X:

$$cl_E(E_j) = I_X(E_j, *) \in (H_2X)^\vee = H^2X, \qquad * \in H_2X.$$

The map $\iota_E : H_2E \to H^2E$ is the compositum $i^* \circ cl_E$ is the first appearance in these lectures of a key player in the decomposition theorem, in fact in the basic theory of perverse sheaves:
it is an incarnation of the refined intersection form associated with $E \subseteq X$.
There are two other equivalent ways to view this map ι_E:
as the refined cup product in relative cohomology (first row)
and as the refined intersection product associated with the compact E on the smooth oriented X (second row)

$$\begin{tikzcd}
H^2(X,U) \times H^2(X,U) \arrow[r] \arrow[d, "LD\times LD"', "\cong"] & H^4(X,U) \arrow[d, "LD"', "\cong"] \\
H_2(E) \times H_2(E) \arrow[r] & H_0(E) = \mathbb{Q},
\end{tikzcd}$$

once we observe that bilinear maps on the bottom correspond to linear maps $H_2E \to H^2E$.
Note that while H^2E and H_2E are dual to each other,
there is no natural map between them until we view E as embedded in the oriented manifold X and we observe that E is compact.

1.0.7 Explicit description of the boundary and restriction maps

1. Le et us first deal with Example 1.0.1.1 with $C = \mathbb{P}^1_{\mathbb{C}}$.
   (The reader can deal with the case of arbitray genus for Examples 1.0.1.1 and 2 as an exercise.)
   We have

$$\begin{tikzcd} H^1U \arrow[r, "b", "\cong"'] & H_2E \arrow[d, "\iota_E=0"] \\ & H^2E, \end{tikzcd} \qquad \begin{tikzcd} H^2X \arrow[r, "j^*", "\cong"'] \arrow[d, "i^*"', "\cong"] & H^2U \\ H^2E. & \end{tikzcd}$$

   Due to the explicit and simple nature of the example, this is info we can see directly.
   Let us work this out.
   We have $(U \subseteq X) \;\; = \;\; (\mathbb{C}^* \times S^2 \subseteq \mathbb{C} \times S^2)$:

   - The map b is the transposed of the map (clearly an iso):

$$H_1U \xleftarrow{\cong} H_2(X,U): \qquad S^1 \times \{p\} \xleftarrow{\delta} D \times \{p\}.$$

     We can also view the map b directly via via Poincaré duality

$$H_*^{BM} \stackrel{PD}{=} H^{4-*}$$

     as follows:

$$H_3^{BM}U = H^1U \xrightarrow{b} H^2X, U$$

     where a generator for the lhs is $\mathbb{R}^{>0} \times S^2$ which is the restriction of $\mathbb{R}^{\geq 0} \times S^2$ and b sends said generator to the boundary of $\mathbb{R}^{\geq 0} \times S^2$ which is $0 \times S^2$ i.e. the fundamental class of E (via Lefschetz duality)
   - The map j^* is an iso by direct calculation involving Künneth:
     a generator for the lhs, i.e. a section $\mathbb{C} \times p$ of $\mathbb{C} \times \mathbb{P}^1_{\mathbb{C}}$, goes to
     a generator for the rhs, i.e. the section $\mathbb{C}^* \times p$ of $\mathbb{C}^* \times \mathbb{P}^1_{\mathbb{C}}$.
   - The map $cl_E = 0$ since the class of E in $H_2^{BM}X$ is trivial as it is the boundary of $\mathbb{R}^{\geq 0} \times S^2$.

2. Let us nw deal with the complex algebraic contractions.
   We look at the special case of Example §1.0.1.2 in the case of genus zero and $degL = -1$.
   (As before, the other cases are left as exercises).

   We have:

$$0 \cong H^1U \xrightarrow{b=0} H_2E, \quad H_2E \stackrel{\iota}{\cong} H^2E, \quad H^2X \xrightarrow{j^*=0} H^2U \cong 0.$$

Of course, we can work things out explicitly.

We have $U \sim_{htp} S^3$ and this gives the first two items below

- $b = 0$.
- $j^* = 0$.
- We have $cl_E =$ iso (standard fact about the blow up).

1.0.8 Compactifying the examples

We can compactify the examples in §1.0.1 by adding a copyC_∞ of C at infinity.
We obtain contractions

$$\overline{f} : (\overline{X}, E) \longrightarrow (\overline{Y}, v).$$

and what was complex algebraic is still complex algebraic and what was not so, it is still not so.
Note that this operation does not affect the map ι_E, for nothing happens around E.
On the other hand, something interesting happens to cl_E (which has the new target $H^2\overline{X}$):

1. In the two non complex algebraic examples compactified, we have:
   $cl_E =$ mono (unlike before the compactification),
   $\iota_E = 0$ (like before the compactification).
2. In the two complex algebraic examples compactified, we have:
   $cl_E =$ mono (like before) and
   $\iota_E =$ iso (like before).

Of course, in all the examples, ι_E cannot change after compactifying: it is computed in a neighborhood W of E in X.

- What is interesting, and ultimately a simple instance of the decomposition theorem for proper maps of complex algebraic varieties, is that in our two complex algebraic examples, the $\mathrm{rk}\, cl_E$ does not depend on W and

  $\mathrm{rk}\, cl_E$ *can be expressed in sheaf-theoretic terms locally near* v

  because it contributes the same amount to all $H^2 f^{-1}U$ (U neighborhoods of y)
- On the other hand, in the non complex algebraic examples, the rank of $cl_E : H_2E \to H^2 f^{-1}U$ changes with U, so that:

  $\mathrm{rk}\, cl_E$ *cannot be expressed in sheaf-theoretic terms locally near* v.

It is a bit premature now, but what I mean is that we have maps in the derived category, in fact of perverse sheaves (**????**):

$$(H_2E)_v \xrightarrow{cl_E} Rf_*\mathbb{Q}_X[2] \xrightarrow{p} (H^2E)_v$$

the composition of which is ι_E and that

- in the complex algebraic case it splits $H_2 E_v$ (and hence $H^2 E_v$) off $Rf_* \mathbb{Q}_X$
- in the non complex algebraic case it does not split off either summand and $H^2 E_v$ appears as a non split quotient of $Rf_* \mathbb{Q}_X$.

1.0.9 The Leray-Hirsch theorem

1.0.10 Statement of Leray-Hirsch

We state a version of this classical result in the context of smooth fiber bundles so that we can use differentail forms. Later, we revisit this approach and sheafify it ans use it as a working example to introduce derived cateogries, especially the notion of splitting of the derived direct image.
Let $f : M \to B$ be a C^∞ fiber bundle with fiber F of dimension l and let, for $b \in B$, $M_b := f^{-1}b$.
The cohomology $H^*M, \mathbb{R}$ is an $H^*B, \mathbb{R}$-module via $f^*(-) \cup (-)$.
Let $\{\alpha_{ij}\}$, $0 \leq i \leq l$, $j \in J_i$, be a collection of cohomology classes in $H^iM, \mathbb{R}$ with the following property: for every i, the classes $\alpha_{ij|M_b}$, $j \in J_i$ form a basis for $H^iM_b, \mathbb{R}$.

Theorem 1.0.10.1 *The H^*B-module structure on H^*M is free with basis the α_{ij}'s, i.e.:*

$$A \,:\, \bigoplus_{0 \leq i \leq l} H^*(B, \mathbb{R})^{J_i}[-i] \xrightarrow{\sum_{i,j} f^*\alpha_{ij} \cup -} H^*(M, \mathbb{R})$$

is an iso.

1.0.11 Leray-Hirsch via Mayer-Vietoris

A standard reference for this approach is [3].
The proof below looks a bit different. This is because we later turn it into a proof of a splitting in the derived cateogry.
We use the de Rham model for real cohomology cohomology:

$$H^iM = H^i(\Gamma(M, E_M) = \frac{\text{closed } i\text{-forms}}{\text{exact } i\text{-forms}},$$

where E_M is the sheafified de Rham complex and the rhs is the cohomology of the complex of global sections on M (M can be replaced by any of its open subsets).
Choose closed representatives $a_{ij} \in \Gamma(M, E^i_M)$ for each of the α_{ij}.
Define a map of complexes

$$A \,:\, \bigoplus_i \Gamma(B, E_B)^{J_i}[-i] \xrightarrow{f^*a_{ij} \wedge -} \Gamma(M, E_M).$$

Recall that if if $C = (C^l, d^l)$ is a complex, then $C[k]$ is the complex with $C[k]^l := C^{k+l}$ and $d[k]^l = (-1)^k d^l$ (visually: if $k > 0$ you translate it back by k units).
Note that the above is a map of complexes precisely because the forms a_{ij} are closed.
Note also that if we change the representatives a_{ij}, then the new map A' is homotopic to A and therefore induces the same map on the cohomology of the two sides.
Recall that $f, g : C \to D$ are homotopic, if there is a collection $t_i : C^i \to D^{i-i}$ with $f - g = t \circ d + d \circ t$.
The cohomology of the complex domain of the map A is a free module over H^*B of the type predicted by Leray-Hirsch.
Clearly, the two complexes are not isomorphic (in general).
The proof of the Leray-Hirsch theorem we have in mind, consists of showing that A induces an iso on the cohomologies of the two complexes, i.e. that A is a quasi-isomorphism (qis).

Let us now sketch the Mayer-Vietoris-type argument.
Cover B with a good covering (U_λ), i.e. U_λ and $U_\lambda \cap U_\mu$ diffeomorphic to $\mathbb{R}^{\dim B}$.
For simplicity, assume the covering is finite.
Let $M_\lambda := f^{-1} U_\lambda$.
We have a commutative diagram of ses of complexes (let us omit some decorations) (the exactness is an argument with partitions of unity for the difference maps on the rhs).

$$\begin{tikzcd}
0 \arrow[r] & \oplus_i \Gamma(B_\lambda \cup B_\mu)^{J_i}[-i] \arrow[r] \arrow[d] & \oplus_i (\Gamma(B_\lambda) \oplus \Gamma(B_\mu))^{J_i}[-i] \arrow[r] \arrow[d] & \oplus_i \Gamma(B_\lambda \cap B_\mu)^{J_i}[-i] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(M_\lambda \cup M_\mu) \arrow[r] & \Gamma(M_\lambda) \oplus \Gamma(M_\mu) \arrow[r] & \Gamma(M_\lambda \cap M_\mu) \arrow[r] & 0
\end{tikzcd}$$

The hypothesis of Leray-Hirsch imply the conclusion for the (necessarily) trivial bundles over B_λ and $B_\lambda \cap B_\mu$.
The map of les and the five lemma imply the conclusion for the bundle over $B_\lambda \cup B_\mu$.
We conclude by induction:

$$B_1 \subseteq (B_1 \cup B_2) \ldots \subseteq (B_1 \cup B_2 \cup \ldots \cup B_N) = B.$$

2 Leray-Hirsch re-visited and the language of derived categories

2.1 Summary

This section is a kind of warm-up in view of Deligne theorem on smooth projective maps (§3). This theorem can be considered as the precursor to the decomposition theorem which is a about a splitting taking place in the derived category.

We introduce some of the necessary language and, in order to be a bit concrete, we state and prove the Leray-Hirsch theorem and the Künneth formula as splittings in the derived category. We first do so rather directly, using differential forms, and then we do it again using some of the language of derived categories.

The outline of this section is as follows. A review of sheaf cohomology using soft sheaves, Weil's proof of the de Rham theorem, push-forward of complex of differential forms, Leray-Hirsch and Künneth, review of derived categories and derived functors, Rf_* ????, re-formulation of Leray-Hirsch in the derived category.

2.2 Mini-review: cohomology groups and direct image sheaves

A standard reference is [4]. A short one is [5] (uses soft sheaves). Others: [6, 7, 8, 2].

We work with sheaves of Abelian groups on a topological space T.
Let F be a sheaf T.

Complexes $\ldots \to C^{i-1} \to C^i \to C^{i+1} \to \ldots$ are often denoted by C.
As we have seen earlier, they can be shifted $C[k]$.
An object F can be promoted to a complex with 0-th entry F and thus shifted, $F[k]$.
A quasi-isomorphism (qis) is a map of complexes inducing iso on cohomology.

There is the Godement resoulution of F:

$$0 \longrightarrow F \xrightarrow{\epsilon} G^0(F) \longrightarrow G^2(F) \longrightarrow G^2F \longrightarrow \ldots$$

which is a les of sheaves constructed canonically starting with F.
This construction is functorial.
It is often convenient to look at $F \to G(F)$ as qis of complexes.
Define the cohomology groups of T with coefficients in F by setting:

$$H^i(T, F) := H^i(\Gamma(T, G(F)) \quad \text{(cohomology of the complex of global sections).}$$

Each $G^i(F)$ is a flabby sheaf (sections extend from open subsets).

The flabbiness, implies that the functor $F \mapsto G(F)$ is exact (ses of sheaves $\mapsto$ ses of complexes).
If follows that, given a ses $0 \to F' \to F \to F'' \to 0$, we get the ses of complexes involving the $\Gamma(T, G(-))$ and the usual snake lemma yields the funcotrial les:

$$\ldots \longrightarrow H^i(T, F') \longrightarrow H^i(T, F) \longrightarrow H^i(T, F'') \longrightarrow H^{i+1}(T, F') \longrightarrow \ldots.$$

A flabby sheaf F is $\Gamma(U, -)$-acyclic, i.e. for every open set $U \subseteq T$, we have $H^{>0}(U, F) = 0$ (that is because a bounded below les of flabby sheaves yields a les of global sections).
Given any $\Gamma(T, -)$-acyclic resolution $a : F \to A$ (a is a qis, A^i are $\Gamma(T, -)$-acyclic), there is a canonical iso

$$H^i(\Gamma(T, A)) \xrightarrow{=} H^i(T, F)$$

obtained by composing the natural maps below:
(all isos by acyclicity of the A's) ($K^i := \operatorname{Ker} A^i \to A^{i+1}$):

$$H^i(\Gamma(T, A)) = \frac{\Gamma(T, K^i)}{\operatorname{Im} \Gamma(T, A^{i-1})} \overset{\partial\cong}{\to} H^1(T, K^{i-1}) \overset{\partial\cong}{\to} H^1(T, K^{i-1}) \overset{\partial\cong}{\to} \ldots \overset{\partial\cong}{\to} H^i(T, K^0 = F).$$

(N.b.: if we take the Godement resolution it is a fundamental (and non-tautological) fact that the resulting map is the identity.)
This construction tells that cohomology can be computed canonically using *any* $\Gamma(T, -)$-acyclic resolution.

Example 2.2.0.1 By the Poincaré lemma, the resolution $\mathbb{R}_M \to E_M$ of the constant sheaf on a smooth manifold M via the sheafification of the de Rham complex is a $\Gamma(M, -)$-acyclic resolution which is not a flabby resolution. In fact, it is a $\Gamma(U, -)$-acyclic resolution for every open $U \subseteq M$.

Example 2.2.0.2 The complex of singular cochains C_T on a metrizable topological space T is a flabby resolution of the sheaf $\mathbb{Z}_T$. One can see that there are canonical identifications:

$$H^i_{\text{sing}}(T, \mathbb{Z}) = H^i(\Gamma(T, C_T)) = H^i(T, \mathbb{Z}_T)$$

(caution: the first equality is not tautological! [9], p.26), and similarly, for the differentiable cochains. This is important as it tells us that sheaf cohomology computes singular cohomology. Similarly, for the differentiable cochains. A picky remark: if T is only paracompact, then the complex of singualar cochains is only soft (soft = sections lift from closed subsets), but since soft implies $\Gamma(T, -)$-acyclic it is still ok. I do not know much about this is T is not paracompact.

Let $f : T \to S$ be a continuous map.
The direct image pre-sheaf below is a sheaf (clearly flabby, if F is flabby):

$$f_*F : U \longmapsto F(f^{-1}U).$$

We can repeat all of the above by replacing the left-exact $\Gamma(T, -)$ with the left-exact $f_*(-)$ and obtain the complex $f_*G(F)$ on S and the canonical direct image sheaves on S

$$R^i f_* F := H^i(f_*G(F)).$$

They can also be computed using $f_*(-)$-acyclic resolutions (e.g. flabby, or, if T paracompact, soft).

Exercise 2.2.0.3 Prove that the sheaf $R^i f_* F$ above is canonically isomrophic to the sheaf associated with the pre-sheaf

$$U \longmapsto H^i(U, F).$$

Find many examples of this presheaf not being a sheaf for $i > 0$.

Example 2.2.0.4 Let $f : M \to S$ be a continuous map, with M a manifold. The sheaves E^i_M of i-differential forms are soft (sections lift from closed subsets) on every open subset $U \subseteq M$. It follows that they are $\Gamma(U, -)$-acyclic for every U and thus that they are $f_*(-)$-acyclic.

All of the above works with a bounded below ($C^i = 0$, $\forall i \ll 0$) complex of sheaves C replacing F: we get the groups $H^i(T, C)$ and the sheaves $R^i f_* C$ on S.
In this case $G(C)$ is the single complex of the double complex $G^i(C^j)$.

Exercise 2.2.0.5 Prove that the cohomology sheaf $H^i C$ of a complex of sheaves is the sheaf associated with the pre-sheaf

$$U \longmapsto H^i(U, C).$$

Find many examples of this presheaf not being a sheaf.

Remark 2.2.0.6 We showed that cohomology and direct image sheaves are definied up to canonical iso, independently of the chosen $\Gamma =$ or f_*-acyclic resolutions. If $C \to A$ is a $\Gamma(T, -)$-acyclic ($f_*(-)$-acyclic, resp.) resolution, then we obtain the complex $\Gamma(T, A)$ (f_*A, resp.). If we have two resolutions the resulting complexes are not iso as complexes. In fact they are not even qis. In order to state that these complexes are well-defined up to canonical iso, we need the language of derived categories (where qis=iso by decree). We come back to this point later.

2.3 Leray-Hirsch via sheaf cohomology

2.3.1 The complex E_X: Weil's proof of de Rham's theorem

Let M be a C^∞ m-manifold.
Weil's proof of de Rham's theorem goes as follows ([5]):

- One proves (Exercise 2.2.0.2) that $H^i_{\text{sing}}(M, \mathbb{R}) = H^i(M, \mathbb{R}_M)$, where the lhs is singular cohomology and the rhs is sheaf cohomology.
- One sheafifies the de Rham complex and obtains the complex $E_M = (E^\bullet_M, d)$, where E^i_M is the sheaf of smooth $\mathbb{R}$-valued differential i-forms on M, and the differential is the sheafified exterior derivation.
- One forms the de Rham complex of sheaves (2.2.0.1):

$$0 \longrightarrow \mathbb{R}_M \longrightarrow E^0_M \xrightarrow{d^0} E^1_M \xrightarrow{d^1} \ldots \xrightarrow{d^{m-1}} E^m_M \longrightarrow 0$$

  which is a soft, hence $\Gamma(U, -)$-acyclic, resolution of $\mathbb{R}_M$, for every U open in M: i.e.

$$H^q(U, E^i_U) = 0, \quad \forall i, \ \forall q > 0.$$

- By splicing the resolution into ses, Weil deduces, by what is now a classical inductive argument (§2.2), the de Rham's theorem:

$$H^i_{\text{sing}}(M, \mathbb{R}) = H^i(M, \mathbb{R}_M) = H^i(\Gamma(M, E_M), d) = \frac{\text{global closed } i\text{-forms}}{\text{global exact } i\text{-forms}}.$$

2.3.2 The complex f_*E_M and the sheaves $H^i f_* E_M$

Let $f : M \to B$ be a map of C^∞-manifolds.
If F is soft, then f_*F is soft.
The pushed-forward complex on B:

$$f_*E_M := \left[0 \to f_*E^0_M \stackrel{f_*d^0}{\to} f_*E^1_M \stackrel{f_*d^1}{\to} \ldots \stackrel{f_*d^{m-1}}{\to} f_*E^m_M, \to 0\right]$$

is a complex of soft sheaves, therefore we have canonical identifications:

$$H^i(B, f_*E_X) = H^i(\Gamma(B, f_*E_M)) = H^i(\Gamma(M, E_M)) = H^i(M, E_M) = H^i(M, \mathbb{R}_M).$$

i.e. *the complex of global sections of f_*E_M on B computes the cohomology of M.*
The usual pull-back in cohomology $f^* : H^i(B, \mathbb{R}) \to H^*(M, \mathbb{R})$ can be viewed as follows:

take the natural pull-back map of complexes

$$E_B \longrightarrow f_* E_M, \qquad \omega \longmapsto f^* \omega$$

and take the map induced on the cohomology of the complexes of global sections. Note that the pull-back in cohomology is rarely injective.
Since soft sheaves are $f_*(-)$-acyclic, we can use $f_* E_M$ to compute $R^i f_* \mathbb{R}_M$ and we have canonical isos:

$$H^i f_* E_M = R^i f_* \mathbb{R}_M, \qquad \forall\, i.$$

Remark 2.3.2.1 (**Maps induced by cohomology classes on** M)

1. Let $u \in \Gamma(M, E_M^k)$ be a *closed* k-form on M. The cup product map with u defines a map of complexes

$$u : E_M \longrightarrow E_M[k], \quad a \longmapsto u \wedge a,$$

   where $E_M[k]$ is the complex E_M moved to the left k steps (so that E_M^i is aligned with E_M^{i+k}).
   The induced map in cohomology is of course the cup product map with $[u]$.
2. Similarly, we have

$$u : f_* E_M \longrightarrow f_* E_M[k],$$

   inducing the same map as above in cohomology.
   It also induces the cup product map on the cohomology sheaves:

$$[u] : H^i f_* E_M \longrightarrow H^{i+k} f_* E_M.$$

   This map is important in the context of the relative hard Lefscehtz theorem.
3. We also have the map ($E_B \to f_* E_M$ seen above is a special case):

$$u : E_B \longrightarrow f_* E_M[k], \qquad b \longmapsto u \wedge b$$

   inducing $b \mapsto [u] \cup f^* b$ is cohomology ($H(M, \mathbb{R}_M)$ is an $H(B, \mathbb{R}_B)$-module).
   This plays an important role in the Leray-Hirsch theorem.

Remark 2.3.2.2 Recall the notion of homootpy in §1.0.11): a map of complexes $f : C \to D$ is homotopic to zero if there are $t^i : C^i \to D^{i-1}$ s.t. $f^i = t \circ d_C + d_D \circ t$. Two homotopic maps induce the same map in cohomology. What happens if we take $u' = u + dv$ in what above? Easy, we get that the two resulting maps u and u' are homotopic to each other (i.e. $u' - u$ is homotopic to zero).

2.3.3 Leray-Hirsch via sheaf cohomology

Let $f : M \to B$ be a fiber bundle with typical fiber $M_b = F$ of dimension l.
Let

$$\{a_{ij}\} \in \Gamma(M, E_M^i), \quad 0 \leq i \leq l, \quad j \in J_i$$

be a collection of closed forms on M.
Each closed form a_{ij} gives rise (see Remark 2.3.2.1, part 3) to a map

$$\mathscr{A}_{ij} \, : \, \mathbb{R}_B \xrightarrow[qis]{\epsilon_B} E_B \xrightarrow{a_{ij} \wedge f^* -} f_* E_M[i].$$

We shift each such map forward by i steps, and assemble all the maps into a single map of complexes and obtain: (denote the sum of the ϵ_B by $\widetilde{\epsilon_B}$)

$$\mathscr{A} = \textstyle\sum_{i,j} \mathscr{A}_{ij} \; : \qquad \bigoplus_{i \geq 0} \mathbb{R}_B^{J_i}[-i] \xrightarrow[qis]{\widetilde{\epsilon_B}} \bigoplus_{i \geq 0} E_B^{J_i}[-i] \xrightarrow[?qis?]{\sum a_{ij} \wedge f^* -} f_* E_M.$$

Up to homotopy, the map $\mathscr{A}$ depends only on the cohomology classes $[a_{ij}]$ (Remark 2.3.2.2).

It is clear that in order to use the map $\mathscr{A}$ to prove the Leray-Hirsch theorem, we need to prove that $\mathscr{A}$ is a soft resolution.
The map $\widetilde{\epsilon_B}$ is a soft resolution of its source.
The target of $\mathscr{A}$ is a soft complex.
We are left with showing that $\sum_{ij} \mathscr{A}_{ij}$ is a qis.
This can be verified locally on B, i.e. by looking at maps induced on the cohomology sheaves.
Take

$$a_{ij} \wedge f^* - : E_B[-i] \longrightarrow f_* E_M.$$

It induces a map of i-th cohomology sheaves:

$$[a_{ij}] \, : \, \mathbb{R}_B \longrightarrow H^i f_* E_M.$$

Exercise 2.3.3.1 Verify that the i-th cohomology sheaf $H^i(f_* E_M)$ is a sheaf with stalk at $b \in B$ canonically isomorphic to the de Rham cohomology of the corresponding fiber M_b:

$$H^i(f_* E_M)_b = H^i(f^{-1}(U_b), \mathbb{R}) = H^i(M_b, \mathbb{R})$$

(here $U_b \subseteq B$ is any contractible open neighborhood of $b \in B$).

According to Exercise 2.3.3.1, at the level of stalks at $b \in B$, the map of cohomology sheaves $[a_{ij}]$ sends $1_b \to [a_{ij}]_{|M_b} \in H^i(M_b, \mathbb{R})$.
Since we are assuming that these classes form a basis, we have reached the desired conclusion: $\mathscr{A}$ is a soft resolution.
Clearly, the map induced by $\mathscr{A}$ in cohomology is the Leray-Hirsch map associated with the classes $[a_{ij}]$ and we are done.

We have proved something a little stronger that is quite important in view of the subjects of these lectures:
we have showed that, up to a qis, the complex f_*E_X splits into a direct sum of complexes of type $\mathbb{R}_B[-i]$.
In the language of derived categories, we have proved that $Rf_*\mathbb{R}_M$ is isomorphic in the derived category to the direct sum of its shifted cohomology sheaves, all of which are constant sheaves

$$Rf_*\mathbb{R}_M \cong \bigoplus_{i\geq 0} R^if_*\mathbb{R}_M[-i].$$

Exercise 2.3.3.2 Let $f : M = B \times F \to B$ be the projection. Show that we do not need to make any choice of generating classes and can obtain a canonical qis

$$\mathscr{K} \ : \ \bigoplus_{i\geq 0} \underline{H^i(F)}[-i] \longrightarrow f_*E_M.$$

where $\underline{H^i(F)}$ denotes the constant sheaf with stalk $H^i(F, \mathbb{R})$.
Deduce the classical Künneth formula.

2.4 Derived Leray-Hirsch

2.4.1 Mini-review: derived categories and functors

An excellent reference is [10].
First a few words. Given a complex C, we have constructed different qis $\epsilon_i : C \to D_i$. We owuld like a framework in which it is meaningful to invert the qis, so that they become iso and we can happily state that $D_i \cong D_j$. This can be done, almost trivially, by means of an elementary universal construction. The problem is that one looses control of the calculus of the arrows. We do not only need the qis to be iso, we need some important ses of complexes to retain enough of their exactness in the new category so that they stay useful and possible to work with. This is unclear, at best, if we use that universal construction. One really needs to have a triangulated categoriy (where a ses has become something retaining the essence of the exactness). One construction that achieves this is the homotopy category

of complexes. However, the qis are still not iso. The miracolous fact is that if we perform on the homotopy category the inversion of the qis construction, the calculus of the arrows is greatly simplified, the result is equivalent to the qis-to-iso construction performed on the cateogry of complexes that resulted in the "bad" arrows, and is still triangulated! This is funny: qis-to-iso on complexes forces a factorization via the homotopy category.
The resulting cateogry is called the derived category. It is the natural framework for defining derived functors togehter with their universal proeprty. Many of the classical objects of algebraic topology can be interpreted via derived funcotrs, many of the classical operations between them can be handled efficiently via the arrows of the derived cateogry, and many of the classical long exact sequences are now encoded into the triangles that give the name to a triangulated category.
Verdier, whose thesis under Grothendieck's guidance, introduced this concept, used to say (more or less) that they are a paradise where dozens of identities make for an efficient handling of many complicated concepts. The introduction to his thesis makes this case very clearly for the iterated change of coefficients "formulæ" when dealing with complexes.

Let $\mathcal{A}$ be an Abelian category and $\mathcal{C}(\mathcal{A})$ be the category of complexes.
The derived category $\mathcal{D}(\mathcal{A})$ is obtained from the category of complexes $\mathcal{C}(\mathcal{A})$ by imposing that qis become isos. There is a canonical way to do this in a very general setting. Unfortunately, the ensuing description of the arrows is more or less useless in practice.
There is a construction that yields to a better description of the arrows: take the homotopy category $\mathcal{K}(\mathcal{A})$ first (maps of complexes modulo homotopies) and then localize wrt qis (this localization procedure is not possible in $\mathcal{C}(\mathcal{A})$, for you need certain diagrams to commute, and they do only in $\mathcal{K}(\mathcal{A})$):

$$\mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{K}(\mathcal{A})_{qis} =: \mathcal{D}(\mathcal{A}).$$

The objects are the same in all three categories; the arrows $K \to K'$ in $\mathcal{D}(\mathcal{A})$ can be described as equivalence classes roofs in $\mathcal{K}(\mathcal{A})$:

$$\begin{tikzcd} & K'' \arrow[dl, "qis"'] \arrow[dr, "f"] & \\ K & & K'. \end{tikzcd}$$

where two roofs are equivalent if there is a commutative diagram:

$$\begin{tikzcd} & K'' \arrow[dl, "qis"'] \arrow[dr, "f"] & \\ K & \overline{K} \arrow[l, "qis"'] \arrow[u, "qis"] \arrow[d, "qis"] & K' \\ & K'''. \arrow[ul, "qis"] \arrow[ur, "g"'] & \end{tikzcd}$$

Composition requires the basic verifications that suitable diagrams can be completed (omitted).

It is clear then that, at the price of replacing K with the source K'' of a qis $K'' \to K$, we can represent an arrow in $\mathcal{D}(\mathcal{A})$ with an arrow in $\mathcal{K}(\mathcal{A})$.

Caution: arrows in $\mathcal{D}(\mathcal{A})$ yield maps in cohomology. An arrow is then as iso iff it induces isos in cohomology; however a map inducing the zero maps in cohomology, is not necessarily the zero map (the simplest example I know with field coefficients for the sheaves comes from the normalization of two complex lines meeting at one point).

Caution: the derived category is not Abelian, for one can show that kernels would have to be direct summands. This applies to the homotopy cateogory as well.

If $\mathcal{A}$ has enough injectives, then we can give a better description of the morphisms in $\mathcal{D}^+(\mathcal{A})$ (+ stands for complexes bounded below, e.g. Godement resolutions of sheaves are typically infinite on the right, but are clearly bounded below).

Briefly:

- An object I of $\mathcal{A}$ is said to be inejctive, if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact;
- we say $\mathcal{A}$ has enough injectives if every object in $\mathcal{A}$ embeds into an injective one;
- if $\mathcal{A}$ has enough inejctives, then there is an equivalence of categories

$$\mathcal{K}^+(\mathcal{I}) \xrightarrow{\cong} \mathcal{D}^+(\mathcal{A}),$$

  where $\mathcal{I} \subseteq \mathcal{A}$ is the full subcategory (not Abelian, not a problem: it is additive and that is enough) of injective objects.

  An inverse consists of chosing injective resolutions (which exist, by the second bullet).

Exercise 2.4.1.1 Let $I \to I'$ be a qis in $\mathcal{K}^+(\mathcal{I})$. Show directly that it is an iso.

Via this equivalence, the arrows in the derived category can be seen as bona-fide maps of complexes modulo homotopy.

The category of (sheaves of) Abelian groups has enough injectives (any coefficients OK!).

From now on, assume that $\mathcal{A}$ has enough injectives.

Given $C \in \mathcal{D}^+(\mathcal{A})$, there is an injective resolution $C \to I(C)$ ($I \in \mathcal{C}^+(\mathcal{I})$, the arrow a qis).
Any two injective reolutions are canonically qis in the homotopy category and are thus canonically isomorphic in the derived category: this is why derived categories are so useful.
For every $C \in \mathcal{D}^+(\mathcal{A})$ fix an injective resolution $C \to I_C$.
Given a left-exact functor of Abelian cteogries

$$G : \mathcal{A} \longrightarrow \mathcal{B}$$

(a ses $0 \to A \to A' \to A'' \to 0$ is sent to an es $0 \to G(A) \to G(A') \to G(A'')$),
we have the right derived functor:

$$RG : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{B}), \qquad C \longmapsto G(I_C).$$

(it is well-defined up to unique iso of functors subject to a certain universal property).

Remark 2.4.1.2 Note that the functor above is actually defined with source $\mathcal{K}^+(\mathcal{I})$. In order to verify that it is indeed a functor we need

Exercise 2.4.1.3 (i) Prove that a ses of injectives splits. (ii) Use this fact and splicing to prove that if $I \in \mathcal{C}^+(\mathcal{I})$ is acyclic (i.e. $H^*(I) = 0$) and G is left-exact, then $G(I)$ is acyclic. (iii) Prove that if $u : I \to I'$ is a qis in $\mathcal{C}^+(\mathcal{I})$, then $G(u)$ is a qis in $\mathcal{C}^+(\mathcal{B})$; (hint: use the cone $C(u)$ and (ii)). (iv) Let $C_i \in \mathcal{C}^+(\mathcal{A})$, $i = 1, 2$, and $C_i \to I_i$ be two resolutions in $\mathcal{C}^+(\mathcal{I})$. Prove that a map $u : C \to C'$ in $\mathcal{C}^+(\mathcal{A})$ yields a canonical map of the corresponding resolutions in $\mathcal{K}^+(\mathcal{I})$ and that this map is an iso iff u is a qis.

We have the i-th right derived functors:

$$R^iG : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{B}, \qquad C \longmapsto H^i(RG(C)).$$

A ses in $\mathcal{C}^+(\mathcal{A})$ gives rise to a les of R^iG's in $\mathcal{B}$
(the right generality for this statement requires the notion of cones and distinguished triangles; we do not dwell on this in these lectures; if this makes you nervous, just

keep in mind, that a distinguished triangle is isomorphic in the derived category to a ses of complexes. Moreover, a ses yields a canonical distinguished triangle.
When working in the derived cateogry, I will use, by a serious abuse of language, the term ses in palce of distinguished triangle.

Remark 2.4.1.4 (**Not enough inejctives?**) It happens. You can still get by if you have a fixed left-exact functor $G : \mathcal{A} \to ?$ and you have a subclass $\mathcal{R} \subseteq \mathcal{A}$ of objects adapted to G: i.e. (i) if $R \in \mathcal{C}^+(\mathcal{R})$ is acyclic, then $G(R)$ is acyclic, (ii) every $A \in \mathcal{A}$ embeds in some $R \in \mathcal{R}$. Then you still have an equivalence $\mathcal{K}^+(\mathcal{R})_{qis} \cong \mathcal{D}^+(\mathcal{A})$. (If there are enough injectives, the class of injective objects is adapted to any left-exact functor and we have $\mathcal{K}(\mathcal{I}) \cong \mathcal{K}(\mathcal{I})_{qis}$, hence the importance of inejctives. However, the categories of constructible sheaves and of perverse sheaves do not have enough inejctives!) This is good for two reasons: (i) the more concrete description via this equivalence and (ii) we can construct the derived functor of G using these resolutions exactly as it was done above. Note that once this is done, it makes sense to define G-acyclic objects of $\mathcal{A}$ ($R^{>0}GA = 0$). In this case, the objects of $\mathcal{R}$ are automatically G-acyclic and the collection of G-acyclic objects is adapted to G. Moreover, if we have two G-acyclic resolutions $C \to J_i$, where the J_i have G-acyclic entries, then the two objects $G(J_i)$ are canonically isomorphic in $\mathcal{D}^+(\mathcal{B})$. The advantage of this is that once we know RG exists, we have some freedom in choosing $\mathcal{R}$. Compare this with Weil's argument outlined in §2.2. Caution: given another left-exact G', the functor RG' may not exist and, even if it does, there is no reason why it should be computed using objects adapted to G.

Example 2.4.1.5 (**Classes adapted to some funcctors**) ([4, 7]) Flabby sheaves are adapted to the global sections $\Gamma(T, -)$ and push-forward $f_*(-)$ functors. Ditto for soft sheaves on paracompact spaces, where flabby implies soft. A useful fact is that on a metrizable space, the restriction of flabby to *any* subset is flabby. Coherent sheaves are adapted to $\Gamma(X, -)$, X an affine variety (this was one of many Serre's stunning discovery). Flat sheaves (:= tensor product with them is an exact functor) are adapted to the (right-exact) tensor product (essential, since in general the category of sheaves does not have enough projectives). For locally compact spaces which are countable at infinity, soft sheaves are adapted to $\Gamma_c(T, -)$ (sections with compact support) and $f_!(-)$ (direct image with proper support). This is because, in this case soft = c-soft (can lift sections from a compact subset) ([7]). Note also that $f_!$ preserves c-softness.

2.4.2 The functors $R\Gamma$, Rf_*, f^*, $R\Gamma_c$, $Rf_!$, $R\Gamma_Z$, $R\mathrm{Hom}(K, -)$, $R\mathcal{H}om(K, -)$

Well, this is a lot of functors. The good news is that the recipe to define them is the same in each case (except for f^* which is exact). The bad news is that to have

a good working knowledge of these objects, one should add a few more (say f^*, $f^!$, derived tensor product duality, vanishing and nearby cycles) and then work through a long list of the wonderful proeprties that these derived functors enjoy, especially when combined (for a short list, see [12]; for a longer one see [7]).
In these lectures, I think I get by with the following list:

$$Rf_*, f^*, i^!(\text{i closed embedding}), R\mathrm{Hom}, R\mathcal{H}om, \mathbb{D}.$$

Let $f : T \to S$ be continuous and C be a bounded below complex of sheaves of Abelian groups.
The functors $\Gamma(T, -)$ and $f_*(-)$ are left exact.
The construction in §2.4.1, i.e. apply the functor to an injective resolution, gives rise to:

$$R\Gamma(T,C), \quad R^i\Gamma(T,C) =: H^i(T,C); \qquad Rf_*C, \quad R^if_*.$$

Remark 2.4.2.1 What is the relation between this definition of cohomology and the one via Godement resolutions (§2.2)? They are canonically iso in view of the fact that the Godement resolution is flabby (Example 2.4.1.5 and Remark 2.4.1.4). Ditto for $R\Gamma(T,-)$, Rf_* and R^if_*. Similar remarks hold for the other functors and will not be made explicit.

Exercise 2.4.2.2 (Cohomology on T and S and direct image presheaves) Show that

$$H^i(S, Rf_*C) = H^i(T,C).$$

Prove that

$$R^if_*C \text{ is the sheaf associated with the presheaf } V \longmapsto H^i(f^{-1}(V), C_{|}).$$

Look back to your solution to Exercises 2.2.0.3 and 2.3.3.1. This is not the case for $R^if_!C$: why?

Exercise 2.4.2.3 (Pull back f^*) Let G be a sheaf on S and define f^*G to be the sheaf associated with the presehaf

$$T \supseteq U \longmapsto \lim_{W \supseteq f(U)} G(W).$$

Show that the presehaf above is almost never a sheaf. Show that there is a natural map of sheaves $G \to f_*f^*G$. Deduce that there is a natural map $G \to Rf_*f^*G$ in the derived category. The induced map in cohomology $H(S,G) \to H(T, f^*G)$ is called pull-back. If $G = \mathbb{Z}_S$, then it is the usual pull-back in cohomology (for paracompact spaces §2.2). The functor f^* is exact and it extends in an obvious way to an exact functor on complexes and to a functor of the corresponding derived categories (no boundedness needed).

Since the i-th direct image sheaf $R^i f_* C$ is the sheaf on Y associated with the presheaf:

$$U \longmapsto H^i(f^{-1}U, C_|),$$

restriction (=pull-back) to the fiber induces the natural base change map:

$$(R^i f_* C)_s \longrightarrow H^i(T_s, C_{|T_s}).$$

Give examples where this map is neithr mono, nor epi.
It is an iso if f is a fiber bundle over a locally contractible space, or if f is a proper map of locally compact spaces (proper base change theorem).

Exercise 2.4.2.4 (**Cohomology with compact supports and** $Rf_!$) (See [10, 7]). Let T be a locally compact space. Define $\Gamma_c(T, -)$ the functor of sections with compact supports. Show that it is left-exact. Define the cohomology groups with compact supports $H^i_c(T, F)$. Let f be a map of locally compact spaces. Let $f_!$ be the functor direct image with proper supports:

$$f_! F(V) := \{s \in F(f^{-1}(V)) \mid \text{supp(s)} \to V \text{ is proper}\}.$$

We have a map $f_! \to f_*$ which is an iso iff f is proper. The functor $f_!$ is left-exact. Define $Rf_!$, $R^i f_!$. Observe that if f is proper, then $Rf_! = Rf_*$. Show that $H^i_c(T, C) = H^i_c(S, Rf_! C)$. Show that there is a natural isomorphism $(R^i f_! C)_s \cong H^i_c(T_s, C_{|T_s})$ This is the base change iso: note the different, better, behavior with the base change map for the ordinary direct image sheaves seen above. Find family of sheaves which are Γ_c-acyclic and/or $f_!$-acyclic.

If S and T are locally compact, then we also have

$$R\Gamma_c(T, C), \quad R^i\Gamma_c(T, C) =: H^i_c(T, C); \qquad Rf_! C, \quad R^i f_! C,$$

$$H_c(T, C) = H_c(S, Rf_! C).$$

However, $R^i f_! C$ is not the sheaf associated with the pre-sheaf $U \mapsto H^i_c(f^{-1}U, C_|)$ (it is not a pre-sheaf, for there is no restriction to open subsets!).

Example 2.4.2.5 The map $\mathbb{R}_M \to E_M$ (§2.3.1) is a qis of complexes, hence (gives rise to) an iso in the derived category. Since E_M is soft, hence $\Gamma(U, -)$-acyclic and thus $f_*(-)$-acyclic, we deduce (Remark 2.4.1.4) that the complex $f_* E_M$ (§2.3.2) is canonically isomrophic to $Rf_* \mathbb{R}_M$ in the derived category. A bit more explicitely: given $\mathbb{R}_M \to E_M$ and $\mathbb{R}_M \to I_{\mathbb{R}_M}$, by the proeprties of injectives, there is a map of resolutions $E_M \to I_{\mathbb{R}_M}$ which is a qis and unique up to homotopy. Apply f_* and show that the resulting map is a qis (see §2.2) and thus gives the desired canonical iso in the derived category.

Exercise 2.4.2.6 Let $Z \subseteq X$ be locally closed. (Reminder: the support of a section is always a closed set, but the support of a sheaf F is defined to be the closure of the not necessarily closed set where $F_x \neq 0$.) Define the functor $\Gamma_Z(X, -)$, from sheaves on X to Abelian groups, by setting $F \mapsto \Gamma(X, F)$, the sections of F on X supported on Z. Show that $\Gamma_Z(X, -)$ is left-exact and yields $R\Gamma_Z(X, -)$. Define a functor $F \mapsto \Gamma_Z(F)$, from sheaves on X to sheaves on X, by the assignement $U \mapsto \Gamma_Z(U, F)$ (sections of F on U supported on Z). Show that Γ_Z is left-exact so that we get $R\Gamma_Z$. Prove that $H^i(X, R\Gamma_Z(C)) = H^i(R\Gamma_Z(X, C)) =: H^i_Z(X, C)$ (cohomology with supports on Z).

Exercise 2.4.2.7 (**f_* and injectivity, $f_!$ and c-softness** [10]) Show that (f^*, f_*) is an adjoint pair of functors (Hom groups):

$$\mathrm{Hom}(f^*G, F) = \mathrm{Hom}(G, f_*F).$$

Deduce that f_* preserves injectivity. Find examples showing that $f_!$ does not preserve injectivity. Deduce the formula $R(g \circ f)_* = Rg_* \circ Rf_*$ (= means can iso). Prove that $f_!$ preserves c-softness so that we get the formula $R(g \circ f)_! = Rg_! \circ Rf_!$.

In order to discuss $R\mathrm{Hom}$ etc., one should really discuss derived functors of bi-functors (see [7]). Instead of doing that, let us just outline what ensues.
Let $\mathcal{A}$ be an Abelian category. Given $K \in \mathcal{C}^-(\mathcal{A})$ and $C \in \mathcal{C}^+(\mathcal{A})$, define, functorially, a complex in $\mathcal{C}^+(\mathcal{A})$:

$$\mathrm{Hom}^l(K, C) := \Pi_{p\in\mathbb{Z}}\mathrm{Hom}_{\mathcal{A}}(K^p, C^{p+l}), \quad [d^l f]^p := d^{l+p} \circ f^p + (-1)^{l+1} f^{p+1} d^p.$$

Note that an l-cocycle is a map of complexes $K \to C[l]$ and that an l-coboundary is a map $K \to C[l]$ homotopic to zero, i.e.

$$H^l(\mathrm{Hom}(K, C)) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(K, C[l]).$$

The functor descends to the respective homotopy categories (this is not obvious).
Define: ($C \to I_C$ an injective resolution) (^{o} is for opposite category):

$$R\mathrm{Hom}(K, C) := \mathrm{Hom}(K, I_C), \qquad R\mathrm{Hom} : (\mathcal{D}^-(\mathcal{A}))^o \times \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathbb{Z}\text{-}\mathcal{M}\mathrm{od}).$$

(Note the pleasant fact that we do not need to modify K.)
This turns out to be the right derived functor of the left-exact bi-functor Hom : $\mathcal{A}^o \times \mathcal{A} \to \mathbb{Z}$-mod, whatever that is.
We have

$$H^n(R\mathrm{Hom}(K, C)) = H^n(\mathrm{Hom}(K, I_C)) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(K, I_C).$$

We are free to replace K with anything qis to it ([2], I.6.2; there K is bounded below, but what is important is that I_C is bounded below and injective), without changing the rhs.
Using the description of arrows in $\mathcal{D}(\mathcal{A})$ as equivalence classes of roofs in the homootpy category, we see that we have the natural map

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(K, I_C) \longrightarrow \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(K, I_C).$$

This map is surjective (use the roof description and [2], I.6.2).
The map is injective: use the roof descritpion and deduce that if a map goes to zero, then we get $K' \stackrel{u\,qis}{\longrightarrow} K \stackrel{f}{\to} I_C$ giving the zero map in homotopy and we use [2], I.6.2 again.
We thus have the important formula:

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(K, C[i]) = H^i(R\mathrm{Hom}(K, C)) = H^0(R\mathrm{Hom}(K, C[i]))$$

Example 2.4.2.8 Let $K = \mathbb{Z}_T$. Then $\mathrm{Hom}(\mathbb{Z}_T, C) = \Gamma(T, C)$. It follows that since $R\mathrm{Hom}(\mathbb{Z}, -) = R\Gamma(T, -)$, so that

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{Z}_T\text{-mod})}(\mathbb{Z}_T, C[i]) = H^i(T, C),$$

i.e. an i cohomology class with coefficients in C is the same thing as a map $\mathbb{Z}_T \to C[i]$ in the derived category.

Remark 2.4.2.9 In particular, we obtain a different way of viewing the cup product maps 1 and 3 in Remark 2.3.2.1:

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{Z}_T\text{-mod})}(\mathbb{Z}_T, \mathbb{Z}_T[i]) = H^i(T, \mathbb{Z}_T) \ni a \,:\, \mathbb{Z}_T[-i] \longrightarrow \mathbb{Z}_T.$$

$$\mathrm{Hom}_{\mathcal{D}(\mathbb{Z}_T\text{-mod})}(\mathbb{Z}_S[-i], Rf_*\mathbb{Z}_T) = H^i(T, \mathbb{Z}_T) \ni a \,:\, \mathbb{Z}_S[-i] \longrightarrow Rf_*\mathbb{Z}_T.$$

Let us talk about $R\mathcal{H}om$.
There is the presheaf $\mathcal{H}om(F, G) : U \mapsto \mathrm{Hom}(F_{|U}, G_{|U})$.
It is a sheaf and its global sections are $\mathrm{Hom}(F, G)$.
Repeating the constructions mentioned above for $R\mathrm{Hom}$, we obtain $R\mathcal{H}om(K, C)$, the result being a complex of sheaves (vs. Abelian groups).

Remark 2.4.2.10 (**Non zero maps which are zero in cohomology**) Example 2.4.2.8 shows inequivocably that maps in derived categories $K \to C$ that induce the zero maps $H^iK \to H^iC$ need not to be zero.

Remark 2.4.2.11 (**Maps do not glue properly**) The same example shows that a map in the derived category of sheaves that becomes zero on the elements of an open covering, needs not to be zero. This latter fact shows we cannot glue uniquely maps in the derived category (which is thus not a stack).

Exercise 2.4.2.12 Prove that

$$R\mathrm{Hom}(K, C) = R\Gamma(T, R\mathcal{H}om(K, C)).$$

(Hint: if F and G are sheaves and G is injective, then $\mathcal{H}om(F, G)$ is flabby, hence $\Gamma(T, -)$-injective).
"Derive" the adjunction relation of Exercise 2.4.2.7, i.e. show that

$$Rf_* R\mathcal{H}om(f^*K, C) = R\mathcal{H}om(K, Rf_*C)$$

and

$$\mathrm{Hom}(f^*K, C) = \mathrm{Hom}(K, Rf_*C).$$

Apply this to $f^*K = f^*K$ and to $Rf_*C = Rf_*C$ to get the natural adjunction maps

$$K \longrightarrow Rf_*f^*K, \qquad f^*Rf_*C \longrightarrow C.$$

(Hint: see previous hint.) (See also §4.2.3.)

Remark 2.4.2.13 (**Yoneda Ext-groups**) ([10]) Do not assume there are enough inejctives. We have that the Yoneda $\mathrm{Yon.Ext^i_{\mathcal{A}}(A,B)}$ groups classifying i-extensions of B by A, $i \geq 0$, are canonically iso with $\mathrm{Hom_{\mathcal{D}(\mathcal{A})}(A, B[i])}$. If there are enough injectives, then these can be computed using

$$\mathrm{Yon.Ext}^i_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, B[i]) = H^i(R\mathrm{Hom}(A, B)) = H^i(T, R\mathcal{H}om(A, B)).$$

2.4.3 Derived of Leray-Hirsch and s-splitting

We have defined the functor Rf_* in the previous section.
It is thus clear that the proof of the Leray-Hirsch theorem in §2.3.3 gives something stronger, namely that there is an isomorphism

$$\mathscr{A} = \sum_{i,j \in J_i} [a_{ij}] \, : \, \bigoplus_{i \geq 0} R^i f_* \mathbb{R}_M[-i] \xrightarrow{\cong} Rf_*\mathbb{R}_M.$$

in the derived category of sheaves on B whose induced map in cohomology is the map A of Theorem 1.0.10.1.
Note that in view of Remark 2.4.2.9, we do not need to use differential forms.
In fact we do not need field coefficients.
The reader can formulate and prove the $\mathbb{Z}$-version of Leray-Hirsch.

Caution: One has to exercise caution with the Künneth formula.

Exercise 2.4.3.1 Let $f : X = F \times Y \to Y$ be the projection. Find a class of spaces for which you can solve this exercise with the same method of proof of "derived" Leray-Hirsch outlined in this section. Prove that there is a natural isomorphism in the derived category

$$\bigoplus_i \underline{H^i(F,\mathbb{Z})}[-i] \xrightarrow{\cong} Rf_*\mathbb{Z}_X,$$

where $\underline{H^i(F,\mathbb{Z})}$ is the constant sheaf with stalk $H^i(F,\mathbb{Z})$. Deduce that

$$H^a(X,\mathbb{Z}) \cong \bigoplus_i H^{a-i}(Y, \underline{H^i(F,\mathbb{Z})})$$

and give examples showing that in general, we have

$$H^{a-i}(Y, \underline{H^i(F,\mathbb{Z})}) \neq H^{a-i}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^i(F,\mathbb{Z}).$$

Definition 2.4.3.2 A complex $K \in \mathcal{D}(\mathcal{A})$ is said to be s-plit if there is an iso in $\mathcal{D}(\mathcal{A})$

$$\bigoplus_{i\in\mathbb{Z}} H^i(K) \xrightarrow{\cong} K.$$

We call such an iso an s-splitting (s stands for standard, as in standard truncation, standard t-structure ...).

Clearly, inthe Leary-Hirsch and Künneth situations, the complex $Rf_*\mathbb{Z}_M$ is s-split.

2.4.4 s-splitting for blowing-up nonsingular centers

Let us show that the method of proof of the derived Leray-Hirsch theorem outlined in §2.4.3 applies to blowing-ups with nonsingular centers.

Let $f : Bl_Z Y =: X \to Y$ be the blowing up of a noningular complex manifold Y along a nonsingular closed submanifold $Z \subseteq Y$.

Let $c := \dim Y - \dim Z$ (note that the cases $c \leq 1$ are trivial).
Let $E \in H^2(X,\mathbb{Z})$ be the class of the exceptional divisor.
We have that $p := f_{|E} : E \to Z$ is a $\mathbb{P}^{c-1}$-bundle (i.e. the cohomology of the fibers of f is known).
The powers $E^j \in H^{2j}(X,\mathbb{Z})$, $0 \leq j \leq c-1$ satisfy the hypothesis of Leray-Hirsch in the sense that, even though f is not a fiber bundle, we have that the collections of classes $\{E^j\}$ forms a basis of the cohomology, when restricted to any fiber of f (points, or $\mathbb{P}^{c-1}_{\mathbb{C}}$).
By Remark 2.4.2.9, we have maps in the derived category:

$$\mathscr{E}^0 : \mathbb{Z}_Y \longrightarrow Rf_*\mathbb{Z}_X, \qquad \mathscr{E}^j : \mathbb{Z}_Z[-2j] \longrightarrow Rf_*\mathbb{Z}_X,\ 1 \leq j \leq c-1,$$

which we can add up to yield a map

$$\mathscr{E} \ : \ \mathbb{Z}_Y \bigoplus \bigoplus_{1 \leq j \leq c-1} \mathbb{Z}_Z[-2j] \xrightarrow{\cong} Rf_*\mathbb{Z}_X$$

which is an iso since it induces isos on all cohomology sheaves.

3 Decomposition theorem for smooth projective maps

The main goal of these lectures is to discuss the decomposition theorem [16] for proper maps of complex algebraic varieties, the archetype of which is the combination of two theorems of Deligne's which are the subject of this section.

3.1 Notation

Unless otherwise stated, we work with complex quasi projective varieties endowed with the classical topology.
Our main interest is

$$\boxed{\text{smooth projective maps of complex quasi projective varieties } f: X \longrightarrow Y.}$$

For ease of exposition only, we assume that X (and thus Y) is <u>nonsingular</u>.
Projective: f can be factored as

$$pr_Y \circ i \ : \ X \longrightarrow \mathbb{P}^N \times Y \longrightarrow Y,$$

with i a closed embedding; this is automatic, in our case.
Smooth: df has maximal rank, so that f is a proper holomorphic submersion and, in particular, all fibers have the same dimension called the relative dimension of f:

$$n := \text{ relative dimension of } f := \dim X - \dim Y.$$

Unless otherwise stated, we work with <u>rational</u> cohomology:

$$H^i X := H^i(X, \mathbb{Q}).$$

The theme here is that

$$H^i X = H^i\, Y, Rf_* \mathbb{Q}_X \qquad \text{(as for any continuous map)}$$

and we want to understand a bit better $Rf_*\mathbb{Q}_X$ and $R^i f_* \mathbb{Q}_X$ using the properties of smooth projective map of complex algebraic varieties.

3.2 Deligne's two theorems

Theorem 3.2.0.1

1. (s-**splitness**) ([18]) *There is an isomorphism in the derived category of sheaves on* Y*:*

$$Rf_*\mathbb{Q}_X \cong \bigoplus_{i\geq 0} R^i f_*\mathbb{Q}_X[-i].$$

In particular, the Leray spectral sequence $H^p Y, R^q f_*\mathbb{Q}_X \implies H^{p+q}X$ *is* E_2*-degenerate.*

2. (**Semisimplicity**) ([20]) *The direct image sheaves* $R^i f_*\mathbb{Q}_X$ *are semsimple local systems.*

As we shall see the decomposition theorem is a far-reaching generalization of this theorem to proper maps.

Remark 3.2.0.2 As the proof shows, part 1 (s-splitting) of Deligne theorem holds for projective maps of analytic varieties. Part 2 (semisimplicity) does not (Example 3.3.2.2.2). We choose to state the two parts together to emphasize the connection with the decomposition theorem.

3.3 Ehresmann lemma, local systems

3.3.1 Ehresmann lemma: f is a fiber bundle

For a proof (of a strengthening) of the following classical fact in the context of proper holomorphic submersions, see ([22]).

Fact 3.3.1.1 (**Ehresmann lemma**) *If* $p : M \to B$ *is a* C^∞ *proper submersion of smooth manifolds, then* f *is* C^∞*-bundle. In particular, all fibers of* f *are diffeomorphic to each other.*

Corollary 3.3.1.2 *Smooth projective maps* f *are* C^∞ *fiber bundles.*

The proof is not hard, lift a local unit vector field on the base to the top and flow at constant speed along it. On the other hand it affords the following beautiful classical

Corollary 3.3.1.3 *All nonsingular hypersurfaces of fixed degree* d *in a projective space* $\mathbb{P}^N_{\mathbb{C}}$ *are diffeomorphic to each other.*

Of course, properness is essential to the Ehresmann lemma. Let me quote a master: "puncturing inflates the topology."

3.3.2 Fiber bundles and local systems

Let $\mathcal{L}$ a local system on a good connected space (B, b_o).
View $\mathcal{L}$ as a covering space $(\mathcal{L}, 0_{b_o}) \to (B, b_o)$.
Let $(\widetilde{B}, \widetilde{b_o}) \to (B, b_o)$ be a covering space and $(\widetilde{\mathcal{L}}, 0_{\widetilde{b_o}})$ be the pull-back.
Let $\pi := \pi_1(B, b_o)$ be the fundamental group (á la Deligne: $a \cdot b$ is b *followed* by a).
Then π acts on the LEFT on $(\widetilde{\mathcal{L}}, 0_{\widetilde{b_o}}) \to (\mathcal{L}, 0_{b_o})$ and

$$(\mathcal{L}, 0_{b_o}) = \left(\widetilde{\mathcal{L}}, 0_{\widetilde{b_o}}\right) / \pi.$$

We have that π acts on the left on $\mathcal{L}_{b_o}$ (pull-back, act, project), hence a representation

$$\rho : \pi \longrightarrow \mathrm{Aut}(\mathcal{L}_{b_o}).$$

Moreover, since $\widetilde{B}$ is simply connected, the local system $\widetilde{\mathcal{L}}$ on $\widetilde{B}$ is (iso to) the constant local system with stalk $\mathcal{L}_{b_o}$.
The discussion above leads to the well-known classification of local systems on (B, b_o) with a given stalk L in terms of conjugacy equivalence classes of representations $r : \pi \to \mathrm{Aut}(L)$, where r gives rise to the local system on B:

$$\left(\left(\widetilde{B}, \widetilde{b_o}\right) \times L\right) / \pi.$$

Recall some basic terminology for representation:

- irreducible = simple: no (non trivial) subrep;
- indecomposable: not a direct sum of subrep (in a non trivial way);
- completely reducible = semisimple: every subrep has a complementary subrep.

Accordingly, we use the same terminology for the corresponding properties of the associated local systems.

Remark 3.3.2.1 The constant local system $\mathbb{Z}_B$ on any space is not simple: $2\mathbb{Z}_B \subseteq \mathbb{Z}_B$. This shows that the notion of simplicity is useful in the context of, for example, field coefficients (we are interested in $\mathbb{Q}$).

Example 3.3.2.2 Let $\pi = \langle e \rangle$ be infinite cyclic.

1. Let $L = \mathbb{Z}^2$ and $r(e)$ be the matrix:

$$\begin{matrix} 0 & 1 \\ 1 & 0. \end{matrix}$$

Then r is indecomposable and not simple (hence not semisimple) so that so is the associated local system $\mathcal{L}$ which fits into the non split ses obtained by taking invariants:

$$0 \longrightarrow \mathbb{Z}_B \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $\mathcal{M}$ is associated with $r(e) =$ multiplication by -1.

If $L = \mathbb{Q}^2$, then r it is semsimple and decomposable

(direct calculation and also because it factors through a finite group)

and the ses analogous to the one above splits (use the trace).

Geometrically, the associated $\mathcal{L} = f_* \mathbb{Z}_{\mathbb{C}^*}$, where $f : \mathbb{C}^* \to \mathbb{C}^*$ is the square map, which is a smooth projective map of quasi-projective varieties.

2. Let $L = \mathbb{Q}^2$ and $r(e)$ be the matrix:

$$\begin{matrix} 1 & 1 \\ 0 & 1. \end{matrix}$$

Then r is indecomposable and not simple.

Geometrically, this is the local system $R^1 f_* \mathbb{Z}_{\mathcal{E}_{D^*}}$, where $f : \mathcal{E} \to \mathbb{P}^1$ is a Lefschetz pencil of elliptic plane curves, D^* is a punctured disk about a critical value and $\mathcal{E}_{D^*} := f^{-1} D^*$ (Picard-Lefschetz formula).

Both of the examples arise in connection with proper holomorphic submersions. The difference is that the former is also a map of algebraic varieties, while the latter is not (it is only a small Euclidean "piece" of one).

Let $f : M \to B$ be a fiber bundle.
Then each direct image sheaf $R^i f_* \mathbb{Z}_M$ is a local systems on B with fiber $H^i(M_b, \mathbb{Z})$. In fact, $R^i f_* \mathbb{Z}_M$ is constant on the open subsets of a trivializing open cover made of contractible open sets.

Exercise 3.3.2.3 Prove the last assertion. (Hint: the presheaf direct image (Exercise 2.4.2.2) is not locally constant; however, on each of the contractible open sets there is a natural map of presehaves from the appropriate constant sheaf into this presehaf ...)

These are the local systems we have met in Exercise 2.3.3.1, albeit with $\mathbb{R}$-coefficients.

3.3.3 Remarks on semsimplicity

Since I have no particular insight to offer concerning Deligne's proof of the semisimplicity statement via his theory of mixed Hodge structures, I simply refer to his proof.
Let me make the following remarks.

- Remark 3.3.2.1 tells us that there is no hope to have semisimplicity if we use integral coefficients. See also Example 3.3.2.2.1.
- Example 3.3.2.2.1 is an example coming from a smooth projective map. We note that the hypothesis of the Leary-Hirsch theorem are not met (look at H^0), but we trivially have $Rf_*\mathbb{Z} = R^0 f_*\mathbb{Z}$ and the E_2-degeneration of the LSS. In particular, E_2-degeneration is not the same thing has having enough global classes generating the cohomology of the fibers.
- Example 3.3.2.2.2 can be seen as coming from a smooth projective map of complex manifolds, yet semsimplicity fails. If we take the Zariski-dense open subset U of $\mathbb{P}^1_{\mathbb{C}}$ of the regular values (instead of just D^*), then the resulting map is smooth and projective of quasi projective varieties. The semisimplicty theorem applies. Logically, this forces the local invariants about D^* to be non $\pi_1(U, u_o)$-invariant, a fact that can be verified directly.
- It is a general fact (due to Landman), that local monodromies in complex algebraic geometry are quasi-unipotent (:= power of monodromy operator minus the identity is nilpotent = all eigenvalues are roots of unity).

3.4 The Leray spectral sequence LSS

3.4.1 The Leray spectral sequence (LSS): take I

Some references (for fiber bundles) are: [24], p.473; [**?**], Theorem 1.3, for constant coefficients; for locally constant coefficients, see page 17 and the section on local coefficients in [26].

Let S be a topological space and $S_* := \emptyset \subseteq S_0 \subseteq S_1 \subseteq \ldots \subseteq S_d = S$ be an increasing sequence (flag) of closed subspaces.
There is a spectral sequence

$$E_1^{pq} = H^{p+q}\, S_p, S_{p-1} \Longrightarrow H^{p+q}S, \qquad d_1 = \text{the coboundary of the triples } (S_l, S_{l-1}, S_{l-2}),$$

abutting to the decreasing filtration given by $F^pH^{p+q}S := \mathrm{Ker}\{\, H^{p+q}S \to H^{p+q}S_{p-1}\} \subseteq HS$ (this means $E_\infty^{p,q} = F^pH^{p+q}S/F^{p+1}H^{p+q}S$).

Example 3.4.1.1 (**Cohomology of cell complexes**) If S_* is a cell complex, then $E_1^{p,q} = 0$ for every $q > 0$, the spectral sequence is then just the complex $(E^{*,0}, d_1)$ (lying on the p-axis) and $E_2^{p,0} = H^p S$ and the filtration is essentially trivial: $F^p H^p S = H^p S$ and $F^{p+1} H^p S = 0$.

Let $f : T \to S$ be continuous and set $T_p := f^{-1} S_p$. We thus have the spectral sequence

$$E_1^{pq} = H^{p+q} T_p.T_{p-1} \Longrightarrow H^{p+q} T, \qquad F^p H^{p+q} T := \mathrm{Ker}\{ H^{p+q} T \to H^{p+q} T_{p-1} \}.$$

Exercise 3.4.1.2 Compute the cohomology of the Klein bottle T using the usual structure of cell complex on T (1 2-cell, 2 1-cells, 1 0-cell); let $f : T \to S$ be the usual bundle projection onto $S := S^1$; compute the spectral sequence and abutment that result by taking the obvious cell complex structure on S^1. Do the same for $T = S^1 \times S^1$ and $f = pr : S^1 \times S^1 \to S^1$. Do the same for the Hopf fibration $S^3 \to S^2$ and deduce that, in this case, the spectral sequence is not E_2-degenerate, i.e. $d_2 \neq 0$ (in this case it will be $d_2^{0,1} \neq 0$).

Let $f : M \to B$ be a C^∞ fiber bundle with fibers $F = M_b$.
Let $\emptyset \subseteq B_0 \subseteq B_1 \subseteq \ldots B_n = B$ be a cell complex decomposition for B.
There is the spectral sequence ($M_p := f^{-1} B_p$): $E_1^{pq} = H^{p+q} M_p, M_{p-1} \Longrightarrow H^{p+q} M$, mentioned above.
Since B_p / B_{p-1} is a bouquet of p-spheres, one has that:

$$E_2^{pq} = H^p B, R^q f_* \mathbb{Q}_M.$$

If $E = B \times F$, then the formula above is clear: in fact, $H^{p+q} M_p, M_{p-1} = H^p(B_p, B_{p-1}) \otimes H^q F$ and the differential effects only the first factor; the spectral sequence is then just a complex, the complex computing the cohomology of the cell complex B_* with the extra factor, $H^* F$, and we find (a weak version of) the Künneth formula.

From E_2 on, this is the Leray spectral sequence (LSS) for π as we define is in §3.4.3.

If the LSS is E_2-degenerate, then we have a non canonical isomorphism:

$$H^a M \cong \bigoplus_{i \geq 0} H^{a-i} B, R^i f_* \mathbb{Q}_M.$$

If not, then the lhs is non canonically isomorphic to a subquotient of the rhs, however, without extra information, one does not know much beyond this (i.e. which subquotient?).
Now we start over, and introduce the LSS after Grothendieck. In order to do this (§3.4.3), let us first discuss truncations.

3.4.2 Truncations

Every complex K in an Abelian category can be truncated by taking the subcomplexes:

$$\tau_{\leq i}K := \ldots \longrightarrow K^{i-2} \longrightarrow K^{i-1} \longrightarrow \mathrm{Ker}\{d^i\} \longrightarrow 0 \longrightarrow \ldots$$

and we have a system of maps

$$\ldots \tau_{\leq i-1}K \longrightarrow \tau_{\leq i}K \longrightarrow \ldots \longrightarrow K.$$

We can also take the quotient subcomplexes

$$\tau_{\geq i}K := \ldots 0 \longrightarrow \mathrm{Coker}\, d_{i-1} \longrightarrow K^{i+1} \longrightarrow K^{i+2} \longrightarrow \ldots$$

with the system of maps analogous to the one above (reverse arrows andinequalities). By construction: $\tau_{\leq i}\tau_{\geq i}K = \tau_{\geq i}\tau_{\leq i}K = H^iK[-i]$ the i-th cohomology object of the complex, placed in cohomological degree i.
The truncations define functors, denoted by the same symbol, in the derived category:

$$\tau_{\leq i} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}^{\leq i}(\mathcal{A}), \qquad \tau_{\leq i} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}^{\geq i}(\mathcal{A}),$$

where $\iota_{\leq i} : \mathcal{D}^{\leq i}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is the full subcategory of complexes K for which the natural arrow $\tau_{\leq i}K \to K$ is an iso. Similarly, for $\iota_{\geq i} : \mathcal{D}^{\geq i}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.
Note that $\mathcal{D}^i(\mathcal{A}) = \mathcal{D}^{\leq 0}(\mathcal{A})[-i]$ and similarly for $\geq i$.
We have functorial ses (distinguished triangles):

$$0 \longrightarrow \tau_{\leq i}K \longrightarrow K \longrightarrow \tau_{\geq i+1}K \longrightarrow 0.$$

We have $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{D}^{\leq i}(\mathcal{A}), \mathcal{D}^{\geq i+1}(\mathcal{A})) = 0$

Remark 3.4.2.1 The notion of t-structure on a triangulated category is obtained by turning some of the properties above into axioms. See **????**.

Exercise 3.4.2.2 Show that $\mathcal{D}^{\leq i}(\mathcal{A}) \cap \mathcal{D}^{\geq i}(\mathcal{A})$ is a full subcategory of $\mathcal{D}(\mathcal{A})$ equivalent to $\mathcal{A}[-i]$. Assume $\mathcal{A}$ has enough injectives. Show that if $K \in \mathcal{D}^{\leq i}(\mathcal{A}) \cap \mathcal{D}^{+}(\mathcal{A})$ and $C \in \mathcal{D}^{\geq i}(\mathcal{A})$, then

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(K, C) = \mathrm{Hom}_{\mathcal{A}}(H^i(K), H^i(C))$$

(this is reasonable, since K "stops" at i and C "starts" at i). (Hint: use $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(K, C) = H^0(\mathrm{Hom}(K, I_C))$ (§2.4.2). One does not need injectives or boudnedness: use the fact that $(\iota_{\leq i}, \tau_{\leq i})$ and $(\tau_{\geq i}, \iota_{\geq i})$ are pairs of adjoint functors (first entry left adjoint to the second entry, second entry, right adjoint to the first) ([7]):

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(K, C) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(K, \tau_{\leq i}C) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(K, H^i(C)[-i]) =$$

$$= \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(\tau_{\geq i}K, H^i(C)[-i]) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(H^i(K)[-i], H^i(C)[-i]) = \mathrm{Hom}_{\mathcal{A}}(H^i(K), H^i(C)).$$

Exercise 3.4.2.3 Let $f : T \to S$ be continuous and $K := Rf_*\mathbb{Q}_T$: you get $R^i f_* \mathbb{Q}_T[-i]$.

Assume there are enough injectives. Let $G : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. Using Cartan-Eilenberg resolutions, i.e. replace K by an injective resoltuion that has the additional property that $\tau_{\leq i}K$ and $H^i(K)$ are also injective (this can bed one! ([27])), we get a filtered complex

$$\ldots \subseteq G(\tau_{\leq i-1}K) \subseteq G(\tau_{\leq i}K) \subseteq \ldots \subseteq G(K).$$

The Grothendieck (or standard) spectral sequence for G with coefficients in K is the spectral sequence of the filtered complex above:

$$R^pG(H^qK) \Longrightarrow R^{p+q}G(K), \qquad F^pR^{p+q}G(K) := \mathrm{Im}\{\, R^{p+q}G(\tau_{\leq -p}K) \to R^*G(K)\}.$$

(Everything is well-defined up to a unique isomorphism in the filtered derived category)

Remark 3.4.2.4 If $K \cong \oplus H^i(K)[-i]$ is s-split, then the Grothendieck spectral sequence above is E_2-degenerate: for so is the spectral sequence of a filtered complex that splits, as a filtered complex, into the sum of its graded complexes endowed with the direct sum filtration $D^i = \bigoplus_{\leq -i}$.

3.4.3 The Leray spectral sequence (LSS): take II

The Leray spectral sequence is a special case of the Grothendieck spectral sequence where $K := Rf_*\mathbb{Q}_T$ is on S and G is the global sections functor $\Gamma(S, -)$:

$$H^p\,S, R^qf_*\mathbb{Q}_T \Longrightarrow H^{i+j}T, \qquad F^pH^*T := \mathrm{Im}\{\, H^*(S, \tau_{\leq -p}Rf_*\mathbb{Q}_T) \longrightarrow H^*T.\}.$$

Remark 3.4.3.1 By Remark 3.4.2.4, the LSS is E_2-degenerate as soon as $Rf_*\mathbb{Q}_T \cong \oplus_i R^if_*\mathbb{Q}_T[-i]$.

Question 3.4.3.2 What is the relation between the definition II above and I in §3.4.1?

Answer.

The answer is not entirely trivial. Let us discuss this a bit.

First of all, I is for bundles over cell complexes, while and II is general (f is a continuous map of topological spaces). So let us first explain the relationship in this case.

As pointed out in §3.4.1, the E_2 terms of the two spectral sequences coincide.
The reference [**?**] does not explain why the differentials $d_{r\geq 2}$ are the same in the two cases I and II (II is not discussed there).
[28] explains a technique (applied there in the context of complex algebraic geometry) that applies to this context.
The main point is that the flag S_* should be *cellular* for the direct image sheaves, i.e.

$$H^j(S_p, S_{p-1}, R^k f_*\mathbb{Q}_X) = 0, \quad \forall k, \ \forall j \neq p.$$

In the case of a fiber bundle this is automatic, since the direct image sheaves are constant on the simply connected cells.
Once this cellularity condition is met, then [28, 29] explains how to identify II with I.
Again, this does not apply only to fiber bundles, but to any map for which there is a flag on the target meeting the cellularity requirements for the direct image sheaves.
I did not think hard enough about this; but algebraic maps of real/complex algebraic varieties OK; ditto for semianalytic, subanalytic, semialgebraic.

3.4.4 Non E_2-degeneration: the Hopf fibration

Let us collect in one place the following related constructions.
Let $\mathbb{C}^{2^*} := \mathbb{C}^2 \setminus \{(0,0)\}$.
It is acted upon by $\mathbb{C}^*$ via dilations.
It is acted upon by $\mathbb{Z}$, generated by the dilation $1/2$.

1. $\mathbb{C}^{2^*} \to \mathbb{C}^{2^*}/\mathbb{C}^* =: \mathbb{P}^1_{\mathbb{C}}$; it is the $\mathbb{C}^*$-bundle of the complex line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)$.
2. The Hopf fibration $S^3 \to S^2$ is the restriction of the map above to the unit 3-sphere.
3. The Hopf surface $\mathfrak{H} := \mathbb{C}^{2^*}/\mathbb{Z}$. There is the natural map $\mathfrak{H} \to \mathbb{P}^1$ which is a proper holomorphic submersion with fibers ellitpic curves.

Fact 3.4.4.1 *The LSS of any of the three maps above is not E_2-degenerate.*

Proof. In all three cases let us write the map as $\pi : M \to S^2$.
In case 1. and 2. we have $b_1(M) = 0$, whereas in case 3. we have $b_1(M) = 1$.
Since the base S^2 is simply connected, the local systems in sight are all constant.
If we had E_2-degeneration, then we would have

$$b_0(S^2) \cdot b_1(F) = b_1(F) \leq b_1(M).$$

In case 1. and 2., $b_1(F) = 1$, and in case 3., $b_1(F) = 2$.

In each on the three cases we have reached a contradiciton.
(See also Exercise 3.4.1.2.) $\square$

The differential d_2 can be seen ([3]) to be the cup product with the Euler class $e \in H^2 S^2$ (= first Chern class) of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$.

Remark 3.4.4.2 In these examples, all the local systems involved are constant, so that E_2-degeneration is not related to how twisted the local systems $R^i f_* \mathbb{Q}_M$ are.

IMPORTANT. These examples are in sharp contrast with the realm of smooth projective complex algebraic maps, where, according to Deligne theorem, E_2-degeneration is the norm:

- the map 2. is a real algebraic proper submersion,
- the map 3. is a holomorphic proper submersion and
- the map 1. is a complex algebraic map which is even Zariski locally trivial over the base (but it is not proper).

UPSHOT: E_2-degeneration does not hold for proper bundle maps in the realms of smooth manifolds, real algebraic geometry and complex geometry
(as we shall see, neither does the decomposition theorem).

Remark 3.4.4.3 Example 3.3.2.2.2, i.e. the smooth projetive map $f : \mathbb{C}^* \to \mathbb{C}^*$ given by squaring, shows that the E_2-degeneration for smooth projective maps predicted by Deligne theorem is not a Leray-Hirsch-type phenomenon, as it may occur in the absence of enough global classes to generate the cohomology of the fibers. Compare with Remark 3.4.4.2.

Example 3.4.4.4 Let $f : X \to Y$ consider the family of smooth elliptic curves in $\mathbb{P}^2$. Deligne theorem applies. In particular, we have the ses

$$0 \longrightarrow H^1 Y, R^0 f_* \mathbb{Q}_X \longrightarrow H^1 X \longrightarrow H^0 Y, R^1 f_* \mathbb{Q}_X = H^1 X_y^{\pi_1(Y,y)} \longrightarrow 0.$$

Note that $H^1 X \to H^1 X_y$ factors through the subspace of global invariants.
One can show that the global invariants $(H^1 X_y)^{\pi_1(Y,y)} = 0$, but we need less: the global invariants are a subspace of the local invariants about a critical value of a Lefscehtz pencil, and we know, by Picard-Lefschetz (i.e. the local monodromy representation is as in Example 3.3.2.2.2) that this is a 1-dimensional space, whereas $H^1 X_y$ is two dimensional and there are not enough 1-classes on X to plug into Leray-Hirsch.

What is responsible for the E_2-degeneration of the LSS for smooth projective maps of complex varieties?

3.5 The easy relative Hard Lefschetz theorem (eRHL)

3.5.1 Hard Lefschetz (HL) and primitive decomposition (PLD)

Let Z be a projective manifold of dimension n and $\eta \in H^2 Z$ be the first Chern class of an ample line bundle on Z.
The hard Lefschetz theorem (HL) is the statement that we have isomorphisms:

$$\eta^k : H^{n-k} Z \xrightarrow{\cong} H^{n+k} Z.$$

Note that this implies $b_{n-k}Z = b_{n+k}Z$, a conclusion that can be reached also by Poincaré duality (which is an entirely different statement).
HL is a deep theorem, even for Z a surface: how to prove $\eta : H^1 Z \cong H^3 Z$?
The primitive spaces relative to η are defined by setting

$$P_\eta^{n-k} := \mathrm{Ker}\,\{\eta^{k+1} : H^{n-k} Z \longrightarrow H^{n+k+2} Z\}.$$

By reasons of elementary linear algebra, we get the primitive Lefschetz decomposition (PLD)

$$H^{n-k} Z = P_\eta^{n-k} \bigoplus \eta H^{n-k-2} = \bigoplus \eta^j P_\eta^{n-k-2j}, \qquad H^{n+k} Z = \bigoplus_j \eta^{k+j} P_\eta^{n-k-2j}.$$

Example. Let $Z = \mathbb{P}^1 \times \mathbb{P}^1$, $\eta = E + F$ (the two rulings) and $\eta' = E + 2F$. Then

$$P_\eta^2 = \langle E - F \rangle, \qquad P_{\eta'}^2 = \langle E - 2F \rangle.$$

3.5.2 eRHL: HL for the fibers of a smooth projective map

Remark 2.3.2.1 part 2 tells us that the choice of a closed 2-form η' on X representing $\eta \in H^2(X, \mathbb{R})$ yields a map of complexes

$$\eta' : f_* E_X \longrightarrow f_* E_X[2].$$

If η'' is another closed representative, then the map η'' is homotopic to the map η' (this is a simple exercise that is probably at the very root of the notion of maps of complexes being homotopic to zero),
so that, we get a well-defined map $f_* E_X \to f_* E_X[2]$ in the derived category.
Recalling Example 2.4.2.5, this is the same as a map (in the derived category):

$$\eta : Rf_* \mathbb{R}_X \longrightarrow Rf_* \mathbb{R}_X[2].$$

By Exercises 2.3.3.1 and 2.4.2.2, this map yields maps of cohomology sheaves which, stalk by stalk are the cup product map with the restriction of η to the fibers of f:

$$\eta : R^i f_* \mathbb{R}_X \longrightarrow R^{i+2} f_* \mathbb{R}_X, \quad \eta_y : H^i X_y \longrightarrow H^{i+2} X_y, \quad u \longmapsto \eta_{|X_y} \cup u.$$

The HL theorem applies to each pair (X_y, η_y) so that we obtain what we may call the relative hard Lefschetz theorem (RHL) for the smooth projective map f: we have isos:

$$\eta^k : R^{n-k} f_* \mathbb{R}_X \xrightarrow{\cong} R^{n+k} f_* \mathbb{R}_X, \quad \forall\, k \geq 0.$$

We have chosen to explain what above using the de Rahm complex, so that we expressed everything using real coefficients.
Of course all of the above remains valid with $\mathbb{Q}$-coefficients (but not integrally, for even the HL may fail integrally): one can take the flabby resolution of $\mathbb{Q}_X$ by the complex of sheaves of cochains on X and use a closed cochain representing η and repeat what above.
We can also use Remark 2.4.2.9, which gives $\eta : \mathbb{Q}_X \to \mathbb{Q}_X[2]$, apply Rf_*, take cohomology sheaves and then argue that the map on the stalks of the cohomology sheaves is the direct limit of the cup product maps $H^i f^{-1}(U) \to H^{i+2} f^{-1}(U)$ and that, in view of the fact that the map is proper, this is the cup product map on the cohomology of the fibers.

At any rate, we apply HL to the fibers of the smooth projective map f and we get isomorphisms of local systems

$$\eta^k : R^{n-k} f_* \mathbb{Q}_M \xrightarrow{\cong} R^{n+k} f_* \mathbb{Q}_M,$$

and PLDs for the local systems

$$R^{n-k} f_* \mathbb{Q}_M = \bigoplus_j \eta^j P_\eta^{n-k-2j} \quad (P_\eta \text{ sheaves of primitive spaces}).$$

3.6 Proof of Deligne Theorem

3.6.1 E_2-degeneration for smooth projective maps

Note that the s-splitting of $Rf_* \mathbb{Q}_X$ implies at once the E_2-degeneration of the standard spectral sequence $H^p Y, R^q \Longrightarrow H^{p+q} X$:
in fact, the filtered complex in §3.4.2 splits accordingly and the differentials are zero (Remark 3.4.3.1).
In this section, we prove this E_2-degeneration directly and without invoking derived categories.

Recall that all the fibers of smooth projective map are diffeomorphic to each other and that we have the local systems $R^i f_* \mathbb{Q}_X$ on the base Y.

Theorem 3.6.1.1 (Deligne 1968) *Let $f : X \to Y$ be a smooth projective map. Then the LSS*

$$H^p\,Y, R^q f_*\mathbb{Q}_X \Longrightarrow H^{p+q}X$$

is E_2-degenerate, so that

$$H^a X \cong \bigoplus_i H^{a-i}\,Y, R^i f_*\mathbb{Q}_X.$$

Sketch of proof. This can be found in textbooks (e.g. [11]). We sketch a proof below. Let us deal with $d_2 : H^pY, R^q f_*\mathbb{Q}_X \to H^{p+2}Y, R^{q-1}f_*\mathbb{Q}_X$ (the case of d_r, $r \geq 3$ is similar).
By using the PLD's on the local systems, it is enough to show that

$$d_2 : H^p\,Y, P^{n-k}_\eta \longrightarrow H^{p+2}\,Y, R^{n-k-1}f_*\mathbb{Q}_X$$

is zero.
We conclude by looking at the indicated properties of the following commutative diagram,

$$\begin{tikzcd}
H^p\,Y, P^{n-k}_\eta \arrow[r, "d_2"] \arrow[d, "0"', "\eta^{k+1}"] & H^{p+2}Y, R^{n-k-1}f_*\mathbb{Q}_X \arrow[d, "injective"', "\eta^{k+1}"] \\
H^{p+2(k+1)}Y, R^{n+k+2}f_*\mathbb{Q}_X \arrow[r, "d_2"] & H^{p+2+2(k+1)}Y, R^{n+(k+1)}f_*\mathbb{Q}_X,
\end{tikzcd}$$

which imply that the top $d_2 = 0$, i.e. the desired conclusion. □

The key observation that makes the above work is that q goes down when applying differentials so that the target of the differential survives more application of η's than the source and this, together with the injectivity statement of HL, forces the triviality of the differentials.

3.6.2 Proof of Deligne's theorem in relative dimension one.

The main idea emerges already for a smooth projective $f : X^{n+1} \to Y^n$ of relative dimension one.
In fact, let us fix ideas and set $n = 1$: $f : X^2 \to Y^1$.
Recall that we can use f_*E_X in place of $Rf_*\mathbb{R}_X$.
Note that f_*E_X is a complex with non-trivial entry in cohomological degrees in $[0, 4]$ (X is a of real dimension 4) and that $R^i f_*\mathbb{R}_M = 0$ for $i \notin [0, 2]$ (fibers have real dimension 2 and the proper base change theorem).
Choose a 2-form $\Gamma(X, E^2_X)$ representing $\eta \in H^2X$ and denote it by η.

We have the cup product map $\eta : E^i_X(U) \to E^{i+2}(U)$ and this induces a map of complexes
$$\eta : f_*E_X \to f_*E_X[2].$$
We have the commutative diagram which is a qis of complexes:

$$\begin{tikzcd}
0 \arrow[r] & f_*E^0_X \arrow[r] \arrow[d, "="] & f_*E^1_X \arrow[r] \arrow[d, "="] & \mathrm{Ker}\, f_*d_2 \arrow[r] \arrow[d, "\text{inclusion}"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_*E^0_X \arrow[r, "f_*d_0"] & f_*E^1_X \arrow[r] & f_*E^2_X \arrow[r, "f_*d_2"] & f_*E^3_X \arrow[r] & f_*E^4_X \arrow[r] & 0
\end{tikzcd}$$

Since we are working in the derived category, qis=iso and we can replace the bottom row, with its suitably truncated versions: the top row
We have the (non commutative!) diagram: (denote by e the map induced by η)

$$\begin{tikzcd}
 & R^0f_*\mathbb{R}X = \mathrm{Ker}\, f_*d^0 \arrow[rr, "e", "\cong"'] \arrow[d, "\iota"', "\text{inclusion}"] & & R^2f_*\mathbb{R}_X = \frac{\mathrm{Ker}\, f_*d_2}{\mathrm{Im} f_*d_1} & \\
0 \arrow[r] & f_*E^0_X \arrow[r] & f_*E^1_X \arrow[r] & \mathrm{Ker}\, f_*d_2 \arrow[r] \arrow[u, "\text{quotient}", "\pi"'] & 0
\end{tikzcd}$$

We thus get a map
(rem $\eta : f_*E_X \to f_*E_X[2]$ and in the derived category we can replace f_*E_X with the truncated bottom row above):

$$R^0[0] \oplus 0 \oplus R^2[-2] \xrightarrow{\iota + 0 + \eta\circ\iota\circ e^{-1}} [0 \to f_*E^0_X \to f_*E^1_X \to \mathrm{Ker}\, f_*d_2 \to 0]\,.$$

Exercise 3.6.2.1 Find an explicit splitting map going the other way. Deduce the desired conclusion: we have found that there is an isomorphism in the derived category $R^0[0]\oplus R^1[-1]\oplus R^2[-2] \cong Rf_*\mathbb{R}_X$. Note that while the 0-th and 2nd component are given above, this method only shows the existence of the 1st component, i.e. it does not yield a distinguished isomorphism (hint: what happens to the map from $R^1f_*\mathbb{R}_X$ if you change the closed form representing the class η?).

The case of arbitrary relative and total dimensions is very similar and left to the reader.

3.6.3 Proof of Deligne theorem

The following proof is not the original one in [18] (nor is the one in [30]). On the other hand, it puts into action some of the definitions we have seen (e.g. truncations §3.4.2).

Note that $\tau_{\leq 1}\tau_{\geq 2n-1}Rf_*\mathbb{Q}_X$ is a complex (think of it as a subquotient of $Rf_*\mathbb{Q}_X$) which shares cohomology sheaves with $Rf_*\mathbb{Q}_X$ for $i \in [1, 2n-1]$, the remaining ones being zero.
We have the commutative diagram

$$\begin{array}{ccc} \tau_{\leq 0}Rf_* = R^0 & \xrightarrow{\iota} & Rf_* \\ {\scriptstyle e^n}\downarrow{\scriptstyle \cong_{HL}} & & \downarrow{\scriptstyle \eta^n} \\ \tau_{\geq 0}(Rf_*[2n]) = R^{2n} & \xleftarrow{\pi} & Rf_*[2n] \end{array}$$

It is easy to show that

$$R^0 \oplus R^{2n}[-2n] \xrightarrow{\iota+\eta^n\iota(e^n)^{-1}} Rf_*$$

induces an iso on the 0-th and $2n$-th cohomolgy sheaves and that it splits (exercise: find a splitting).
In particular, the cone $C \cong \tau_{\geq 1}\tau_{\leq 2n-1}Rf_*$ splits off (noncanonically).
Verify that the RHL hypothesis works just as well for C:
the cohomolgoy sheaves $H^iC = 0$ for $i \notin [1, 2n-1]$, $H^iC = R^i$ $i \in [1, 2n-1]$ and C inherits the map $\eta : C \to C[2]$.
Repeat what above for the cone C together with the induced map η, and conclude by descending induction. $\square$

Note that while the map into Rf_* in the first step is rather natural (it depends only on the cohomology class of η, not on the chosen representative), the map from the cone into Rf_* depends on the representative (or at least, I was not able to show that is not the case), thus making the proceedings above highly noncanonical.
On the other hand there are procedures that allow to choose distinguished splittings. This is explained in [30].

3.6.4 The direct image complexes for the Hopf fibration and surface

Since the LSS for the Hopf fibration $S^3 \to S^2$ is not E_2-degenerate, there can be no s-splitting for $Rf_*\mathbb{Q}_{S^3}$.
Since we know about truncations, let us see what we get here.
We have the short exact sequence (more precisely a distinguished triangle)

$$0 \longrightarrow (\tau_{\leq 0}Rf_*\mathbb{Q}_{S^3} =)R^0f_*\mathbb{Q}_{S^3}[0] \longrightarrow Rf_*\mathbb{Q}_{S^3} \longrightarrow R^1f_*\mathbb{Q}_{S^3}[-1](= \tau_{\geq 1}RF_*\mathbb{Q}_{S^3}) \longrightarrow 0.$$

These are classified by the maps in the derived category (recall that $R^0 f_* \mathbb{Q}_S^3 \cong R^1 f_* \mathbb{Q}_{S^3} \cong \mathbb{Q}_{S^2}$.

$$\mathrm{Ext}^2(\mathbb{Q}_{S^2}, \mathbb{Q}_{S^2}) = H^2 S^2 \cong \mathbb{Q}.$$

Since the extension is not trivial, it is classified by a non zero class in $H^2 S^2$ and one sees that this class is the Euler class of the oriented S^1-bundle.
This is another way of viewing the differential d_2 of the LSS.
In short, $Rf_* \mathbb{Q}_{S^3}$ is (up to iso in the derived category) the unique complex fitting into a non-splitting ses $0 \to \mathbb{Q} \to ? \to \mathbb{Q}[-1]$ of complexes on S^2.

Let us turn to the Hopf surface.
Write $\mathbb{C}^2 - (0,0) = S^3 \times \mathbb{R}^{>0}$ and notice that the group of homotheties preserves the punctored lines through the origin which, in turn, we view as $e^{i\theta}(z_o, w_o)\rho$ with $\theta \in [0, 2\pi)$, $\rho \in \mathbb{R}^{>0}$ and $(z_o, w_o) \in S^3$.
With this description, it is clear that we have a holomorphic proper submersion

$$q : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}, \qquad e^{i\theta}(z_o, w_o)\rho \longmapsto (z_o : w_o),$$

with typical fiber an elliptic curve, with q diffeomorphic, as a fiber bundle to

$$S^1 \times S^3 \longrightarrow S^2, \qquad (a, b) \mapsto p(b).$$

In other words, in the smooth category, the Hopf surface is a trivial S^1-thickening of the Hopf fibration.
The cohomology sheaves of $Rq_* \mathbb{Q}_X$ are $R^0 f_* \mathbb{Q} \cong \mathbb{Q} = R^2 f_* \mathbb{Q}$, $R^1 f_* \mathbb{Q} \cong \mathbb{Q}^2$.
Since $b_1(X) = 1$ is odd, X is not Kähler. We have seen that the LSS is not E_2-degenerate, so $Rq_* \mathbb{Q}_X$ is not s-split.
Still, in thsi case, due to the Künneth formula [7] we do have a splitting of sorts for $Rq_* \mathbb{Q}_X$:
consider the Cartesian diagram (r the projection to S^2):

$$\begin{tikzcd}
S^1 \times S^3 \arrow[r] \arrow[d] \arrow[dr, "q"] & S^3 \arrow[d, "p"] \\
S^1 \times S^2 \arrow[r, "r"'] & S^2;
\end{tikzcd}$$

we have

$$Rq_* \mathbb{Q}_X = Rp_* \mathbb{Q} \otimes Rr_* \mathbb{Q} = Rp_* \mathbb{Q}_{S^3} \otimes (\mathbb{Q}_{S^2} \oplus \mathbb{Q}_{S^2}[-1]) = Rp_* \mathbb{Q}_{S^2} \oplus Rp_* \mathbb{Q}_{S^2}[-1].$$

3.7 Summary and failures for projective maps with singularities

We have met the following concepts in the context of smooth projective maps:

- local systems on the target; they form an Abelian category which is clearly Noetherian and Artinian, but not semisimple;
- s-splitting for $Rf_*\mathbb{Q}$ and some of the machinery behind it

  truncation functors: $\ldots \to \tau_{\leq i-i} \to \tau_{\leq i} \to \ldots \to \mathrm{Id}$,

  cohomology sheaves functors: $H^i(-) := \tau_{\leq i}/\tau_{\leq i-i}(-)[i]$,
- the map $\eta : Rf_*\mathbb{Q}_X \longrightarrow Rf_*\mathbb{Q}_X[2]$ and the induced maps $\eta^k : R^{n-k} \longrightarrow R^{n-k}$.

We have also met the following theorems for these maps (n the relative dimension):

1. s-splitting of $Rf_*\mathbb{Q}_X$;
2. semisimplicity of R^i;
3. RHL: $\qquad \eta^k : R^{n-k}f_*\mathbb{Q}_X \cong R^{n+k}f_*\mathbb{Q}_X$.

The various examples we have met have shown us how all of this fails if we do not assume that the map is a smooth and projective map of quasi projective complex varieties:
Let us now observe that these results fail completely if the map is projective, with singular fibers.

1. A natural source of examples for which $Rf_*\mathbb{Q}_X$ is not s-split are resolution of the singularities $f : X \to Y$ of normal projective varieties admitting at least one cohomology group H^iY whose mixed Hodge structure is not pure: in that case it is known that $H^iY \to H^iX$ is not injective, so that $\mathbb{Q}_Y = R^0f_*\mathbb{Q}_X$ cannot split off.

   A concrete example: blow up $\mathbb{P}^2$ along 10 suitable points along an elliptic curve (this is X) and then blow down the strict transform of the elliptic curve (the target is Y). One then shows that H^2Y has non zero classes in weight 1.

2. This is easy: if the map is not a fiber bundle, in general $R^i\mathrm{f}_*\mathbb{Q}_X$ is not locally constant. One may ask if the *sheaf* $R^i f_*\mathbb{Q}_X$ is semisimple. But the category of sheaves on Y has too few simple objects:

   for example the constant sheaf $\mathbb{Q}_C$ on a curve always has unbounded descending chains of subsheaves: choose a sequence of distinct points on C and look at

   $$\ldots \subseteq j_!\mathbb{Q}_{C\setminus\{p_1,p_2\}} \subseteq j_!\mathbb{Q}_{C\setminus\{p_1\}} \subseteq \mathbb{Q}_C,$$

   where we have denoted all open immersions by j.

   This shows that the category of sheaves, unlike the one of finite rank rational local systems, is not Artinian.

   This makes it clear that we cannot have semisimplicity statements for the direct image sheaves sheaves arising from singular proper maps.

3. First of all, it is not clear hot to formulate a RHL for a map f: what do we take as n? On the other hand for the map $X \to pt$ it is clear that we should take $n = \dim X$. Take a (necessarily singular) projective variety X for which the palindromic symmetry of its Betti numbers, ($b_{n-k} = b_{n+k}$) predicted by Poincaré duality fails. Then HL must fail as well (for every line bundle!).

   A concrete example is the projective come over $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$, for which we have that $b_2 = 1 \neq 2 = b_4$.

These examples seem to point to the fact that it is hopeless to have some kind of generalization of Deligne theorem to singular projective maps.
In fact, there is such a generalization, i.e. the decomposition theorem.
Before getting to it, which requires that we discuss perverse sheaves etc., let us discuss some examples.

4 Introduction to perverse sheaves

4.1 Contracting curves on complex surfaces re-visited

4.2 Review: the functor $i^!$

An excellent reference is [2], II.6.
Let $i : Z \to X$ be a locally closed embedding and F be a sheaf on X.
Recall that $\Gamma_Z F$ is the sheaf of local sections of F supported on Z, so that its stalks are zero outside Z.
Define the functor:

$$^o i^! := i^* \circ \Gamma_Z \qquad \text{(sheaves on } X \text{ to sheaves on } Z\text{)}.$$

We have $i_! i^! F = \Gamma_Z F \to F$.
If G is a sheaf on X whose stalks are zero outside Z, then every $G \to F$ factors uniquely through as $G \to i_! \, {}^o i^! F \to F$.
It follows that if E is a sheaf on Z, then (recalling that $i_!$ is extension by zero):

$$\mathrm{Hom}(i_! E, F) = \mathrm{Hom}(i_! E, i_! i^! F) = \mathrm{Hom}(E, i^! F),$$

so that we have the adjoint pair $(i_!, i^!)$ (for sheaves).
The functor $i^!$ is left-exact and preserves injectives.
We have $i^! := R({}^o i^!) = i^* R\Gamma_Z$ and it is calculated by taking ${}^o i^!$ of injective resolutions.
As above, if K is supported on Z, then any $K \to C$ factors uniquely through $i_! i^! C$.
By "deriving" the adjunction above, we get the adjoint pair with the adjunction map:

$$(i_!, i^!), \qquad i_! i^! C \longrightarrow C.$$

From now on, let $i : Z \to X \leftarrow U : j$ be two complementary closed/open embeddings.
Then $j^! = {}^o j^! = j^*$ and it preserves injectives.
We have a ses of sheaves

$$0 \longrightarrow j_! j^! F \longrightarrow F \longrightarrow i_* i^* F \longrightarrow 0$$

where all the functors involved are exact, thus giving us a functorial ses (distinguished triangle) in the derived category

$$i_1 i^! C \longrightarrow C \longrightarrow Rj_* j^* C \longrightarrow$$

whose les is the les of relative cohomology for the pair (X, Z):

$$H^*_U X = H^* X, Z \longrightarrow H^* X \longrightarrow H^* Z \xrightarrow{+}.$$

There is an es of sheaves

$$0 \longrightarrow i_!{}^o i^! F \longrightarrow F \longrightarrow j_* j^* F$$

which, if F is injective is exact on the right also It follows that we have a ses (distinguished triangle):

$$i_! i^! C \longrightarrow C \longrightarrow R j_* j^* C \longrightarrow 0,$$

whose les is the les of relative cohomology for (X, U):

$$H^*_Z X = H^* X, U \longrightarrow H^* X \longrightarrow H^* U \xrightarrow{+}.$$

Note that $Ri^! C = i^* R\Gamma_Z$ and that, since $R\Gamma_Z$ is supported on Z:

$$i_! i^! = i_* R i^! = R\Gamma_Z.$$

Finally, if $f : X \to Y$ is continuous, $Z' \subseteq Y$ is closed, $Z := f^{-1}Z$, and $g : Z \to Z'$ is the induced map, then we have a natural identification (this is completely general ([7], p.**????**), but for this special case, the un-derived version can be found in [2], p. 111):

$$i^!_{Z'} R f_* = R g_* i^!_Z.$$

4.2.1 Trying to split Rf_* when contracting curves on surfaces

Let things be as in any of the four examples in 1.0.1.
We have the natural surjection of complexes $p : Rf_* \mathbb{Q}_X \to R^2 f_* \mathbb{Q}_X[-2] = (H^2 E)_v$.
Let us make the notation lighter: $Rf \to R^2 f[-2]$.
We want to know if $R^2 f[-2]$ splits off.
This ie equivalent to having a map $\sigma : R^2 f[-2] \to Rf$ inducing $R^2 f = R^2 f$.
Since $R^2 f[-2]$ is supported on v, the map σ must factor uniquely $\sigma' : R^2 f[-2] \to i_{v!} i_v{}^! Rf$.
We obtain the following commutative diagram of maps in the derived category and of corresponding cohomology sheaves:

$$\begin{tikzcd}
& R^2 f[-2] \arrow[dl, "\sigma'"'] \arrow[d, "\sigma"] \\
i_! i^! Rf \arrow[r, "a"] \arrow[dr, "i"'] & Rf \arrow[d, "p"] \\
& R^2 f[-2]
\end{tikzcd}
\qquad
\begin{tikzcd}
& R^2 f = H^2 E \arrow[dl, "\sigma'^2"'] \arrow[d, "\sigma^2"] \\
H_2 E = H^2(i_! i^! Rf) \arrow[r, "cl_E"] \arrow[dr, "\iota_E"'] & R^2 f = H^2 E \arrow[d, "="] \\
& R^2 f = H^2(i_* i^* Rf) = H^2 E.
\end{tikzcd}$$

Note that $i_! i^! Rf$ is supported on v and, we have

Exercise 4.2.1.1 Use the last formula in §4.2 to prove that $i_! i^! Rf$ has cohomology sheaves $H^k \neq 0$ only for $k \in [2, 4]$:

$$i_! i^! Rf = H_4 E[0] \oplus H_3[-1] \oplus H_2 E[-2] \oplus H_1 E[-3] \oplus H_0 E[-4].$$

By **????** and the above exercise, to give a map σ, and thus equivalently to give a map σ', is the same thing as giving ${\sigma'}^2$.
We thus have:

Fact 4.2.1.2 $R^2 f[-2]$ splits off IFF the refined intersection form in degree 2 $\iota_E =$ iso.

Fact 4.2.1.3 (**First appearance of the decomposition theorem**) Let the contraction be as in the complex algebraic examples §1.0.1.(2),(4). Then we have

$$\tau_{\leq 1} Rf_* \mathbb{Q}_X = \tau_{\leq 1} Rj_* \mathbb{Q}_U,$$

and we have our first instance of the decomposition theorem:

$$Rf_* \mathbb{Q}_X \cong \tau_{\leq 1} Rj_* \mathbb{Q}_U \oplus R^2 f_* \mathbb{Q}_X[2].$$

Proof. We know that $\iota_E =$ iso, hence $R^2 f$ slits off and $\tau_{\leq 1} Rf$ is what is left.
We only need to show the first equality.
Note that, by adjunction, we have $Rf_* \to Rj_* j^* Rf_* = Rj_* \mathbb{Q}_U$.
By truncating this map, we have the map

$$\tau_{\leq 1} Rf_* \mathbb{Q}_X \longrightarrow \tau_{\leq 1} Rj_* \mathbb{Q}_U.$$

By inspecting cohomology sheaves, we need to show $R^1 f \to R^1 j$ is an iso.
This map is $j_1^* : H^1 X \to H^1 U$.
We have already observed in §1.0.1 that $j_1^* =$ iso IFF $\iota_E =$iso. $\square$

Remark 4.2.1.4 (**First appearance of the intersection complex**) *At the appropriate time, we shall introduce the intersection cohomology complex of any variety: (in the present case)*

$$IC_v = \mathbb{Q}_v; \qquad IC_Y = \tau_{\leq -1}(Rj_* \mathbb{Q}_U[2]), \quad \textit{equivalently } \mathcal{IC}_Y := \tau_{\leq 1} Rj_* \mathbb{Q}_U.$$

Then the splitting above takes the following form:

$$Rf_* IC_X \cong IC_Y \oplus IC_v^{b_2 E}.$$

Exercise 4.2.1.5 (**The non complex algebraic contractions**) Let the contraction be as in the non complex algebraic examples 1.0.1.(1),(2). Show that:

1. Rf does not split off $R^2f[-2]$, so that it does not split off $\tau_{\leq 1}Rf$.
2. $\tau_{\leq 1}Rf$ splits off $R^1f[-1]$ IFF $H_1E = 0$, in which case $R^if = 0$.
3. $\tau_{\leq 1}Rf \to \tau_{\leq 1}Rj$ is not an iso and it does not split.
4. $\mathbb{Q}_Y \to \tau_{\leq 1}Rf$ splits IFF $H_1E = 0$, $\mathbb{Q}_Y \to \tau_{\leq 1}Rj$ does not split.

Remark 4.2.1.6 (**First appearance of Jordan-Hölder for perverse sheaves**) At the appropriate time, we shall see that in all four examples $Rf[2]$ is a perverse sheaf. Every perverse sheaf has a Jordan-Hölder filtration. The work done in Fact 4.2.1.3 and Exercise 4.2.1.5 can be interpreted as follows: in the complex algebraic case, Rf is semisimple with simple quotients $IC_Y = \tau_{\leq 1}Rj\mathbb{Q}_U[2]$ and b_2 copies of IC_v; in the non complex algebraic case, we have a filtration:

$$0 \subseteq R^1j/R^1f \subseteq \tau_{\leq 1}Rf \subseteq Rf[2]$$

with simple quotients the perverse sheaves on Y:

$$(H^1U/H^1X)_v \quad IC_Y = \tau_{\leq -1}Rj_*\mathbb{Q}_[2], \qquad (H^2E)_v.$$

4.2.2 Review: the adjunction map $a : \mathbb{Q}_Y \to Rf_*f^*\mathbb{Q}_X = Rf_*\mathbb{Q}_X$

We want to view the content of §4.2.1 in a different way: we want to see what happens when we try to splits the adjunction map $a : \mathbb{Q}_Y \to Rf_*\mathbb{Q}_X$, i.e. we want to see what happens if we try and start splitting Rf_* from the opposite end!
The point of doing this is that, as we try and do this we first meet the intersection form. If it is an iso, then we can map Rf into $\mathcal{IC}_Y$ which thus appears naturally here as an hurdle as we try to split a.

Let $f : X \to Y$ be a continuous map.
There is the map of sheaves:

$$\mathbb{Q}_Y \to f_*f^*\mathbb{Q}_Y = f_*\mathbb{Q}_X$$

where a local section (U, s) Y gives a section upstairs, i.e. $(f^{-1}U, f^*s)$ of $f^*\mathbb{Q}_Y = \mathbb{Q}_X$ which is, by definition, a section of $f_*\mathbb{Q}_X$ on U.
Replace $\mathbb{Q}_X$ with an injective resolution and get

$$a : \mathbb{Q}_Y \longrightarrow Rf_*f^*\mathbb{Q}_Y = Rf_*\mathbb{Q}_X$$

it is called the adjiunction maps since it arises in the context of (f^*, f_*) and (f^*, Rf_*) being adjoint pairs (Exercise 2.4.2.12).
By taking cohomology, we find that the adjunction map is the familiar pull-back in cohomology:

$$H(a) = f^* : H^*Y \longrightarrow H^*X.$$

Exercise.

- Let $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$. Show that a splits. (Hint: use the trace).
- Let $f : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. Prove that a splits.
- Let $f : S^3 \to S^2$ be the Hopf map. Prove a does not split. (Hint: look at f^* on H^2).
- Let $f : X \to Y$ be a map of algebraic varieties such that H^jX is a pure Hodge structure in some degree j, but H^jY is not. Deduce that a does not split. (Hint: functoriality of the mixed Hodge structures involved.)

4.2.3 Trying to split the adjunction map

We want to address the following

Question 4.2.3.1 Does a split in the Examples 1.0.1.(1-4). (contracting curves on surfaces)?

Let $j : U := (Y - v) \longrightarrow Y$ be the open embedding.
We have the adjunction map for j: (noting that $j^*Rf_*\mathbb{Q}_X = \mathbb{Q}_Y$)

$$Rf_*\mathbb{Q}_X \longrightarrow Rj_*j^*Rf_*\mathbb{Q}_X = Rj_*\mathbb{Q}_U.$$

The sheaves $R^if_*\mathbb{Q}_X$ are:

$$R^0f_*\mathbb{Q}_X = \mathbb{Q}_Y, \quad R^kf_*\mathbb{Q}_X = (H^kE)_v.$$

They are zero for $k > 2$, so that:

$$Rf_*\mathbb{Q}_X = \tau_{\leq 2}Rf_*\mathbb{Q}_X.$$

Since truncation is a functor, we get

$$Rf_*\mathbb{Q}_X \xrightarrow{\mathcal{J}} \tau_{\leq 2}Rj_*\mathbb{Q}_U.$$

The complex $Rj_*\mathbb{Q}_Y$ has the following cohomology sheaves:

$$R^0 j_*\mathbb{Q}_U = j_*\mathbb{Q}_U = \mathbb{Q}_Y; \qquad R^k j_*\mathbb{Q}_U = H^k(U)_v \quad \forall k > 0.$$

Look at the following diagram:

$$\begin{tikzcd}
\mathbb{Q}_Y \arrow[rr, "a"] & & Rf_*\mathbb{Q}_X \arrow[dll, "?_0"'] \arrow[dl, "?_1"'] \arrow[dr, "\mathcal{J}"] \arrow[ddl, "?_1^1"] \arrow[ddr, "\mathcal{J}^2"'] & \\
\mathbb{Q}_Y = \tau_{\leq 0}Rj_*\mathbb{Q}_{Y^*} \arrow[r] & \tau_{\leq 1}Rj_*\mathbb{Q}_{Y^*} \arrow[rr] \arrow[d] & & \tau_{\leq 2}Rj_*\mathbb{Q}_{Y^*} \arrow[d] \\
& R^1j_*\mathbb{Q}_{Y^*}[-1] = (H^1U)_v & & R^2j_*\mathbb{Q}_{Y^*}[-2] = (H^2U)_v
\end{tikzcd}$$

The question whether a splits has become:

can we lift the map $\mathcal{J}$ to $\tau_{\leq 0}Rj_*\mathbb{Q}_{Y^*} = \mathbb{Q}_Y$?

Remark 4.2.3.2 (**First appearance of the intersection complex**) The complex

$$\mathcal{IC}_Y := \tau_{\leq 1}Rj_*\mathbb{Q}_U$$

has emerged as a natural actor in this play: it is the Deligne-Goresky-MacPherson intersection cohomology complex of the pseudomanifold Y with an isolated singularity v (here and always: middle perversity).

The map $\mathcal{J}^2 : R^2f_*\mathbb{Q}_X \to R^2j_*\mathbb{Q}_{Y^*}$ can be seen as the pull-back in cohomology

$$j_2^* : H^2X \longrightarrow H^2U.$$

We have the les: (shorthand: $(Rf_*, Rj_*) = \mathrm{Hom}(Rf_*\mathbb{Q}_X, Rj_*\mathbb{Q}_U)$, etc.)

$$\ldots (Rf_*, R^2j_*[-3]) = 0 \longrightarrow (Rf_*, \tau_{\leq 1}Rj_*) \longrightarrow (Rf_*, \tau_{\leq 2}Rj_*) \longrightarrow (Rf_*, R^2j_*[-2]) \longrightarrow \ldots$$

where $\mathcal{J} \mapsto \mathcal{J}^2$.
Clearly

$\exists\, ?_1$ *lift IFF* $j_2^* = 0$ *and, if this lift exists, then it is unique.*

The first lift $?_1$ exists (and is then unique) in the two complex algebraic contractions and it does not in two non complex algebraic ones.
Let us deal with the complex algebraic contractions, where we have $?_1$ and thus the natural $?_1^1$.

This map fits into the les

$$H_3E = 0 \longrightarrow H^1X (\cong H^1E) \xrightarrow{?_1^1 = j_1^*} H^1U \xrightarrow{0} H_2E$$

and is thus an iso.
The same les of Hom argument above, shows that the lift $?_0$ exists and is unique IFF the genus of E is zero.
Note that $g(E) = 0$ IFF $\mathcal{IC}_Y = \mathbb{Q}_Y$.

Therefore, the first point we wish to make about these four examples is that:
the adjunction map does not split;
there is a lift $?_1 : Rf_*\mathbb{Q}_X \to \mathcal{IC}_Y$ *of* $\mathcal{J} : Rf_* \to \tau_{\leq 2}Rj_*$ *only in the complex algebraic case.*

Let us now show that the map $?_1$ splits in the derived category.
We take the cone C of the map (pretend you are working with a ses):

$$Rf_*\mathbb{Q}_X \longrightarrow \tau_{\leq 1}Rj_*\mathbb{Q}_U \xrightarrow{\beta} C(?_1) \longrightarrow .$$

The les of cohomology sheaves strarts with $\mathbb{Q}_Y = \mathbb{Q}_Y$ so we can ignore it. The remaining part is made of stuff supported on the singular point v and it boild down to the following:

$$0 \longrightarrow H^0C \longrightarrow H^1X \xrightarrow{j_1^*} H^1U \xrightarrow{\beta_1} H^1C \longrightarrow H^2X \longrightarrow 0.$$

We know that $j_1^* =$ iso.
This forces $H^0C = 0$, $H^1C = (H^2E)_v$ and $H^1U \to H^1C$ to be the zero map.
We thus have that $\tau_{\leq 1}Rj_*$ stops in degree 1, while C starts in degree 1 so that the maps between them are classified (**????**) by the maps $H^1U \to H^1C$.
We conclude that $\beta = 0$ and $Rf_*\mathbb{Q}_X \to \mathcal{IC}_Y$ splits (canonically) and we have a (canonical) iso:

$$Rf_*\mathbb{Q} \cong \mathcal{IC}_Y \oplus (H^2E)_v[-2].$$

• This is our first example of the decomposition theorem.

Remark 4.2.3.3 (**Key role of** ι_E) Note that the splitting occurred because of two occurrences, namely $j_2^* = 0$ and $j_1^* =$ iso. In fact, we have observed that these two fact are equivalent to each other and equivalent to the refined intersection form ι_E = iso. Therefore, in these four examples, the refined intersection form is the agent responsible for the splitting above. As it turns out, refined intersection forms are always the agents controlling the splitting of the perverse sheaves occurring in the decomposition theorem **????**.

Exercise 4.2.3.4 (**Special case when** $E = \mathbb{P}^1$) Let X be the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$. Let $f : X \to Y$ be the (holomorphic!, in fact algebraic!) contraction of the zero section E. Show that Y is the affine cone over the twisted rational curve of degree n in $\mathbb{P}^n$. Show that we have $\mathcal{IC}_Y = \mathbb{Q}_Y$ and thus

$$\mathcal{IC}_Y = \mathbb{Q}_Y, \qquad Rf_* \mathbb{Q}_X \cong \mathbb{Q}_Y \oplus \mathcal{IC}_v[-2]$$

where $\mathcal{IC}_v = \mathbb{Q}_v$ is the intersection complex of the manfold v. (N.b. rational homology spheres $S^3/(\mathbb{Z}/n\mathbb{Z})$ will arise). Observe that $Rf_*\mathbb{Q}_X$ is s-split.

Exercise 4.2.3.5 (**Contracting curves on surfaces**) Let $(X, \cup E_j)$ be a nonsingular complex surface and E_j be finitely many compact irreducible curves on X. Let $Y := X/\cup E_j$ and $f : X \to Y$ be the natural map contracting the collection of curves to a single point v. Show that we have a splitting as in the previous example $Rf_*\mathbb{Q}_X \cong \mathcal{IC}_Y \oplus \mathcal{IC}_v^{\#}[-2]$ where $\#$ is the number of curves IFF the intersection matrix $I = ||E_j \cdot E_k||$ is nondegenerate. Observe that this happens on $(\mathbb{P}^2_{\mathbb{C}}, E)$, E any irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$, but it does not happen for $E_1 \cup E_2$ two distinct irreducible curves. (N.b.: Grauert proved that the configuration of curves is contractible in the category of complex analytic spaces IFF I is negative definite.)

Example 4.2.3.6 ($\mathcal{IC}_Y$ **is not** $\mathbb{Q}_Y$) Let $f : X \to Y$ be blowing up of the vertex of the affine cone Y over a projective nonsingular curve $E \subseteq \mathbb{P}^N_{\mathbb{C}}$ of positive genus (if the genus is zero, then we are essentially in the situation of Exercise 4.2.3.4). Use the Leray spectral sequence for the $\mathbb{C}^*$-bundle $X \setminus E \to E$ to show that

$$H^1(\mathcal{IC}_Y) = (H^1E)_v.$$

This shows that $\mathcal{IC}_Y$ is seldom equal to $\mathbb{Q}_Y$ (as it was the case in Exercise 4.2.3.4). This is exactly the obstruction to split the adjunction map.

Exercise 4.2.3.7 (**Currents**) Let things be as in Exercise 4.2.3.4. Consider the soft resolution $\mathbb{R}_X \to D_X$ (where D_M is the complex of sheaves of currents on the C^∞ manifold underlying X. Then $f_*D_X = Rf_*\mathbb{R}_M$ (canonical iso in the derived category. Define a map $\mathbb{R}_Y \to f_*D_X$ by sending, for every V open in Y, 1_V to the current of integration [11] $\int_{f^{-1}V}$. Define a second map $\mathbb{R}_v[-2] \to f_*D_X$ by sending 1_v to $\int_E$. Show that we get an iso in the derived category

$$\mathbb{R}_Y \bigoplus \mathbb{R}_v[-2] \xrightarrow{\cong} Rf_*\mathbb{R}_X.$$

Discuss why this does not work in the situation of Example 4.2.3.6 where one indeed has the conclusion of Exercise 4.2.3.5, i.e. $Rf_*\mathbb{R}_X \cong \mathcal{IC}_Y \oplus \mathbb{R}_v[-2]$. (Hint: $H^1E \neq 0$.)

Remark 4.2.3.8 I do not want to extrapolate too much, at this point, from the DT for Example 2. I will say that the intersection form ι was the key to splitting the direct image. This is an important ingredient and I will discuss it in detail by providing a splitting criterion for perverse sheaves that uses the correct generalization of this form. The second important ingredient is the relative hard Lefschetz theorem. It has not appeared yet: the example we have discussed are trivial with respect to this theorem.

4.3 5 Examples

4.3.1 The blow up of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ at a point t

Consider the following commutative diagram:

$$\begin{tikzcd} X := \mathrm{Bl}_t\, \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "b"] \arrow[dr, "f"'] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "p"] \\ & \mathbb{P}^1. \end{tikzcd}$$

We already know that

$$Rb_*\mathbb{Q}_X \cong \mathbb{Q}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathbb{Q}_t[-2].$$

We have

$$\begin{aligned} Rf_*\mathbb{Q}_X &\cong Rp_*Rb_*\mathbb{Q}_X \\ &\cong Rp_*\mathbb{Q}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus Rp_*\mathbb{Q}_t[-2] \\ &\cong^1 [R^0 = \mathbb{Q}_{\mathbb{P}^1}] \oplus \left[R^2 = \mathbb{Q}_{\mathbb{P}^1} \oplus \mathbb{Q}_{p(t)}\right][-2] \\ &\cong^2 \mathcal{IC}_{\mathbb{P}^1} \oplus \mathcal{IC}_{\mathbb{P}^1}[-2] \oplus \mathcal{IC}_{p(t)}[-2]. \end{aligned}$$

The third row tells us that $Rf_*\mathbb{Q}_X$ decomposes as the direct sum of its shifted cohomology sheaves;
it also tells us that the second direct image sheaf splits into two summands:

$$R^2f_*\mathbb{Q}_X \cong \mathbb{Q}_{\mathbb{P}^1} \oplus \mathbb{Q}_{p(t)}.$$

I want to point out that these two splittings are really different and I will part them by iintroducing a kind of symmetry that was inisible in the examples considered so far: *the relative hard Lefschetz theorem.*

4.3.2 The 5 Examples

In the following five examples $f : X \to Y$ is a projective map of quasi porjective varieties of the indicated dimensions.
In the previous sections we have proved that, in each example, $Rf_*\mathbb{Q}_X$ splits. We this information below, taking care to write each term with its support as a subscript.

1. $f : X^n \to Y^0$ ($v := Y^0$):

$$Rf_*\mathbb{Q}_X \cong \bigoplus_{i=0}^{2n} H^i(X)_v[-i].$$

2. $f : X^{m+n} \to Y^m$ projective smooth (example 1 is a special case of this):

$$Rf_*\mathbb{Q}_X \cong \bigoplus_{i=0}^{2n} R^i_Y[-i] \qquad (R^i_Y := R^i f_*\mathbb{Q}_X \text{ local system on } Y)$$

3. $f : X^2 \to Y^1$ ($\mathrm{Bl}_t\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ (§**??**):

$$Rf_*\mathbb{Q}_X \cong \mathbb{Q}_Y \oplus \mathbb{Q}_Y[-2] \oplus \mathbb{Q}_{p(t)}[-2].$$

4. $f : X^n = \mathrm{Bl}_Z Y \to Y^n$ blow up with smooth center (§2.4.4) (set $r := c - 1$):

$$Rf_*\mathbb{Q}_X \cong \mathbb{Q}_Y \bigoplus \bigoplus_{1 \le j \le r} \mathbb{Z}_Z[-2j].$$

5. $f : X^2 \to Y^2$ contraction of a curve E on a surface to a point v:

$$Rf_*\mathbb{Q}_X \cong \mathcal{IC}_Y \bigoplus H^2(E)_v.$$

In order to not get distracted by details, we write each term above simply as $\mathcal{I}_{support}[-]$.
Now, set
$$\mathscr{I}_{support} := \mathcal{I}_{support}[\dim support].$$
We get

1. $f : X^n \to Y^0$ ($v := Y^0$):

$$Rf_*\mathscr{I}_X \cong \bigoplus_{i=-n}^{n} \mathscr{I}_v[-i]$$

2. $f : X^{m+n} \to Y^m$ projective smooth (example 1 is a special case of this):

$$Rf_*\mathscr{I}_X \cong \bigoplus_{i=-n}^{n} \mathscr{I}_Y[-i].$$

3. $f : X^2 \to Y^1$ ($\mathrm{Bl}_t \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ (§**??**):
$$Rf_* \mathscr{I}_X \cong \mathscr{I}_Y[1] \oplus \mathscr{I}_{p(t)} \oplus \mathscr{I}_Y[-1].$$
4. $f : X^n = \mathrm{Bl}_Z Y \to Y^n$ blow up with smooth center (§2.4.4) (set $r := c - 1$):
   (a) if r is odd:
   $$Rf_* \mathscr{I}_X \cong \mathscr{I}_Z[r] \oplus \ldots \oplus \mathscr{I}_Z[1] \oplus \mathscr{I}_Y \oplus \mathscr{I}_Z[-1] \oplus \ldots \oplus \mathscr{I}_Z[-r];$$
   (b) if r is even:
   $$Rf_* \mathscr{I}_X \cong \mathscr{I}_Z[r] \oplus \ldots \oplus \mathscr{I}_Z[2] \oplus (\mathscr{I}_Y \oplus \mathscr{I}_Z) \oplus \mathscr{I}_Z[-2] \oplus \ldots \oplus \mathscr{I}_Z[-r].$$
5. $f : X^2 \to Y^2$ contraction of a curve E on a surface to a point v:
$$Rf_* \mathscr{I}_X \cong \mathscr{I}_Y \oplus \mathscr{I}_v.$$

Something remrkable in now in plain sight:

the direct sum decompositions are palindromic!

There is the Grothendieck-Verdier generalization of Poincaré duality. It applies here and it predicts, *once* $Rf_* \mathbb{Q}_X$ is known to split that the splitting should be palindromic (up to duality).
So it seems that what above is expected.
But a closer look shows that the terms that correspond to each other under the symmetry $-i \leftrightarrow i$ are in fact isomorphic, whereas the duality would predict that they are dual to each other.
Again, this seems remarkable.
On the other hand, the terms in the example are self-dual, so this could be a coincidence.
But wait, this is what happens with the HL theorem: $H^{n-k} \cong (H^{n+k})^*$ (PD),H$^{n-k} \cong H^{n+k}$ (PD), so the terms are self-dual:
$$H^{n-k} \times H^{n-k} \longrightarrow \mathbb{Q}(a,b) \longrightarrow \int_X \eta^k \cup a \cup b$$
is a perfect pairing.
If there is a kind of HL among the terms corresponding in the symmetry, then we should be onto something.

Exercise 4.3.2.1 Verify that in each case there is η ample on X (or even just f-ample, i.e. ample on the fibers) such that, for every $i \geq 0$, η^i yields isos between the terms in cohomological degree $-i$ and i.

So are we onto something?
The answer is yes.
In order to start explaining why, I need to explain the meaning of the dimensional shift we have used above.
It is time to talk about perverse sheaves.

4.4 The perverse t-structure

4.4.1 t-structures

The context is the one of a triangulated category $\mathcal{D}$.
Instead of saying what that is, let us be re-assured by the fact that if $\mathcal{A}$ is an Abelian category, then the homotopy and derived categories, as well as they bounded variants, $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ are triangulated. So the reader can keep these models in mind.
As the various definitions below unfold, it is a good idea to keep in mind §3.4.2 on truncations.
A t-structure p on $\mathcal{D}$ is the datum of two full subcategories ${}^p\mathcal{D}^{\leq 0}$ and ${}^p\mathcal{D}^{\geq 0}$ of $\mathcal{D}$ subject to three axioms: (for a class $\mathcal{R} \subseteq \mathcal{D}$, set $\mathcal{R}[i]$ to be the class obtained by translating the elements of $\mathcal{R}$; set ${}^p\mathcal{D}^{\leq i} := {}^p\mathcal{D}^{\leq 0}[-i]$, ${}^p\mathcal{D}^{\geq i} := {}^p\mathcal{D}^{\geq 0}[-i]$):

1. ${}^p\mathcal{D}^{\leq -1} \subseteq^p \mathcal{D}^{\leq 0}$, ${}^p\mathcal{D}^{\geq 1} \subseteq^p \mathcal{D}^{\geq}0$.
2. $\mathrm{Hom}_{\mathcal{D}}({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 1}) = 0$.
3. Every $K \in \mathcal{D}$ gives rise to a distinguished triangle
$$K' \longrightarrow K \longrightarrow K'' \longrightarrow K'[1], \qquad \text{with } K' \in {}^p\mathcal{D}^{\leq 0} \text{ and } K'' \in {}^p\mathcal{D}^{\geq 1}.$$

The first remark is that the distinguished triangle is functorial, thus defining functors $K \mapsto K'$ and $K \mapsto K''$ which is natural to call truncations ${}^p\tau_{\leq 0}$ and${}^p\tau_{\geq 1}$ and a a natural transformation ${}^p\tau_{\geq 1} \to [1] \circ {}^p\tau_{\leq 0}$.
One sets ${}^p\tau_{\leq i}K = ({}^p\tau_{\leq 0}K[i])[-i]$. Similarly, for ${}^p\tau_{\geq i}$.
The standard truncations §2.4.4 define the standard t-structure on the homotopy and derived categories.
All the properties (and more) discussed in §2.4.4 for the (standard) truncation hold for any t-structures.
We have that the full subcategory ${}^p\mathcal{C} := {}^p\mathcal{D}^{\leq 0} \cap {}^p\mathcal{D}^{\geq 0}$ (the heart of the t-structure) is an Abelian category.
We have cohomological (this means a distinguished triangle yields a les) functor ${}^pH : \mathcal{D} \to {}^p\mathcal{C}$ called the cohomology functor of the t-structure: ${}^pH(K) = {}^p\tau_{\leq 0}]{}^p\tau_{\geq 0}K$.
One sets ${}^pH^i := {}^pH \circ [i]$, so that ${}^pH^iK[-i] = {}^p\tau_{\leq i}{}^p\tau_{\geq i}K$.

Let $e : K \to K[r]$ be an arrow in $\mathcal{D}$.
We get, completely formally:

$$ {}^{p}\!H^{i}(e) : {}^{p}\!H^{i} K \longrightarrow {}^{p}\!H^{i+r} K. $$

We say that K is p-split if there is an iso

$$ \bigoplus_{i} {}^{p}\!H^{i} K[-i] \stackrel{\cong}{\longrightarrow} K. $$

4.4.2 Wouldn't it be nice?

We are in the position to state what would be desirable in terms of a generalization of Deligne theorem to proper maps of compelx algebraic varieties:
For every variety X there should be a full subcateogry $\mathcal{D}_X$ of the derived category of the category of sheaves of rational vector spaces on X, a t-structure p on $\mathcal{D}_X$ such that, for every proper map $f : X \to Y$ of complex algebraic varieties, with X nonsingular and for every f-ample $\eta \in H^2X$, we have that:

1. $Rf_*\mathbb{Q}_X$ is p-split in $\mathcal{D}_Y$.
2. ${}^{p}\!H^{i} Rf_*\mathbb{Q}_X$ is semisimple in ${}^{p}\mathcal{C}_Y$.
3. $\eta^i : {}^{p}\!H^{-i}(Rf_*\mathbb{Q}_X[\dim X]) \stackrel{\cong}{\longrightarrow} {}^{p}\!H^{i}(Rf_*\mathbb{Q}_X[\dim X])$ in ${}^{p}\mathcal{C}_Y$, for every $i \geq 0$.

This can all be done. In fact, more can be done.
Let us first say what the categories $\mathcal{D}_X$ are.
below instead duality ....

4.4.3 $f^!$ and duality

Recall that (f^*, Rf_*) are an adjoint pair.
It is a theorem (due to Verdier) that there is a functor $f^!$ (exceptional inverse image, "f upper-shriek") which is the right adjoint to $Rf_!$, $(Rf_!, f^!)$ (Hom's are in the respective derived categories):

$$ \mathrm{Hom}(Rf_!K, C) = \mathrm{Hom}(K, f^!C), \qquad Rf_*R\mathcal{H}om(Rf_!K, C) = R\mathcal{H}om(K, f^!C). $$

As in Exercise §2.4.2.12, we get, functorially, the adjonction maps

$$ Rf_!f^!K \longrightarrow K, \qquad C \longrightarrow f^!Rf_!C. $$

If $f = j$ is an open immersion (or smooth of relative dimension zero), then $j^! = j^*$.
If it is smooth in relative dimension d, then $f^! = f^*[d]$.

If $f = i$ is a locally closed immersion, then $f^! = f^* R\Gamma_X$ (if F is a sheaf on Y, then $\Gamma_X F$ is the sheaf (on Y!) of sections of F supported on X, Γ_X is left-exact, $R\Gamma_X$ is the right derived funcctor; $H(Y, R\Gamma_X(C)) = H_X(Y, C)$ (cohomology with supports on X).

Exercise 4.4.3.1 (**Relative cohomology**) Let $X = U \coprod Z$, where $i : Z \to X$ is a closed immersion and $j : U \to X$ is an open one. Let F be a sheaf on X. Show that there is a functorial exact sequence

$$0 \longrightarrow j_! j^* F \longrightarrow F \longrightarrow i_* i^* F \longrightarrow 0$$

Apply this to an injective resolution of F and deduce that there is an funcotrial exact sequence (more precisely, a distinguished triangle): (here, $Rj_! = j_!$, $j^* = j^!$, $Ri_* = i_*$)

$$j_! j^! C \longrightarrow C \longrightarrow i_* i^* C \longrightarrow .$$

Note that the first two arrows are the adjonction maps for $(Rj_!, j^!)$ and (i^*, Ri_*). The corresponding les is the les of relative cohomology $HX, Z \to HX \to HZ \stackrel{+1}{\to}$. Show that there is a funcotrial ses

$$0 \longrightarrow i_* i^! F \longrightarrow F \longrightarrow j_* j^* F \longrightarrow 0.$$

Derive it as above to obtain

$$i_* i^! C \longrightarrow C \longrightarrow Rj_* i^* C \longrightarrow .$$

Note that the first two arrows are the adjunction maps for $(Ri_!, i^!)$ and (j^*, Rj_*). The corresponding les is the les of relative cohomology $HX, U \to HX \to HU \stackrel{1}{\to}$. Moreover, $HX, U = H_Z X$.

Exercise 4.4.3.2 Let Let $i : o \to \mathbb{R}^n$ be the closed immersion of a point. Use the les of cohomology sheaves of the "ses" $0 \to i_* i^! \to \mathrm{Id} \to Rj_* j^* \to 0$ to prove that $i^! \mathbb{Q}_{\mathbb{R}^n} \cong \mathbb{Q}_o[-n]$.

Exercise 4.4.3.3 Let $i : Z \to X$ be a closed embedding. Prove that $i_* = Ri_*$ is fully faithful (map on Hom's is an iso). In particular, show that $\mathrm{Hom}(i_* K, i_* C) = \mathrm{Hom}(K, C)$.

Exercise 4.4.3.4 (**The map** $i^! \to i^*$) Let $i : Z \to X$ be a closed embedding. Use the natural adjunction maps to get the natural map

$$i^! C \longrightarrow i^* C$$

inducing

$$H^k(Z, i^! C) \longrightarrow H^k(Z, i^* C), \qquad H^k_c(Z, i^! C) \longrightarrow H^k_c(Z, i^* C).$$

Observe that if Z is compact, then the two maps coincide.

Define the dualizing complex ω_X of X by setting

$$\omega_X := \gamma^! \mathbb{Q}_{pt}, \qquad \gamma : X \to pt.$$

Since it is a fact that $(g \circ f)^! = g^! \circ f^!$, we have that, if $f : X \to Y$, then $\omega_X = f^! \omega_Y$. The dualizing complex is canonically isomorphic (in the derived category) to the shifted complex $\mathfrak{D}_X[2\dim_{\mathbb{C}} X$ of Borel-Moore chainson X and we have

$$H^k(X, \omega_X) = H^{BM}_{2n-k} X.$$

[13] explains this statement and illustrates that ω_X has cohomology sheaves only in the interval $[-2\dim_{\mathbb{C}} X, 0]$, shows how to compute this complex using links so that one can be convinced that if X is nonsingular, then $\omega_X = \mathbb{Q}_X[2n]$.

Define the duality functor $\mathbb{D} : C \to C^\vee := R\mathcal{H}om(C, \omega_X)$.
Off the bat, note that, by adjunction $\omega_X = \mathbb{Q}_X^\vee$.
We have: $[k] \circ \mathbb{D} = \mathbb{D} \circ [-k]$.
We have: $\mathbb{D}^2 = \mathrm{Id}$ (canonical iso).
We have the important:

$$\mathbb{D}_Y \, Rf_! \, \mathbb{D}_X = Rf_*, \qquad \mathbb{D}_X \, f^! \, \mathbb{D}_Y = f^*.$$

Exercise 4.4.3.5 (**Poincaré duality**) The importance of the duality functor becomes manifest on X nonsingular, with $K = \mathbb{Q}_X$, $C = \omega_{pt} = \mathbb{Q}_{pt}$. Show (using $(Rf_!, f^!)$) that:

$$\mathrm{Hom}(R\Gamma_c(X, \mathbb{Q}_X), \mathbb{Q}_{pt}) = \mathrm{Hom}(\mathbb{Q}_X, \mathbb{Q}_X[2n]),$$

which gives, after taking i-th cohomology

$$(H_c^{-i} X)^\vee = H^{i+2n} X \qquad \text{(Poincaré duality).}$$

Exercise 4.4.3.6 (**Stalks of $H^i C^\vee$**) Use the argument above to show that:

$$H_c^{-i}(X, C)^\vee = H^i(X, C^\vee).$$

Deduce that (the limit is a *direct* limit) (the first = is general):

$$(H^i(C^\vee)_x =) \;\; H^i(i_x^* \, C^\vee) = \varinjlim_{U \ni x} H_c^{-i}(U, C)^\vee.$$

Exercise 4.4.3.7 (**Lefschetz duality**) Let X be a nonsingular variety of dimension n and $i : Z \to X$ be a closed embedding. Use the composition $Z \to X \to pt$ to deduce that $\omega_Z = i^! \mathbb{Q}_X[-2n]$. Deduce that

$$(H^k X, X \setminus Z =) \;\; H^k(Z, i^! \mathbb{Q}_X) = H^{BM}_{2n-k} Z.$$

Use the intepretation of Poincaré duality in terms of the non degenerate intersection pairing

$$H_k X \times H^{BM}_{2n-k} X \longrightarrow Q$$

to interpret the natural map of Exercise 4.4.3.3

$$H_{2n-k} Z \longrightarrow H^k Z$$

as follows: take a Borel-Moore $(2n-k)$-cycle in Z, push it forward via the (proper) closed embedding i to the "same" cycle viewed in the BOrel-Moore homology of X, apply th eintersection form above on X to this cycle paired with a variable homology cycle on X, thus obtaining a cohomology class on X which, finally, we view as a cohomology class on Z once we pair it with the homology cycles in the image from Z. Make precise the fact that the refined intersection product in §1.0.6 is a special case.

Exercise 4.4.3.8 Let X be contractible, nonsingular manifold of dimension n. Let C be a bounded (may work in general ..., not sure) complex with $L_i := H^i C$ local systems (necessarily constant)

1. Observe that there are no higher extensions among the L_i:

$$\mathrm{Hom}^{>0}(L_i, L_j) = 0$$

2. Prove that C is s-plit: $C \cong \oplus_i L_i[-i]$. (Induction using truncation and the observation above).

3. Show that $C^\vee \cong \oplus L_i^*[2n+i]$ (L^* the dual local system, here iso to L_i).

4. Do not assume that X is contractible anymore. Prove, by working on contractible open sets, that $C^\vee$ is also a bounded complex with locally constant cohomology sheaves $M_i := H^i(C^\vee)$. Show the following: If $L_i = 0$ for $i \notin [a, b]$, then $M_i = 0$ for $i \notin [-2n-b, -2n-a]$. (In fact $M_j = L^*_{-j-2n}$). We may say: if C lives before b, then $C^\vee$ lives after $-2n-b$.

4.4.4 Constructible bounded derived category

A sheaf of finite dimensional rational vector spaces on a variety Z is constructible if there is a partition $Z = \coprod Z_a$ into locally closed subvarieties such that $F_{|Z_a}$ is a local system for every a.
A complex of sheaves of rational vector spaces is said to be constructible if all of its cohomology sheaves are constructible.

Note that the de Rham complex E_Z on a nonsingular Z is constructible, but the E_Z^i are not.
Let $\mathcal{D}_Z$ be the full subcategory of the derived category of the category of sheaves of rational vector spaces whose objects are the bounded constructible complexes.
These categories are stable under all the functors we use (and more):

$$Rf_*, Rf_!, f^*, f^!, R\mathcal{H}om, \otimes, \mathbb{D}, \ldots$$

For example: if $f : X \to Y$ is a map of varieties, then $R^i f_* \mathbb{Q}_X$ is a constructible sheaf on Y.
Another example is that the dualizing complex $\omega_X = \mathbb{Q}_X^\vee$ is constructible. This is clear in view of $\omega_X \cong \mathfrak{D}_X$ (complex of Borel-Moore chains) and the explicit description of the cohohology sheaves $H^i \mathfrak{D}_X$ via links.
Let me try to explain why this is the case.
Let $f : X \to Y$ be a *proper* map of algebraic varieties.
Then by fundamental results of Thom-Mather, there is a partition of $Y = \coprod Y_a$ into locally closed subvarieties, s.t. $f_a : X_a := f^{-1} Y_a \to Y_a$ is topologically locally trivial for every a.
This maps is clear that we have the constructibility statement for $R^i f_{a*} \mathbb{Q}$.
Since the map is proper, by proper base change, $(R^i f_* \mathbb{Q})_{|Y_a} = R^i f_{a*} \mathbb{Q}$, so that $R^i f_* \mathbb{Q}$ is constructible.
If f is not proper, it is a fact that it can be "embedded" into a proper one and can deduce the wanted statement.
The case of C constructible (we discussed $C = \mathbb{Q}_X$ above) is dealt-with similarly (but one needs a bit more technical machinery, namely the local product structure of C in these neighborhoods).
What makes the constructible derived category something we can work with (and the basis for all that above) is that constructible complexes have, locally around the strata of a suitable stratification of X a product structure (see [14], §3.5 and/or [12], §1.5.5 for a summary). This leads to the notion of standard neighborhoods of a point $x \in X$. The most important feature of constructibility is then that, for each k, as U varies among the open neighborhoods of x direct system $H^k(U, C)$ and the inverse system $H_c^k(U_x, C)$ are "eventually constant" ([15], §3), so that the corresponding limits are already attained by a suitably small standard neighborhood $\mathcal{U}$ (and then by any smaller standard neighborhood as well):

$$\varinjlim_{U \ni x} H^k(U, C) = H^k(\mathcal{U}, C), \qquad H_c^k(\mathcal{U}, C) = \varprojlim_{U \ni x} H_c^k(U, C).$$

Compare this situation with the skyscreper sheaf with stalks $\mathbb{Q}$ over the Cantor set.

Exercise 4.4.4.1 (The map $i_x^! \to i_x^*$)

1. Use duality to show that the same two equalities above can be derived from each other. (Hint: it holds for every k and every C, so it holds for $-k$ and $C^\vee$.)

2. Use Exercise 4.4.3.6 and the two equalities above to deduce the following identity for constructible complexes C (not sure how general this is):

$$H^k(i_x^! C) = H^{-k}(i_x^* C^\vee)^\vee.$$

(Hint: we know $H^k(\mathcal{U}, C) \cong H^k(i_x^* C)$; a duality argument shows $H^k(i_x^! C) \cong H_c^k(\mathcal{U}, C)$; now apply Verdier duality.)

3. Show that the natural map $i_x^! \to i_x^*$ of Exercise 4.4.3.4 can be viewed as the natural maps

$$H_c^k(\mathcal{U}, C) \longrightarrow H^k(\mathcal{U}, C), \qquad H^{-k}(i_x^* C^\vee)^\vee \longrightarrow H^k(i_x^* C).$$

4. If C is self dual, conclude that the natural map $i_x^! \to i_x^*$ can be viewed as a natural pairing (map between space and its dual; compare with the refined intersection form in Exercise 4.4.3.7 and in §1.0.6):

$$H^{-k}(i_x^* C)^\vee \longrightarrow H^k(i_x^* C)$$

Remark 4.4.4.2 (**$i_x^! \to i_x^*$ and splitting**) Let $f : X \to Y$ be the non complex algebraic contraction of the zero section (to a point $v \in Y$) of the trivial line bundle on a curve C (Example 1.0.1.1). As seen in **?????**, $P := Rf_* \mathbb{Q}_X[2]$ is a perverse sheaf on Y. The map $i_v^! P \to i_v^* P$ is the zero map between one dimensional vector spaces. The complex P has composition series with exactly 2 constituents (non trivial simple quotients) and it does not split into the sum of its constituents (it is thus reducible but not semisimple). In the complex algebraic contraction of Example 1.0.1.2, instead, the map $i_v^! P \to i_x^! P$ is an iso and P splits and is semisimple. As we shall see, for self-dual perverse sheaves, the map $i_x^! \to i_x^*$ controls exactly the splitting behaviour of a perverse sheaf at x. This turns out to be of fundamental importance for a geometric proof of the decomposition theorem.

Remark 4.4.4.3 (**Whitney stratifications**) We did not mention Whitney stratifications ([17]) in the definition of constructible sheaves/comlexes. That is because of a fact (not easy to prove; need to put together some results in [7])) that the definition employed here ensures that the "stratificatons" employed can be refined to a Whitney stratifications and the complex will be still of course constructible in the sense defined here, but also have a local product structure (mentioned few paragraphs above). Then one proves, for example a precise form of the stability of constructible complexes wrt the functors Rf_* etc.: e.g. if $f : X \to Y$ is stratified in this strong sense and C is constructible (strong sense) on X, then Rf_*C is constructible on Y (strong sense); similarly, for duality etc. See [15].

4.4.5 The definition of perverse sheaves

In what follows X does not need to be irreducible, nor pure-dimensional.
Recall that the support $|F|$ of a sheaf F is the closure of the set of points where $F_x \neq 0$. The support $|K|$ of a complex $K \in \mathcal{D}_X$ is the (Zariski closed) support $|\oplus_i H^i K|$.
Recall that $C^\vee := \mathbb{D}(P)$ is the Verdier dual to $C \in \mathcal{D}_X$.

Definition 4.4.5.1 (**Perverse sheaf**) We say a complex $P \in \mathcal{D}_X$ is a perverse sheaf if:

$$\dim |H^k P| \leq -k, \qquad \dim |H^k P^\vee| \leq -k.$$

The two conditions are called the conditions of support and of co-support.

Since this is the most important definition in these lectures, and it is not easy to digest, let us view it from different angles and give 3 additional equivalent formulations.
There is a partition $X = \coprod X_a$ with X_a locally closed nonsingular subvarieties with $(H^i P)_{|X_a}$ and $(H^i P^\vee)_{|X_a}$ locally constant for all k and all a.

1. The condition of support is equivalent to having:

   $$\forall\, a \;:\; H^k i^*_{X_a} P = 0, \quad \forall k > -\dim X_a$$

   and also to having:

   $$\forall\, a \;:\; i^*_{X_a} P \in \mathcal{D}^{\leq -\dim X_a}_{X_a}.$$

   The condition of co-support is then equivalent to having

   $$\forall\, a \;:\; i^*_{X_a}(P^\vee) \in \mathcal{D}^{\leq -\dim X_a}_{X_a}$$

2. We dualize the above (use the identities $\mathbb{D} i^* \mathbb{D} = i^!$ and $\mathbb{D}^2 = \mathrm{Id}$, apply Exercise 4.4.3.8.(4)) and get that the condition of co-support is equivalent to having

   $$\forall\, a \;:\; i^!_{X_a} P \in \mathcal{D}^{\geq -\dim X_a}_{X_a}.$$

and thus it is also equivalent to having

$$\forall\, a \ : \ H^k i^!_{X_a} P = 0, \quad \forall k < -\dim X_a$$

It follows that we could have defined the category of perverse sheaves by requiring that, for all a: (the two conditions below are exchanged by $\mathbb{D}$)

$i^*_{X_a} P$ *is concentrated in cohomological degrees* $\leq -\dim X_a$

and

$i^!_{X_a} P$ *is concentrated in cohomological degrees* $\geq -\dim X_a$.

3. Recalling that $H^k i_x^{! P = H^{-k}(i_x^* P^\vee)^\vee}$ (Exercise 4.4.4.1.(2), we have that P is perverse iff

$$\dim\{x \mid H^k i^*_x P \neq 0\} \leq -k, \qquad \dim\{x \mid H^k i^!_x P \neq 0\} \leq k.$$

These conditions admit a nice visual exemplication: see [12], p.556, Figure 1, which also displays the analogous conditions of support and co-support for intersection cohomology complexes. That picture was given to us by M. Goresky.

Let $\mathcal{P}_X \subseteq \mathcal{D}_X$ be the corresponding full subcategory of perverse sheaves.
It is clear that the duality functor preserves the category of perverse sheaves.
We denote by $\mathcal{D}_X^{[a,b]}$ the full subcategory of objects C with $H^k C = 0$ for every $k \notin [a, b]$.

Proposition 4.4.5.2 *If* $P \in \mathcal{P}_X$*, then* $P \in \mathcal{D}_X^{[-\dim X, 0]}$.
More precisely, if $|P|$ *is a union of strata* X_b *satisfying* $s \leq \dim X_b \leq d$*, then* $P \in \mathcal{D}_X^{[-d,-s]}$.

Proof. Note that the conditions of support imply at once that $P \in \mathcal{D}_X^{\leq -s}$ so that we consider only the remaining inequality ("$\geq -d$").
$P = 0$?, then nothing left to prove.
We may assume $X = |P|$ and we may assume $\dim X = d$.
Induction on the number # of strata contained in X.
If $\# = 1$, then X is that stratum and it is nonsingular of dimension d.
We have $P \in \mathcal{D}_X^{\leq -d}$ (support).
By Exercise 4.4.3.8.(4), we have $P^\vee \in \mathcal{D}_X^{\geq -d}$ and, by co-support, the opposite inequality.
$P^\vee$ is thus a local system in degree $-d$ and so is P (base case of induction OK).
Assume $(\# - 1)$ is OK.
Let X have # strata. One of them, S, must be closed.

We can write $X = U \coprod S$, where U is the union of the remaining $\# - 1$ strata and $j : U \to X \leftarrow S : i$ are open/closed embeddings.
We have the ses (distinguished triangle):

$$\tau_{\leq -d-1} P \longrightarrow P \longrightarrow \tau_{\geq -d} P \longrightarrow$$

Since $i_S^! = i_S^* R\Gamma_S$, this functor sends $\mathcal{D}_X^{\geq l} \to \mathcal{D}_X^{\geq l}$ for every l.
Apply this to the ses above, take the les and observe that:

$$H^k i_S^! \tau_{\leq -d-1} P \xrightarrow{\cong} H^k i_S^! P = 0, \qquad \forall k < -d,$$

where the $= 0$ is from the cosupport condition.
Using the adjunction maps, we get a natural map $i_S^! \to i_S^*$. This map is iso when we feed it something supported on S.
By the inductive hypothesis, $|\tau_{\leq -d-1} P| \subseteq S$.
It follows that

$$i_S^! \tau_{\leq -d-1} P \xrightarrow{\cong} i_S^* \tau_{\leq -d-1} P$$

so that (using i_S^* is exact):

$$H^k i_S^! \tau_{\leq -d-1} P \xrightarrow{\cong} H^k i_S^* \tau_{\leq -d-1} P = H^k \tau_{\leq -d-1} i_S^* P, \qquad \forall k.$$

The lhs is zero for $k \leq -d - 1$, so is the rhs.
Finally, take the les of cohomology sheaves of

$$j_! P_{|U} \longrightarrow P \longrightarrow i_* i^* P \longrightarrow .$$

By the inductive hypothesis on $P_{|U}$, and by using that $j_!$ and i_{S*} are exact, we get isos:

$$H^k P \xrightarrow{\cong} H^k i_{S*} i_S{}^* P = i_{S*} H^k i_S^* P = i_{S*} H^k i_S^* \tau_{\leq -d-1} P, \qquad \forall k \leq -d - 1.$$

We have proved above that the rhs is zero and we are done. □

Exercise 4.4.5.3

1. Let X be nonsingular of dimension d and L be a local system on it. Show $L[d]$ is perverse. In particular, $\mathbb{Q}_X[d]$ is perverse (and clearly, self-dual).

2. Let $i : o \to \Delta \leftarrow \Delta^* : j$ be the usual thing for the uni disk $\Delta \subseteq \mathbb{C}$. Let L be a local system on Δ^*. Show that $Rj_* L[1]$, $Rj_! L[1] = j_! L[1]$ and $j_* L[1]$ are perverse on Δ.

3. (This is not really an exercise, just a funny observation) As above, with $L = \mathbb{Q}_{\Delta^*}$. In the following ses, nothing is a perverse sheaf, except the quotient:

$$0 \longrightarrow j_! \mathbb{Q}_{\Delta^*} \longrightarrow \mathbb{Q}_\Delta \longrightarrow i_* \mathbb{Q}_o \longrightarrow 0.$$

   Shift by [1]:

$$0 \longrightarrow j_! \mathbb{Q}_{\Delta^*}[1] \longrightarrow \mathbb{Q}_\Delta[1] \longrightarrow i_* \mathbb{Q}_o[1] \longrightarrow 0.$$

   Now the first two are perverse, but not the third. (I wrote the above as a ses, that is fine, it is a ses of complexes. But it is not a ses in the category of perverse sheaves! The category of perverse sehaves is indeed Abelian. A ses in it is a distinguised triangle in the derived category and viceversa a distinguished triangle withwhose entries are perverse sheaves is a ses of perverse sheaves.) Believe that the category of perverse sheaves is Abelian. "Turn" the triangle aboe into

$$i_* \mathbb{Q}_o \longrightarrow j_! \mathbb{Q}_{\Delta^*}[1] \longrightarrow \mathbb{Q}_\Delta[1] \longrightarrow .$$

   Now everything is perverse. Conclusion, the above is a ses in $\mathcal{P}_\Delta$ so that $j_! \mathbb{Q}^*_\Delta$ is a subsheaf of $\mathbb{Q}_\Delta$, but, under the same map!, $\mathbb{Q}_\Delta[1]$ is a quotient of $j_! \mathbb{Q}_{\Delta^*}[1]$ in the category of perverse sheaves. Conclusion: one needs to be careful to use set-theoretic intuitions.

4. Let $f : X \to Y$ be a finite map (= proper + finite fibers) and P be a perverse sheaf on X. Then $Rf_* P = f_* P$ is perverse on Y.

5. Let $f : X \to Y$ be a proper map from X nonsingular of dimension $2m$ contracting exactly one n-dimensional subvariety. Prove that $Rf_* \mathbb{Q}_X[2n]$ is perverse on X.

6. Generalize the above as follows. Assume that $f : X \to Y$ is proper, X is nonsingular of dimension n and that $\dim X \times_Y X \le n$ (it is always $\ge n$ (these maps are called semismall). Show that $Rf_* \mathbb{Q}_X[n]$ is perverse on Y.

7. Same as above, but assume in addition that there is only one irrducible component of $X \times_Y X$ of dimesnion n (these maps are called small; e.g. the blowing up of the cone over the nonsingular quadric in $\mathbb{P}^3$ along one of the planes through the origin). Show that $P := Rf_* \mathbb{Q}_X[n]$ (which is perverse by the previous exercise) satisfies some stronger conditions than the support/co-support ones, namely: $\dim |H^i P| \le -i-1$ and $\dim |H^i(P^\vee| \le -i-1$ for every $i \in [-n+1, 0]$. In particular, $H^0 P = H^0 P^\vee = 0$. Note that the only intersection complex we have met so far, the one of surfaces targets of contractions, satisfy these stronger requirements.

8. Note that the stronger condtions above are not met by $Rj_*\mathbb{Q}_{\Delta^*}[1]$, nor by $j_!\mathbb{Q}_{\Delta^*}[1]$, but that they are met by $j_*\mathbb{Q}_{\Delta^*}[1]$, whic is in fact self-dual. Discuss the analogous situation with a local system L on Δ^*.

9. Let $i : Z \to X$ be a closed subvariety. Show that $i_* = Ri_*$ induces a full embedding $\mathcal{P}_Z \to \mathcal{P}_X$. For this reason, if P is perverse on Z, it is costumary to identify it with Ri_*P (and viceversa).

10. Show that all the complexes $\mathscr{I}$ in §4.3.2 example 1, 2,3 and 4 are perverse. This is the reason why we used the dimensional shifting. The only non trivial one is the 5th example.

11. Do the same for the 5th example. This is not so immediate and it has interesting features. First show that $Rf_*\mathbb{Q}_X$ is perverse by using the support/co-support conditions. Deduce that the summands are perverse. We have just proved that the intersection complex on a singular surfcae is a perverse sheaf.

12. (More about the above) Deduce that there is an iso $a : \mathscr{I}_Y \oplus \mathscr{I}_v \cong \mathscr{I}_Y^\vee \oplus \mathscr{I}_v^\vee$. Consider the 4 components of this iso and show that $\mathscr{I}_Y \to \mathscr{I}_v^\vee$ must be zero (see Exercise 3.4.2.2). By looking at the inverse iso a^{-1}, deduce that the map $\mathscr{I}_v \to \mathscr{I}_Y^\vee$ is also zero. Deduce that a is diagonal so that the summands are self-dual. We have just proved that the intersection complex of a singular surface is self-dual.

13. (More about the above) Prove that the intersection complex $\mathscr{I}_Y$ of the surface Y is a simple object in the category of perverse sheaves on Y.

14. ([13], §4.3.5, 4.3.6) The complex $\mathbb{Q}_C[1]$, C any curve is perverse. The complex $\mathbb{Q}_S[2]$ is perverse if S has only unibranch singularities. Find a natural condition for the perversity of $\mathbb{Q}_Y[\dim Y]$, for Y with isolated singularities in any dimension.

4.4.6 The perverse t-structure on $\mathcal{D}_X$

There is a t-structure $\mathfrak{p}$ on $\mathcal{D}_X$ (constructible bounded derived cateogry of sheaves of rational vector spaces on X) and its heart is the category of perverse sheaves $\mathcal{P}_X$, which is therefore Abelian.
It is easy to define the full subcateogry:

$$^{\mathfrak{p}}\mathcal{D}_X^{\leq 0} := \{K \in \mathcal{D}_X \mid \dim |H^i K| \leq -i, \forall i\}.$$

Set

$$^{\mathfrak{p}}\mathcal{D}_X^{\geq 0} := \mathbb{D}\left({}^{\mathfrak{p}}\mathcal{D}_X^{\leq 0}\right).$$

Axiom 1 in §4.4.1 is verified easily.
Axiom 2 requires some work.

Exercise 4.4.6.1 Prove that axiom 2 holds. Here are the main steps. (This is just an adaptation of [16] to this special case). It is enough to prove the statement for two complexes K and C which are constructible with respect to a fixed stratification $X = \coprod X_a$. On the strata of top dimension K and C are complexes that live in degrees ≤ 0 and ≥ 1, repsectively so we can use Exercise 3.4.2.2. The key point becomes the following. Assume $X = U \coprod S$ with $i : S \to X$ and $j : U \to X$, with U union of strata and S the new stratum (U open, S closed). The inductive hypothesis is that axiom 2 holds on U and S separately (for the complexes constructible with respect to the stratification we are using). Verify that $j^*K \in {}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_U$, $i^*K \in {}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_S$ and that $j^*C \in {}^{\mathfrak{p}}\mathcal{D}^{\geq 1}_U$, $i^!C \in {}^{\mathfrak{p}}\mathcal{D}^{\geq 1}_S$. Apply Hom to the ses (distinguished triangle) $i_*i^*K \to K \to j_!j^*K \to$ and get an es:

$$\mathrm{Hom}(i_*i^*K, C) \longrightarrow \mathrm{Hom}(K, C) \longrightarrow \mathrm{Hom}(j_!j^*K, C).$$

Note that $\mathrm{Hom}(i_*i^*K, C) = \mathrm{Hom}(i^*K, i^!C) = 0$ (inductive hypothesis for S) and that $\mathrm{Hom}(j_!j^*K, C) = \mathrm{Hom}(j^*K, j^*C) = 0$ (inductive hypothesis for U).

The key point is the existence of the distinguished triangle, i.e. the construction of the perverse truncation ${}^{\mathfrak{p}}\tau_{\leq 0}$ and ${}^{\mathfrak{p}}\tau_{\geq 1}$ ([16, 7]).
Let me refer the reader to [19], §4.1 where a short account of the follwoing facts is given: construction of perverse truncation functors, exchanges via duality, i.e.

$$\mathbb{D} \circ {}^{\mathfrak{p}}\tau_{\leq k} \circ \mathbb{D} = \circ\, {}^{\mathfrak{p}}\tau_{\geq -k}, \qquad \mathbb{D} \circ {}^{\mathfrak{p}}H^k = {}^{\mathfrak{p}}H^{-k} \circ \mathbb{D},$$

truncations and shift, i.e. (symbolically):

$$[-l] \circ \tau_k \circ [l] = \tau_{k+l}, \qquad H^k \circ [l] = H^{k+l}$$

The functor ${}^{\mathfrak{p}}H^0$ is cohomological: a distinguished triangle $K \to K' \to K'' \to K[1]$ is in particular an infinite diagram of maps in the derived category with two consecutive arrows composing to zero. This gives a diagram $\ldots \to {}^{\mathfrak{p}}H^j(K) \to {}^{\mathfrak{p}}H^jK' \to {}^{\mathfrak{p}}H^jK'' \to {}^{\mathfrak{p}}H^{j+1}K \to \ldots$ which is automatically a complex by what above. One then verifies it is exact ([16], p.31-32).
We are not going into that in these lectures but let us at least work out one example using the formulae in [19].§4.1.

Example 4.4.6.2 Let $D = D^* \coprod o$ be the unit disk with the indicated stratification Let $K = \mathbb{Q}_D[1] \oplus \mathbb{Q}_D$. Note that $\mathbb{Q}_D[1]$ is perverse and $\mathbb{Q}_D = \mathbb{Q}_D[1][-1] \in \mathcal{P}_D[-1] \subseteq$

${}^{\mathfrak{p}}\mathcal{D}^{\geq 1}_{D}$. We thus exapect that ${}^{\mathfrak{p}}\tau_{\leq 0}K = \mathbb{Q}_D[1]$ (i.e. we lose $\mathbb{Q}_D$). We verify this. Here are the relevant formulæ: $i : o \to D \leftarrow D^* : j$, define $\tau'_{\leq 0}$ and $\tau''_{\leq}0$ as follows:

$$\tau'_{\leq 0}F \longrightarrow F \longrightarrow Rj_* \, {}^{\mathfrak{p}}\tau^{D^*}_{\geq 1} j^* F \longrightarrow, \qquad \tau''_{\leq 0}F \longrightarrow F \longrightarrow i_* \tau_{> -\dim o} i^* F \longrightarrow,$$

$${}^{\mathfrak{p}}\tau^{D}_{\leq 0}F := \tau''_{\leq 0}\tau'_{\leq 0}F.$$

(with the following understanding: this is an inductive procedure from a union of previous strata U to a the adjunciton of a new stratum $U \coprod S$; perverse truncations are assumed to be known on U; the starting point is a one stratum space, which is then smooth and there the complexes have locally constant cohomology sheaves and are truncated using the standard truncation with the dimensional shift ${}^{\mathfrak{p}}\tau_{\leq 0} = \tau_{\leq -\dim}$. Finally, one has to verify that the τ's above are functors.).
Apply $\tau'_{\leq 0}$ to K: (n.b.: ${}^{\mathfrak{p}}\tau^{D^*}_{\geq 1}(\mathbb{Q}_{D^*}[1]) = 0$)

$$? \longrightarrow \mathbb{Q}_D[1] \oplus \mathbb{Q}_D \longrightarrow Rj_* \mathbb{Q}_{D^*} \longrightarrow$$

So that $? = \mathbb{Q}_D[1] \oplus i_! i^! \mathbb{Q}_D$.
Apply $\tau''_{\leq 0}$ to this and verify that you loose $\mathrm{i}_! i^! \mathbb{Q}_D$, so that only $\mathbb{Q}_D[1]$ remains.

Exercise 4.4.6.3 Do the same as above on a d-dimensional disk with $K = \oplus_{i=-d}^{d} \mathbb{Q}_D[d][-i]$.

Once we have the perverse truncations, we have the perverse cohomology functors ${}^{\mathfrak{p}}H^i : \mathcal{D}_X \to \mathcal{P}_X$.
Given a map of perverse sheaves $a : P \to Q$ we form the cone $P \to Q \to C \to$, take the les of perverse cohomology sheaves and get the les of perverse sheaves

$$0 \longrightarrow {}^{\mathfrak{p}}H^{-1}C \longrightarrow P \longrightarrow Q \longrightarrow {}^{\mathfrak{p}}H^0 C \longrightarrow 0.$$

This way one verifies that one has kernels cokernels, etc.
We have maps

$$\ldots \, {}^{\mathfrak{p}}\tau_{\leq i-1} \longrightarrow {}^{\mathfrak{p}}\tau_{\leq i}C \longrightarrow \ldots \longrightarrow C.$$

Using injecitve resolutions as in §**???**, we obtain a filtered complex and hence a spectral sequence: the perverse spectral sequence. If $f : X \to Y$, $C \in \mathcal{D}_X$ and $K := Rf_*C$, then we get the perverse Leray spectral sequence as the perverse spectral sequence for $K := Rf_*C$. (You can avoid using injectives; see [30].)

We say that K is $\mathfrak{p}$-split, if there is an iso in the derived category

$$\bigoplus_i {}^{\mathfrak{p}}H^i K[-i] \xrightarrow{\cong} K.$$

Given $\eta \in H^2X$ ample, we get (§???)

$$\eta : {}^{\mathfrak{p}}H^iRf_*C \longrightarrow {}^{\mathfrak{p}}H^iRf_*C.$$

We are ready to state the decomposition theorem and RHL, including the semisimplicity statement.
However, before doing that, let us first show that like the category of local systems, but unlike the category of sheaves, the category of perverse sheaves (rational coefficients!) is Artinian. Otherwise, semsimplicity would be impossible (see the discussion in ????).

4.4.7 Artinianity and Jordan-Hölder

Exercise 4.4.7.1 (**Glueing sheaves**) Let F be a sheaf on X and $j : U \to X \leftarrow Z : i$ be complementary open/closed embeddings. Show that the category of sheaves on X is equivalent to the category of triples $(A, B, u : B \to i^*j_*A)$, A sheaf on U, B on Z. F itself corresponds to $(j^*F, i^*F, i^*F \to i^*j_*j^*F)$, where the map is given by adjunction.

Exercise 4.4.7.2 (**Category of constructible sheaves: Noetherian, not Artinian**) Use the Noetherianity of varieties to show that the category of constructible sheaves is Noetherian (ascending chain condition ok). Show that it Artinian iff $\dim X = 0$ (descending chain condition fails). With integer coefficients, it fails for $X = pt$. (Hint for Noetherianity: let $F_i \subseteq F$ be the situation; use Exercise 4.4.7.1 and Noetherian induction on X to reduce to proving the statement on a non empty open subvariety; reduct to a non empty irreducible open set where F is a local system L of rank r; there is the notion of generic rank on this open set; the sequence of these generic ranks is stationary; show that the acc is equivalent to the appropriate statements on the quotients; the quotients have stationary ranks; it follows that the increasing kernels of the quotients are eventually supported on a proper closed subset so that they must stabilize by induction.)

Exercise 4.4.7.3 Show that $\mathcal{P}_X$ is Noetherian. Deduce, by using $\mathbb{D}$, that it is Artinian. (Hint (for the Noetherian statement): let $P_i \subseteq P$ be the increasing sequence and let $d := \dim X$; show the sehaves $H^{-d}(P_i)$ form an increasing sequence in $H^{-d}(P)$; deduce that the images ${}^{p}H^0(H^{-d}(P_i))$ inside P_i stabilize; form the increasing sequence of corresponding cokernels and conclude by Noetherian induction).

By the Jordan-Hölder theorem, we have then that every non zero perverse sheaf P on X admits a finite increasing filtration $0 = P_0 \subseteq P_1 \subseteq \ldots \subseteq P_l = P$ with non zero quotients P_i/P_{i-1}, $1 \leq i \leq l$, which are simple perverse sheaves.

The filtration is not canonical, but the set of non zero quotients (called the constituents of P) is canonical.
Since Exercise 4.4.7.3 does not shed light on this structure, let us give another proof.

First of all, let us introduce the intermediate extension functor.
Let $j : U \to X \leftarrow Z : i$ be the usual thing.
Let $Q \in \mathcal{P}_U$.
Take the natural map $a : j_! Q \to Rj_* Q$. We get the natural factorization of ${}^{\mathfrak{p}}H^0(a)$:

$$ {}^{\mathfrak{p}}H^0(j_! Q) \xrightarrow{epi} \operatorname{Im} a \xrightarrow{mono} {}^{\mathfrak{p}}H^0(Rj_* Q). $$

This cosntruction is functorial and we set

$$ j_{!*} Q := \operatorname{Im} a. $$

This functor is left-exact (preserves monos), right-exact (preserves epics), but it is not exact.
We are going to take the following fundamental fact for granted ([16], 1.4.25):
$j_{!*}Q$ is the unique (up to can iso) perverse extension of Q to X which does not admit non trivial perverse subobjects, nor perverse quotients supported on Z.
(It may have non trivial subquotients supported on Z).
If $j : T \to X$ is a locally closed embedding, then by factoring, i.e. $T \to \overline{T} \to X$, we can define the intermediate extension from T to X as the push forward from $\overline{T}$ to X of the intermediate extension from T to $\overline{T}$. This way, it is immediate to see that the intermediate extension functor has the same characterization as above: no subobjets and no quotients supported inside $X \setminus T$.

Exercise 4.4.7.4 Let $j : T \subseteq X$ be a locally closed subvariety (not necessarily irreducible, nor pure-dimensional) and $P \in \mathcal{P}_T$. Prove that $j_{!*}P^\vee = (j_{!*}P)^\vee$. (Hint: dualize the diagram in the definition). In particular, if P is self dual, then so is $j_{!*}(P)$.

Exercise 4.4.7.5 Let $j : U \to X$ be an open immersion. Prove $j^* : \mathcal{P}_X \to \mathcal{P}_W$, i.e it preserves perverse sheaves. Show that is is an exact functor (i.e. ses $\mapsto$ ses).

Theorem 4.4.7.6 *The category of perverse sheaves on X is Artinian and the simple objects are the intermediate extensions $j_{!*}L[\dim V]$ of simple local systems on integral locally closed nonsingular subvarieties V of X.*

(Of course, we view those intermediate extensions, which are perverse sheaves on the closed $\overline{V} \subseteq X$, as perverse sheaves on X.)

Proof. Let us first prove that $j_{!*}L[\dim V]$ is simple.
Let $P \to j_{!*}L[\dim V]$ be mono.
By Exercise 4.4.7.5, $|P| \subseteq \overline{V}$.
By the fundamental fact above, $|P| = \overline{V}$.
There is a non empty open $V' \subseteq V$ s.t. $P_{|W} = M$, with $M \subseteq L_{|V'}$ a local syatem.
Since L is simple, on V, $L_{|V'}$ is also simple ($\pi_1(V', v') \to \pi_1(V, v')$ is epic due to normality).
It follows that $M = L_{|V'}$.
This implies that the quotient $j_{!*}L[\dim V]/P$ is supported indide $\overline{V} \setminus V$ and is thus trivial by the fundamental fact again.
Simplicity follows.
It follows that if L is a local system on V then $j_{!*}L[\dim V]$ admits a composition series with simple quotients: just take a composition series for L and use the fact that $j_{!*}$ is left-exact and what we have just proved.
We now prove that we have a composition series with simple quotients.
We proceed by Noetherian induction, i.e. we start by assuming that the conclusion holds for the perverse sheaves supported on proper closed subsets of X.
Let $P \in \mathcal{P}_X$. Then there is an irreducible open set $U \subseteq X$ s.t. $P_{|U} = L[\dim U]$, for some local system on U. Leet $j : U \to X$ be the embedding.
We have the natural adjunction maps $j_!j^*P \to P \to Rj_*j^*P$. Apply ${}^{\mathfrak{p}}H^0$ and we get the following commutative diagram:

$$\begin{tikzcd}
& P \arrow[dr, "b"] & \\
{}^{\mathfrak{p}}H^0 j_!j^*P \arrow[ur] \arrow[rr] \arrow[dr, "can\,epi"'] & & {}^{\mathfrak{p}}H^0 Rj_*j^*P \\
& j_{!*}L[\dim U] \arrow[ur, "can\,mono"'] &
\end{tikzcd}$$

It follows that the bottom is a subquotient of the top (sub of $\operatorname{Im} b$, image of P).
We get the following commutative diagram:

$$\begin{tikzcd}
& \operatorname{Ker} c \arrow[r, "epi"] \arrow[d, "mono"] & j_{!*}L[\dim U] \arrow[d, "mono"] \\
\operatorname{Ker} b \arrow[ur, "mono"] \arrow[r, "mono"] & P \arrow[r, "epi"] \arrow[dr, "epi", "c"'] & \operatorname{Im} b \arrow[d, "epi"] \\
& & \operatorname{Im} b/j_{!*}L[\dim U]
\end{tikzcd}$$

with $j_{!*}L[\dim U] = \operatorname{Ker} c/\operatorname{Ker} b$.

Hence inclusions

$$\operatorname{Ker} b \subseteq \operatorname{Ker} c \subseteq P$$

With successive quotients

$$\operatorname{Ker} b, \quad j_{!*}L[\dim U], \qquad P/\operatorname{Ker} c$$

where the first and third are supported on $X \setminus U$.
As observed above, $j_{!*}L[\dim U]$ has the required composition series.
By the inductive hypothesis so do the first and third successive quotients so that a composition series of them can be lifted from them and inserted between 0 and $\operatorname{Ker} b$ and betwen $\operatorname{Ker} c$ and P. $\square$

Exercise 4.4.7.7 Use the perverse truncation formulæ and the content of Remark 4.4.4.3 (taken as a black box), to show that if C is constructible wrt a Whitney stratification, then all of its perverse cohomology sheaves are constructible wrt the same Whitney stratification. Show that if $Q \to P$ is a monomorphism of perverse sheaves (perverse subsheaf!), then Q is constructible wrt the same Whitney stratification. Ditto for quotients. Show that these two facts do not hold for constructible sheaves. Formualate and prove the analogous fact for extensions of a P by a P' and note that this remains true even if we do not use Whitney stratifications, but only a partition of X into locally closed subvarieties s.t. both P and P' have locally constant cohomology sheaves on the elements of the partition of X. Show that this holds for constructible sheaves. Show that if $f : K \to C$ is such that K and C have cohomology sheaves which are locally constant on some partition of X as above, then so is the cone of f. Deduce that if in a distinguished triangle any two terms are constructible wrt to a given partition as above, then so is the third.

Exercise 4.4.7.8

1. Let $C = U \coprod o$ be a curve where U is the regular part and o is a singular point.. Use the truncation formulæ in Exercise 4.4.6.2 to show that $\mathbb{Q}_C[1] = {}^{\mathfrak{p}}\tau_{\leq 0}\mathbb{Q}_C$ is perverse. Show that $j_{!*}\mathbb{Q}_U[1] = j_*\mathbb{Q}_U[1]$. Find the composition series for $\mathbb{Q}_C[1]$. Find the composition series for $j_!\mathbb{Q}_U[1]$ and for $Rj_*\mathbb{Q}_U[1]$. Let L be a local system on U. Study the les of perverse cohomology sheaves associated with $0 \to j_!L \to j_*L \to i^*j^*L \to 0$. Do the same for $0 \to j_!L \to Rj_* \to i^*Rj_*L \to 0$.

2. Let $Y = U \coprod v$ be a surface with an isolated singualr point. Show that $\mathbb{Q}_Y[2]$ is perverse iff v is unibranch. Assume v is unibranch (e.g. Y is normal). Use the truncation formulæ to show that ${}^{\mathfrak{p}}\tau_{\leq 0}Rj_*\mathbb{Q}_Y[2] = (\tau_{\leq 2}Rj_*\mathbb{Q}_Y)[2]$. Show that this is perverse, hence it is ${}^{\mathfrak{p}}H^0(Rj_*\mathbb{Q}_U[2])$. Show that ${}^{\mathfrak{p}}H^0(j_!\mathbb{Q}_U[2]) = \mathbb{Q}_Y[2]$.

Use the above and the les of perverse cohomolgoy sheaves of $i_!i^!\mathbb{Q}_Y[2] \to \mathbb{Q}_Y[2] \to Rj_*\mathbb{Q}_Y[2]$ to deduce that $j_{!*}\mathbb{Q}_U[2]$ sits into a ses of perverse sheaves

$$0 \longrightarrow j_{!*}\mathbb{Q}_U[2] \longrightarrow (\tau_{\leq 2}Rj\mathbb{Q}_U)[2] \longrightarrow R^2j_*\mathbb{Q}_U \longrightarrow 0$$

so that

$$j_{!*}\mathbb{Q}_U[2] = (\tau_{\leq 1}Rj\mathbb{Q}_U)[2] = \tau_{\leq -1}(Rj_*\mathbb{Q}_U[2])$$

which is the intersection complex of Y as defined in §**????**. Let L be a local system on U. Study the les of perverse cohomology sheaves associated with $0 \to j_!L \to j_*L \to i^*j^*L \to 0$. Do the same for $0 \to j_!L \to Rj_* \to i^*Rj_*L \to 0$.

3. Let $E^{d-1} \subseteq \mathbb{P}^N$ be an embedded projective $(d-1)$-fold. Let $Y = U \coprod v$ be the corresponding affine cone with vertex v over an E. Verify that

$$j_{!*}\mathbb{Q}_U[\dim Y] = \tau_{\leq -1}(Rj_*\mathbb{Q}_U[\dim Y])$$

and that the stalks

$$H^k(Rj_*\mathbb{Q}_U)_v = P^k_\eta, \qquad 1 \leq k \leq d-1$$

where $P^k_\eta \subseteq H^kE$ is the primitive cohomology wrt the hyperplane bundle. Determine the compostion series for the perverse sheaf $\mathbb{Q}_Y[d]$ (cf. **????**). Let L be a local system on U. Study the les of perverse cohomology sheaves associated with $0 \to j_!L \to j_*L \to i^*j^*L \to 0$. Do the same for $0 \to j_!L \to Rj_* \to i^*Rj_*L \to 0$.

4. Let $P = Rf_*\mathbb{Q}_X[2]$ be as in the four examples in §1.0.1. We know it is perverse (see Exercise 4.4.5.3, parts 5,6,11). Verify that it is uneffected by ${}^{\mathfrak{p}}\tau_{\leq 0}$ (as it should). Determine its composition series (without using the fact that it splits). (See Remark 4.2.1.6.)

Definition 4.4.7.9 Let L be a local system on a non empty open subset U of the regular part of an irreducible varietey X. The Goresky-MacPherson-Deligne intersection cohomology complex of Y with coefficients in L is ($j : U \to Y$ the open immersion):

$$IC_X(L) := j_{!*}L[\dim X].$$

The intersection cohomology complex of a variety X is defined to be

$$IC_X := IC_X(\mathbb{Q}_{X_{reg}}).$$

The cohomology groups of this complex are called the (Goresky-MacPherson) intersection cohomology groups:

$$IH^i(X,\mathbb{Q}) := H^{i+\dim X}(X, IC_X), \qquad IH^i_c(X,\mathbb{Q}) := H^{i+\dim X}_c(X, IC_X).$$

By Exercise 4.4.7.4, IC_X is self-dual and Verdier duality implies that we have Poincaré duality for intersection cohomology:

$$IH^i(X,\mathbb{Q}) \cong IH^{2\dim X-i}_c(X,\mathbb{Q}), \qquad IH^i_c \times IH^{2\dim X-i} \longrightarrow \mathbb{Q} \quad \text{non degenerate}$$

Similarly, for twisted coefficients.
Actually, the original definition [21] of Goresky-MacPherson is for intersection homology groups $IH_{c,*}$ and uses special complexes of geometric chains (i.e. no sheaf theory) with compact support (analogous to singular homology). There is the variant IH_* with locally finite supports (analogous to Borel-Moore homology). (i.e. no sheaf theory). The paper [23] re-builds the theory in sheaf-theoretic terms. In particular, one has $IH_{c,*} = IH^{2\dim X-*}_c$ and $IH_* = IH^{2\dim X-*}$, and the sheaf-theoretic pairing coincides with the one defined geometrically in intersection homology.

Exercise 4.4.7.10 Let (L', U') be a second pair as above. Assume that $L = L'$ on the overlap. Show that $IC_X(L) = IC_X(L')$, i.e. the intersection complex ox X with coefficients above is independent of the regular open set used to define it. (Hint: look at the intermediate extension of from U to $U \cap U'$ and show that it coincides with the local system on $U \cap U'$ obtained by glueing L and L' (shifted by $\dim X$); note that this is a local problem and reduce to computing the intermediate extension of the constant sheaf from $\mathbb{C}^n \setminus \mathbb{C}^{n-k}$ to $\mathbb{C}^n$.)

There is the following formula. Let $X = \coprod S_l$ be a Whitney stratification of X, where S_l is the union of all l-dimensional strata. Let $d := \dim X$. Let $U_l = \coprod_{l' \geq l} S_l$. We have an increasing sequence of open sets $U_d \subseteq U_{d-1} \subseteq \ldots \subseteq U_0 = X$. Let $j_l : U_l \to U_{l-1}$ be the open immersions, $1 \leq l \leq d$. Then

$$IC_X(L) = \tau_{\leq -1} Rj_{1*}\left(\tau_{\leq -2} Rj_{2*}\left(\ldots \tau_{\leq -d+1} Rj_{d-1*}\left(\tau_{\leq -d} Rj_{d*} L[d]\right)\right)\right).$$

This matches with the ad-hoc definition we have used when dealing with contractions of curves on surfaces (see **?????** and also Exercise 4.4.7.8).(2)), where the intersection cohomology complex appeared natrually as an essential ingredient in the study of the topology of the contraction.
It is possible to characterize intersection cohomology complexes using some strengthned conditions of support/co-support. In fact, this was, more or less, the original definition. See [12], p.556 for a visual display of these conditions.

Here, we state the following characterization of intersection cohomolgoy complexes: $IC_X(L)$ is the only (up to iso) perverse sheaf P on X that extends $L[\dim X]$ and has the following property:
(recall that being perverse both P and $P^\vee$ can have non trivial cohomology sheaves of degree $i \in [-\dim X], 0]$ and that $\dim |H^i(P, P^\vee)| \leq -i$)

$$\dim |H^i(P, P^\vee)| < -i, \qquad \forall i \neq -\dim X.$$

E.g.: the intersection complex IC_X of a threefold has $H^i IC_X = 0$ for every $i \notin [-3. -1]$ and $\dim |H^{-2}| \leq 1$, $\dim |H^{-1}| \leq 0$.

Exercise 4.4.7.11 Exercise 4.4.7.4 implies that $IC_X(L)^\vee = IC_X(L^\vee)$. Prove this by testing the support/co-support characterization of $IC_X(L^\vee$ against $IC_X(L)^\vee$.

5 Decomposition and relative Hard Lefschetz theorem

5.1 Statements and features

5.1.1 Statements

Theorem 5.1.1.1 *Let $f : X \to Y$ be a proper map of complex algebraic varieties.*

1. *Then $Rf_* IC_X$ is $\mathfrak{p}$-split:*

$$\bigoplus_i {}^{\mathfrak{p}}H^i(Rf_* IC_X)[-i] \cong Rf_* IC_X.$$

2. *Each ${}^{\mathfrak{p}}H^i(Rf_* IC_X)$ is semisimple, i.e. it splits as a direct sum of the intersection cohomology complexes with simple coefficients of a suitable finite collection of closed irreducible subvarietes of Y:*

$${}^{\mathfrak{p}}H^i(Rf_* IC_X) \cong \bigoplus_a IC_{Z_{i,a}}(L_{i,a}).$$

3. *Assume that f is projective and let $\eta \in H^2X$ be the first Chern class of an f-ample line bundle on X. Then the relative hard Lefschetz theorem holds, i.e. we have the isos*

$$\eta^k : {}^{\mathfrak{p}}H^{-k}(Rf_* IC_X) \xrightarrow{\cong} {}^{\mathfrak{p}}H^k(Rf_* IC_X), \qquad \forall\, k \geq 0.$$

5.1.2 Some features

By Verdier duality, the summation in 1. is symmetric about zero, i.e. $i \in [-r, +r]$ for some $r \geq 0$ and:

$$({}^{\mathfrak{p}}H^i(Rf_* IC_X))^\vee = {}^{\mathfrak{p}}H^{-i}(Rf_* IC_X).$$

Part 1 and Part 3 imply that each perverse cohomology sheaf is self-dual.

The isomorphism 1 is not unique in any way. It is not even in the Leray-Hirsch setting, so it cannot be here. This non uniqueness parallels the fact that once we have a filtration on a vector space, we can split it, but not canonically.

The isomorphism 2 is also not canonical. It can't be: the vector space $\mathbb{R}^2$ splits, but a splitting is the same thing as chosing a basis. However, for every i, we can group together the summands supported on the same subvarieties and get

$${}^{\mathfrak{p}}H^i(Rf_* IC_X) \cong \bigoplus_{Z \subseteq Y} IC_Z(\mathcal{L}_{i,Z}).$$

where the sum is finite and the $\mathcal{L}_{i,Z}$ are semisimple. This decomposition is more canonical as any two such differ by an automorphism of the rhs which is a direct sum map wrt the Z's (essentailly by Schur's lemma). In fact, it is possible to describe the rhs in geometric terms and, with this description of the rhs, a canonical decomposition is possible.

Question. Why bother with this non canonical business?
Answer. Matters of precision aside, let us say that the after taking cohomology both sides of the iso have a mixed Hodge structure. Then one would like to know if one can chose isos compatible with this extra structure.

The RHL iso does not mix the Z-terms above (again by the Schur lemma). In particular, the resulting primitive Lefscehtz decomposition can be performed Z-term by Z-term.

Now we come to a fundamental feature of the decomposition theorem. If we restrict to an open subset (Zariski or even Euclidean!) $U \subseteq V$ an iso 1. we get a corresponding decomposition iso on U. This implies that when we pass from U back to Y we know that the intersection cohomology complexes $IC_{Z\cap U}(L))$ appearing in the decomposition theorem over U also appear automatically, as $IC_X(L)$ of course, in the decomposition theorem over Y! In other words, knowing summands on an open subset tells us that they give corresponding summands globally. This fact is exactly what does not happen in the real algebraic contraction in Example **???** and points to the fact that the decomposition theorem expresses a fundamental rigidity proeprty of complex algebraic maps.
Morevoer, these summands are built using a recipe, intersection cohomology, which is internal to the target Y of the map $f : X \to Y$.

5.1.3 Maps from surfaces onto curves

Let $f : X^2 \to C^1$ be a surjective projective map with connected fibers from a nonsingular surface to a nonsingular curve. Let $j : U \to C$ be the open embedding where U is the set of regular values. For simplicity let $C \setminus U = o$ be one single point. Let $\#$ be the number of irreducible curves in $f^{-1}o$. We have

$$Rf_*\mathbb{Q}_X[2] \cong \{\mathbb{Q}_C[1]\}\,[1] \bigoplus \left\{j_*R^1[1] \oplus \mathbb{Q}_o^{\#-1}\right\}[0] \bigoplus \{\mathbb{Q}_C[1]\}\,[-1].$$

The first summand, in perversity -1, corresponds to the trivial local system on C with canonical generator given by 1 on each fiber.
The RHL predict we should find a twin of this in perversity $+1$ generated by the the cup product of η with 1 restricted to each fiber. And so it is.

In doing so we fail to generate $\# - 1$ classes in $H^2 f^{-1} o = (R^2 f_* \mathbb{Q})_o$ which we find instead in perversity 0! You know this a priori because RHL does not allow for this packet of classes to be placed in any other perversity.
More canonically, it can be shown that that summand is really the non injective! image under the class map of $H_2 f^{-1} o$ in $H^2 X$, follwed by the restriction to $H^2 f^{-1} o$ (recall that this is the refined intersection form on $H_2 f^{-1} o$).
If we apply Poincaré duality to this picture, we deduce that V is self-dual and the intersection form on X (say X compact, for simplicity) induces a non degenerate bilinear form on V. As we shall comment upon later this form is < 0. This is the classical Zariski's lemma.
It follows that we know that all the cohomology classes of the components contribute to the cohomology of X, but in two distinct patterns, one pattern where the refined intersection form is non-degenerate, the other pattern where the class of F contributes as a monodromy invariant to $(j_* R^2)_o = (\mathbb{Q}_C)_o = \langle [F] \rangle$ and on which the refined intersection form is trivial ($F^2 = 0$).
Note that the only important feature of $f^{-1} o$ that appears in the iso above is the number $\#$.
In perversity 0 we find the sheaf $j_* R^1$, where R^1 is the local system on U with fiber H^1 of a regular fiber.
Note that we must have $(R^1 f_* \mathbb{Q}_X)_o \cong (j_* R^1)_o$ and this says that

$$H^1 f^{-1} \Delta_o = R^{1 \pi_1(\Delta_o^*)},$$

i.e. the cohomology classes in the pre-image of a small disk about o are precisely the local monodromy invariants of a regular fiber near o. A priori there is only a map left to right.
This is an instance of the celbrated, and difficult, local invariant cycle theorem.

5.1.4 Resolution of 3-fold singularities

Let $f : X^3 \to Y^3$ be the resolution of the singularities of a 3-fold. For simplicity, we assume there is only one singular point $v \in Y$ away from which f is an iso and that $f^{-1} v = \cup D_j$ a union of $\#$ divisors Then

$$R f_* \mathbb{Q}_X[3] \cong \left\{ \mathbb{Q}_v^{\#} \right\} [1] \bigoplus \left\{ IC_Y \oplus \mathbb{Q}_o^{\%} \right\} [0] \bigoplus \left\{ \mathbb{Q}_v^{\#} \right\} [-1].$$

The first summand in perversity -1 is canonically the injective! image under the class map of $H_2 \cup D_j$, i.e. the divisor classes $[D_j]$ are lineraly inepedent in X and in fact, in any neighborhood of $\cup D_j$.
RHL tells us that the cup product with a hyperplane H produces one curve, $C_j := H \cap D_j$ for any D_j and that their cohomology classes are linearly independent and form a basis for that summand.

Looking at perversity zero, we must have IC_Y as a summand, because it it a summand away from v! This is the fundamental fact discussed earlier (seen also in the previous example, but not commented upon).
This is a general fact, the intersection cohomology groups of a variety are (a canonical) direct summand (of a canonical subquotient) of the cohomology of X.
The piece $\mathbb{Q}_v^{\%}$. This is analogous to the piece $\mathbb{Q}_o^{\#-1}$ in the previous example. In that example it is clear that it is a pure Hodge structure of weight 2 and in fact it is of pure type $(1,1)$. Morevoer, polarized by the intersection form. Here, $\mathbb{Q}_v^{\%}$ is also a polarized PHS, but of weight 3. It is the only contribution to $R^3 f_* \mathbb{Q}_X$ (since IC_Y stops just short of contributing!) so that we must have that $H^3 \cup D_j$ is a PHS. This yields a general contractibility test, which in this case reads that the configuration $\cup D_j$ is contractible only if $H^3 \cup D_j$ is pure and polarized by the intersection form. (Converse is clearly not true: take $D = \mathbb{P}^2$ in $X = \mathbb{P}^3$.

5.2 Decomposition theorem for semismall maps

5.2.1 Semismall maps and perversity of $Rf_* \mathbb{Q}_X$

Let $f : X \to Y$ be a proper map of algebraic varieties.
For ease of exposition, we assume that f is surjective.
Let $n := \dim X$.
Let $Y = \coprod \delta Y_\delta$ be the stratification of Y by the locally closed subsets Y_δ of points over which the fiber has dimension δ.
We say that f is semismall if

$$2\delta \leq n - \dim Y_\delta, \qquad \forall k.$$

Exercise 5.2.1.1 Show that f is semismall iff there is no irreducible subvariety $T \subseteq X$ s.t. $2 \dim T - \dim f(T) > n$, iff $\dim X \times_Y X \leq n$. Note that one always has $\geq n$. We say f is small if, in the definition, we have a strict inequality, except when $\delta = 0$. Show that this is equivalent to having that all but one of the irreducible components of Z of $X \times Y X$ have $\dim Z < n$.

Note that semismall implies generically finite.

Example 5.2.1.2 A surjective map of surfaces is always semismall A blowing up with smooth center of a nonsingular variety is semismall iff the center has codimension 2. The blowing up of the affine quadric cone of dimension 3 over the projective quadric nonsigular surface in $\mathbb{P}^3$ along the divisor of one of the plane rulings is small and not semismall. The blow up of the vertex is not semismall. There are many interesting examples stemming from geometric representation theory [12].

Proposition 5.2.1.3 *Let $f : X \to Y$ be semismall with X nonsingular. Then $Rf_*\mathbb{Q}_X[n]$ is perverse.*

Proof. Note that $\mathbb{Q}_X[n]$ is self-dual:

$$(\mathbb{Q}_X[n])^\vee = \mathbb{Q}_X^\vee[-n] = \omega_X[-n] = \mathbb{Q}_X[2n][-n] = \mathbb{Q}_X[n];$$

since $\mathbb{D} \circ Rf_* = Rf_! \circ \mathbb{D}$, and f si proper, so $Rf_* = Rf_!$, we have that $(Rf_*\mathbb{Q}[n])^\vee = Rf_*\mathbb{Q}_[n]$.
By the very definition of perverse sheaf, it is sufficinent to show that $Rf_*\mathbb{Q}_X[n]$ satisifes the conditions of support:

$$|H^{-i}(Rf_*\mathbb{Q}_X[n])| = |R^{n-i}f_*\mathbb{Q}_X| \leq i.$$

Since f is proper, proper base change gives:

$$(R^i f_*\mathbb{Q}_X)_y = H^i X_y.$$

A simple Noetherian argument shows that any quasi projective variety has no cohomology after degree twice its dimension.
It follows that $R^{n-i}f_*$ is supported on the closure of the union (in fact the union by Chevalley's upper-semicontinuity theorem) of the Y_δ with

$$2\delta \geq n - i.$$

Let Y_{δ_o} be such that $\dim Y_{\delta_o}$ is maximum with that property: this dimension is an upper bound for $\dim |R^{n-i}f_*|$.
we have

$$n - \dim Y_{\delta_o} \geq 2\delta_o \geq n - i,$$

i.e.

$$|R^{n-i}f_*| \leq \dim Y_{\delta_o} \leq i.$$

□

What follows applies to every algebraic map $f : X \to Y$. We limit ourselves to stating what we need (more is true [17, 15]:
there is a finite algebraic Whitney stratification $Y = \coprod Y_a$ (in particular, Y_a are irreducible, locally closed, nonsingular) with $f_a : f^{-1}Y_a \to Y_a$ a topologically locally trivial fibration.
It is a general fact that the top dimensional irreducible components of the fibers of f_a form a locally constant sheaf of sets on Y_a. Since the monodromy factors via a finite group, the resulting local system L_a is self-dual.

Back to f semismall. Note that in the definition of semismall we could have used any stratification as above. In particular, we have the inequality for every Y_a.

say that a stratum Y_a is relevant if we have equality, i.e.

$$2 \dim f^{-1} y_a = n - \dim Y_a.$$

In this case, we have

$$L_a = \text{local system with stalk } H^{BM}_{n-\dim Y_a}(f^-1 y_a)$$

Theorem 5.2.1.4 (**Decomposition theorem for semsimall maps**) *There is a canonical isomorphism*

$$Rf_* \mathbb{Q}_X[n] = \bigoplus_{Y_a \text{ relevant}} IC_{\overline{Y_a}}(L_a).$$

5.2.2 Sketch of proof

Let us work under some strong additional hypothesis that simplify the discussion of the important points:

let us assume that $f : (X^{2n}, E) \to (Y^{2n}, v)$ is a map of projective varieties (X nonsingular) contracting precisely to a apoint $v \in Y$ a closed algebraic subset $E \subseteq X$ with $\dim E \leq n$ and such that the map is of maximal rank over $Y \backslash v$ (hence a covering space with associated local system L).

In this case we have to prove that

$$Rf * \mathbb{Q}_X[n] = IC_Y(L) \oplus (H_{2n}E)_v.$$

Note that we did not assume that v is relevant i.e. that $\dim E = n$. The proof is blind to this fact, as it gives us the direct summand $IC_v(H_2E) = H_2(E)_v$ which is the vector space generated by the irreducible components of E of dimension n and this is non zero iff v is relvant.

The proof consists of showing the following two facts:

1. The complex $Rf_* \mathbb{Q}_X[n]$ splits as predicted iff the refined intersection form

   $$\iota_E : H_{2n}E \to H^{2n}E \qquad \text{is an iso.}$$

   As it turns out, this is a local question and treated using a little formalism from the derrived cateogry.

2. The refined intersection form $\iota_E = \text{iso}$.

   As it turns out, this is proved using a global approach based on the classical Hodge theory of projective manifolds.

Proof of 1. Set $Rf_*\mathbb{Q}_X[n] = P$ and let $i_v : v \to Y \leftarrow U : j$ be the closed/open complementary embeddings, $E = f^{-1}v$ and $U' = f^{-1}U \subseteq X$.
Given any σ as below, we we have the commutative diagram:

$$\begin{tikzcd}
 & (H^0P)_v = (H^{2n}E)_v \arrow[dl, "\sigma'"'] \arrow[d, "\sigma"] \\
i_{v!}i_v^!P \arrow[r, "a"] \arrow[dr, "\mathfrak{i}"'] & P \arrow[d, "p"', "\text{natural truncation map } \tau_{\leq 0}"] \\
 & H^0P = (H^{2n}E)_v
\end{tikzcd}$$

where σ' exists and is unique due to the fact that its source is aupported on v.
We are looking for σ splitting p.
The cohomology hseaves of the taget of σ' are in degree ≥ 0, so that σ' is given precisely by a map in degree zero.
We have $H^0(i_{v!}i_v^!P) = H_{2n}E$.
It follows that to give σ is the same as giving a map of vector spaces $H^{2n}E \to H_{2n}E$.
For σ to be a splitting, we need $p \circ \sigma = \text{Id}$.
Since $\dim H^* = \dim H_*$, this can be done IFF $H^0(\mathfrak{i}) = \iota_E$ is an iso. $\square$

As to the proof that $\iota_E = \text{iso}$, again, let us first look again at the special case of surfaces.

5.2.3 Grauert's sign theorem via mixed Hodge Hodge theory

Recall Grauert criterion for contracting curves on complex surfaces (see Exercise 4.2.3.5).
The harder part of Grauert's result is shwoing that $I < 0$ implies contractibility.
We focus insted on determining the signature of I once we have a contraction. We do so in the projective context.
We want to give a proof of the following

Proposition 5.2.3.1 *Let $f : X \to Y$ be a surjective and projective map of projective surfaces with X nonsingular. Let $\cup E_j$ be a union of distinct curves on X contracted by f to a point $v \in Y$. Then $I = ||E_j \cdot E_k|| < 0$.*

Proof.
CLAIM 1: the class map below is mono:

$$cl : H_2 \cup E_j \longrightarrow H^2 X.$$

Proof of CLAIM 1.
We need to show $cl : H_2 \cup E_j \to H_2 X \cong H^2 X$ is mono.
ETS $H_2 \cup E_j \to H_2 X$ is mono.
Equivalent to showing $H^2 X \to H^2 \cup E_j$ is epi.
By the theory of weights for mHS on the cohomology of complex varieties, if $\cup E_j \subseteq U \subseteq X$ is a Zariski open set, we have that the image of $H^2 X$ into $\mathrm{H}^2 \cup E_i$ coincides with the image of $H^2 U$.
Let V be an *affine* open set containing v and let $U := f^{-1} V$.
It is enough to show that $H^2 U \to H^2 \cup E_j$ is epi.
This follows easily by looking at the E_2 page of the Leray spectral sequence for $f : U \to V$ which shows that all the differentials exiting from $H^0 R^2 f_* \mathbb{Q}_U = H^2 \cup E_j$ land in zero due to the theorem on the cohomological dimension of constructible sheaves on affine varieties: $H^{>\dim_{\mathbb{C}} Z}(Z, F) = 0$, Z affine, F constructible):

$$E_2^{2,1} = H^2 R^1 f_* \mathbb{Q}_U = 0, \qquad E_2^{3,0} = H^3 R^0 f_* \mathbb{Q}_U = 0$$

(in fact R^1 lives on an affine curve in V and R^0 on the affine surface V).
This means that $E_2^{0,2} = E_\infty^{0,2}$ is the last graded piece of the Leray filtration, i.e. $H^2 U \to E_\infty^{0,2}$ is epi.
CLAIM 1 is proved.

CLAIM 2: The intersection form $\int_X$ on the compact oriented X is negative definite when restricted to the image of the class map.
Proof of CLAIM 2.
Let A be the first Chern class of an ample line bundle on Y. Let $L := f^* A$.
Since $L^2 \neq 0$, the hard Lefschetz theorem hold on the even cohomology, i.e.:

$$L^0 = Id : H^2 X = H^2 X, \qquad L^2 : H^0 X \cong H^4 X.$$

We have the primitive Lefschetz decomposition

$$H^2 X = P_L^2 \oplus L \cup H^0 X, \qquad P_L^2 = P_L := \mathrm{Ker}\{L : H^2 X \to H^4 X\}.$$

This decomposition is orthogonal with respect to the intersection form $\int_X$ given by the cup product on X.
In particular, by PD, the restriction of this form to P_L is nondegenerate.
Let η be the first Chern class of an ample line bundle on X.

For convenience only, we swtich to cohomology with real coefficients.
Let $L_\epsilon := L + \epsilon\eta$, $\epsilon > 0$.
The hard Lefschetz theorem holds for L_ϵ.
The Hodge-Riemann bilinear relations tella us that the intersection form $\int_X$ is < 0, when restricted to P_{L_ϵ} for every $\epsilon > 0$.
We have, in the appropriate Grassmannian: (it is essential that the two spaces have the same dimension!)

$$\lim_{\epsilon \to 0} P_{L_\epsilon} = P_L.$$

It follows that $\int_X$ restricted to P_L is ≤ 0.
But it is non degenerate by what above.
It follows it must be < 0.
CLAIM 2 is proved.

Finally:
the intersection form is

$$\int_X H^2 X \times H^2 X \to \mathbb{Q}.$$

We have

$$H^2 X, X - \cup E_j \times H^2 X, X - \cup_j E_j \longrightarrow \mathbb{Q}$$

factoring through $\int_X$ via the class map cl.
Since the class map cl is injective and its image lands in P_L (!), the desired conclusion follows (negative definite form restricted to subspace stays negative definite). $\square$

THe key points of the proof above are

1. $cl_E : H_{2n}E \to H^{2n}X$ is injective.

2. The restriction of the intersection form I_X on $H^{2n}X$, which is non degenerate by Poincaré duality, stays non degenerate when restricted to $\operatorname{Im} cl_E$.

Let us discuss how to generalize these two facts to any dimension in the specail case we have been working with.

Part 1 generalizes to all dimensions with the same proof.

Part 2 is delicate:
we need I_X to define a polarization of the kernel P_L and we need $\operatorname{Im} cl_E$ to be a pure Hodge substructure of the pure Hodge structure P_L.

Let us explain a bit more this point.
$H^{2n}X$ is a PHS;

P_L is a PHSS;
since $H_{2n}E$ is pure, so is its image and $\operatorname{Im} cl_E$ is a PHHS of the PHS P_L.
if we knew I_X that I_X polarizes P_L, then, we would be done by linear algebra:
a polarization restricts to a polarization on any PHSS of P_L, and thus on $\operatorname{Im} cl_E$;
clearly, polarizations are non degenerate: they become positive definite after a simple linear algebra trick (Weil operator) that does not change their rank.

It is now clear that to finish up we need to show that P_L is a PHHS of $H^{2n}X$.

Let us pretend for a second that L ia ample on X. It is not, but ....
Then the Hodge-Riemann bilinear relations would precisely say what we need.
As in our proof of Grauert's criterion, we try to do so with that same limiting argument.
The key to that argument is to have P_L to have the expected dimension $b_{2n}(X) = b_{2n-2}(X)$.
This does not happen if you blow up a point in $\mathbb{P}^4$, for you get the expected dimension to be zero, while P_L is one dimensional.
This happens precisely because the HL does not hold for L.
So one way to show P_L has the right dimension is to prove HL for L.
In the surface case, this aspect was trivial.
In arbitrary dimension we have

Theorem 5.2.3.2 *Let $f : X \to Y$ be a semismall map of projective varieties, X smooth. Let A be ample on Y and $L := f^*A$.*
Then HL for L iff f is semismall.
In this case, we have the PLD and the HRBR.

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Let $Y = U \coprod S$, where S is a d-dimensional closed stratum of a Whitney stratification of Y.

Let $j : U \to Y \leftarrow S : i$ be the open/closed embedding.

Let $P \in P_Y$.

We have the distinguished triangle

$$i_! i^! P \longrightarrow P \longrightarrow Rj_* j^* P \longrightarrow .$$

We have that:

$$\text{either } i_! i^! P = 0, \qquad \text{or } |i_! i^! P| = S,$$

and

$$i_! i^! P \text{ has locally constant cohomology sheaves on } S.$$

The co-support conditions for P say that

$$i_! i^! P \in \mathcal{D}_Y^{\geq -d}.$$

We have the les of cohomology sheaves

$$0 \to H^{-d-1}P \stackrel{j^*_{-d-1}}{\longrightarrow} H^{-d-1}Rj_*j^*P \stackrel{b_{-d-1}}{\longrightarrow} H^{-d} i_! i^! P \stackrel{\mathfrak{i}_{-d}}{\to} H^{-d}P \stackrel{j^*_{-d}}{\to} H^{-d}Rj_*j^*P \stackrel{b_{-d}}{\to} H^{-d+1} i_! i^! P \to 0.$$

In degree $\geq -d$, the above are all local systems on S.

Clearly,

$$(\mathfrak{i}_{-d} = \text{iso}) \quad \text{IFF} \quad (b_{-d-1} = j^*_{-d} = 0) \quad \text{IFF} \quad (j^*_{-d-1} \text{ and } b_{-d} = \text{iso}).$$

Proposition 5.2.3.3 *The perverse sheaf P splits*

$$P = j_{!*} j^* P \oplus H^{-d} P[-d]$$

IFF ι_{-d} is an iso.

In this case, the splitting is canonical.

Proof. We have a diagram

$$\begin{tikzcd}
& H^{-d}P[d] \arrow[dl, "\sigma'"'] \arrow[d, "\sigma"] \\
i_! i^! P \arrow[r, "a"] \arrow[dr, "\mathfrak{i}"'] & P \arrow[d, "\tau"] \\
& H^{-d}P[d],
\end{tikzcd}$$

where a is the adjunction map, τ is the projection induced by truncation and $\mathfrak{i}$ is the composition and where σ is any map and σ' is the unique map that makes the upper part commutative (this is because the top is supported on S).

We have that P splits into $\tau_{\leq -d-1}P \oplus H^{-d}P[d]$ IFF there is σ inducing s.t. $\tau \circ \sigma = \mathrm{Id}$.

Note that giving σ is the same as giving σ' and that, since $H^{-d}P[d]$ (starts and) ends in degree $-d$ and $i_!i^!P$ starts in that same degree, giving σ' is the same as giving $H^{-d}\sigma' : H^{-d}i_!i^!P \to H^{-d}P$ (both local systems on S.

It follows that P splits into $\tau_{\leq -d-1}P \oplus H^{-d}P[d]$ IFF $\mathfrak{i}_{-d} = \text{iso}$. Moreover, this splitting is unique for $H^{-d}\sigma'$ is, so σ' is, so σ is.

It remains to show that, if $\mathfrak{i}_{-d} = \text{iso}$, then

$$\tau_{\leq -d-1}P \longrightarrow \tau_{\leq -d-1}Rj_*j^*P = j_{!*}j^*P$$

is an iso.

The les above (continued on the left) says that we only need to verify that $j^*_{-d-1} = 0$ which is implied by $\mathfrak{i}_{-d}$ =iso. $\square$

Now we show that the map of local systems

$$\iota_S := \mathfrak{i}_{-d} : H^{-d}i_!i^!P \longrightarrow H^{-d}P$$

can be interpreted as the same kind of map associated with the restriction to any normal slice N at any point of $s \in S$ (now a closed stratum in N):

$$\iota_s : H^0 i_{s!}i_s^!P_N \longrightarrow H^0P_N$$

where $P_N := P_{|N}[-d] \in \mathcal{P}_N$.

This amounts to the following:

show that the restriction (i_N^*) to N of the attaching triangle for ($Y = U \coprod S$) is precisely the attaching triangle for $(N, N \setminus s, s)$ (this is a local picture).

This follows by usual base change in view of a double use of $i_N^! = i_N^*[-2\dim S]$ (the double use means here the shifts cancel out).

Note that if $P = Rf_*$ something, then similar base change will relate the above, to a map of type $H^n(X, X \setminus X_s, P) \to H^nX_s, P_{|X_s}$.

(In our typical situation X is smooth and $P = \mathbb{Q}_X[n]$ is self-dual (we did not assume that in what above), and we obtain, for P self-dual, the intersection form on the pre-image of a normal slice).

6 Appendices

6.1 The semisimplicity theorem

6.1.1 Some preliminaries

1. Let S be a topological space.
2. Reminder:
   let M be a local system on S of finite dimensional complex vector spaces;
   let $\mathcal{C} = \mathcal{C}_S$ be the sheaf of $\mathbb{C}$-valued C^0-functions on S;
   let $\mathcal{M} := \mathcal{C}(M) := M \otimes_{\mathbb{C}} \mathcal{C}$
   (identified with the sheaf of continuous sections of the vector bundle $\mathbb{M}$ associated with the local system M);
   there is the natural ses of sheaves
   $$0 \longrightarrow M \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/M \longrightarrow 0; \tag{1}$$
   note that $\mathbb{M}$ admits a trivialization with constant transition functions.
   If S is a smooth manifold, replace $\mathcal{C}$ with the sheaf of smooth functions $\mathcal{A}$;
   Since the transition functions are locally constant, we can define
   $$d \,:\, \mathcal{M} \longrightarrow \mathcal{A}^1(\mathcal{M}) := \mathcal{M} \otimes_{\mathbb{C}} \mathcal{A}^1 \tag{2}$$
   and then
   $$M = \operatorname{Ker} d \tag{3}$$
3. Given a subbundle $\mathbb{N} \subseteq \mathbb{M}$, we have its sheaf of sections.
   We say a subsheaf $\mathcal{N} \subseteq \mathcal{M}$ is a subbundle is $\mathcal{N}$ arises as above.
   We say a subbundle $\mathcal{N}$ is *horizontal* if it is of the form $\mathcal{C}(N)$ for $N \subseteq M$ a sub local system.
   In this case, we have that N is the kernel of the compositum map
   $$\mathcal{N} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/M. \tag{4}$$
   Using the above, it is easy to show that a subbundle $\mathcal{N} \subseteq \mathcal{M}$ is horizontal IFF it admits local frames made of horizontal sections (i.e. sections of M) (for the non trivial implication: take the kernel of (4)).

4. Let $\mathcal{V} \subseteq \mathcal{H}$ be of rank one.

   Then $\mathcal{V}$ is horizontal IFF $\mathcal{V}^{\otimes} \subseteq \mathcal{H}^{\otimes n}$ is horizontal for some $n \neq 1$ IFF it is horizontal for every $n \neq 0$.

   Proof. It is enough to show that $\mathcal{V}^{\otimes n}$ horizontal for some $n \neq 0$ implies $\mathcal{V}$ horizontal.

   Let me do the case $n = 2$ to illustrate.

   There is local frame v for $\mathcal{V}$ and $v \otimes v = h \in H_{\mathbb{C}}$.

   Write $v = \sum_i l_i h_i$ (l_i continuous, (h_i) horizontal local frame).

   Then
$$\sum l_i l_j h_i \otimes h_j \in H_{\mathbb{C}} \tag{5}$$
   so that
$$l_i l_j \in \mathbb{C}, \quad \forall i, j. \tag{6}$$
   This implies exists $c \in \mathbb{C}^n$ (n the rank of $H_{\mathbb{C}}$) and f continuous function s.t. $fc = (l_1, \ldots, l_n)$.

   This means that $(1/f)v$ is a horizontal fram for $\mathcal{V}$ and we are done by what below (4). $\square$

5. In the same vein:

   Let $\mathcal{V} \subseteq \mathcal{H}$ and $\mathcal{V}' \subseteq \mathcal{H}' = H'_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{C}$ be both of rank one. Then $\mathcal{V}$ and $\mathcal{V}'$ are horizontal IFF $\mathcal{V} \otimes \mathcal{V}'$ is horizontal in $\mathcal{H} \otimes_{\mathcal{C}} \mathcal{H}'$.

   (Write $v = \sum l_i h_i$, $v' = \sum l'_j h'_j$ etc.)

6. $\mathcal{N}$ of $\mathcal{M}$ is horizontal IFF the top wedge $\bigwedge^d \mathcal{N}$ is horizontal in $\bigwedge^d \mathcal{M}$.

   For the non-trivial implication $\Longleftarrow$:

   by assumption, there is a local frame $(\nu_1, \ldots, \nu_d)$ of $\mathcal{N}$ such that
$$\nu_1 \wedge \ldots \wedge \nu_d \quad \text{as a section of } \textstyle\bigwedge^d \mathcal{N} \tag{7}$$
   equals some
$$\mu \in \bigwedge^d M \quad \text{n.b.: } M, \text{ not } \mathcal{M}! \tag{8}$$
   As a local section of a local system, μ is determined

   (assume the local frame is on a conencted open set U)

   by its value $\mu(s)$ at a point $s \in U$.

   In particular, we have
$$\mu(s) \;=\; \nu_1(s) \wedge \ldots \wedge \nu_d(s) \tag{9}$$
   Each $\nu_i(s) \in M_s$.

Caution: ν_i is a continuous section of $\mathbb{N}$, hence of $\mathbb{M}$, but is is a discontinuos section of M, in general.

On the other hand: $m_i := \nu_i(s)$ defines a local section of M! and we have the equality of local sections of $\bigwedge^d M$:

$$\mu = m_1 \wedge \ldots \wedge m_d. \tag{10}$$

(The above is the key, for most tensors in $\bigwedge^d \mathbb{C}^n$ do not correspdond to d-dimensional linear subspaces.)

It follows that μ is not only a section of $\bigwedge^d M$, but $\mu(s)$ corresponds to the linear span of the $\nu_i(s)$ in M_s, for every $s \in U$.

In other words: we have that $(m_1, \ldots, m_d)$ is a local frame for $\mathcal{N}$ so, as seen above, $\mathcal{N}$ is horizontal. □

7. Recall that if $H_{\mathbb{Z}}$ is a Hodge structure, then we have

$$H_{\mathbb{C}} = \bigoplus H^{pq} \tag{11}$$

and, for $t \in \mathscr{S} = \mathbb{C}^*$ and $v = \sum v_{pq} \in H_{\mathbb{C}}$:

$$t \star v = t \star \left(\sum v_{pq}\right) = \sum t^p \bar{t}^q v_{pq}. \tag{12}$$

Given $H_{\mathbb{Z}}$ and $H'_{\mathbb{Z}}$ two Hodge structures, we have that $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} H'_{\mathbb{Z}}$ is a Hodge structure and

$$t \star (h_{pq} \otimes h'_{rs}) \;=\; t^{p+r}\bar{t}^{q+s} \, (h_{pq} \otimes h'_{rs})\,. \tag{13}$$

8. A continuous family of Hodge structures on S is:

   - $H_{\mathbb{Z}}$ a local system on S of $\mathbb{Z}$-modules of finite type;
   - $(H_{\mathbb{Z}})_s$ a Hodge structure that varies continuously with $s \in S$.

   The meaning of "continuously" does not seems to be spelled out.

   My interpretations is:

   let $\mathcal{C}$ be the sheaf of $\mathbb{C}$-valued C^0-functions on S;

   let $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$;

   let $\mathbb{H}$ be the associated vector bundle;

   let $\mathcal{H} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{C}$ be the (sheaf of continuosu sections of) the vector bundle ($\mathbb{H}$);

   (we do not distinguish between $\mathcal{H}$ and $\mathbb{H}$, so we use goemetric language: vector bundles instead of sheaves and $H_{\mathbb{C}}$ as the the horizontal sections of $\mathcal{H}$ etc.);

then:

$$\mathcal{H} = \bigoplus \mathcal{H}^{pq} \tag{14}$$

and for every $h \in H_{\mathbb{C}}$, we have $h = \sum h_{pq}$ and, for every s, we have that

$$h(s) = \sum h_{pq}(s) \qquad \text{is the Hodge decomposition (second } \bullet \text{) above)} \tag{15}$$

Be careful: the $\mathcal{H}^{pq}$ are complex continuous subbundles and in general they are not horizontal! else the theory of VHS would be trivial!:
the h_{pq} are not necessarily horizontal sections of $\mathcal{H}$.
Then $\mathscr{S}$ act as in (10) by complex linear isomorphisms.
In particular, if $\mathcal{V} \subseteq \mathcal{H}$ is a subbundle, then we have

$$t\,\mathcal{V} \subseteq \mathcal{H} \tag{16}$$

In general, even if $\mathcal{V}$ is horizontal, there is no reason why $t\,\mathcal{V}$ should be horizontal.
Also, if we take the invariants $H_{\mathbb{C}}^{\pi_1(S,s)}$, there is no reason why the associated horizontal subbundle (which is trivial of some rank) should be, fiber by fiber, given by Hodge substrcutures, for t of an invariant is not necessarily an invariant.
Let us remark that (13) implies at once that if $\mathcal{V}$ is a subbundle on $\mathcal{H}$, then

$$(t\,\mathcal{V})^{\otimes n} = t\left(\mathcal{V}^{\otimes n}\right). \tag{17}$$

9. Such a continuos family is said to be hoomogeneous of weight n is all the fibers are PHS of weight n.
10. The definition of continuous family of $\mathbb{Q}$-Hodge structures on S is analogous.
11. A polarization of a continuos family H of rational structures of weight n is an arrow of local systems

$$\Psi \;:\; H_{\mathbb{Q}} \otimes_{\mathbb{Q}} H_{\mathbb{Q}} \longrightarrow \mathbb{Q}(-n) \tag{18}$$

that defines a polarization in every fiber.
Recall that a polarization .....
12. If H is polarized and H' is a subobject, then H' has a complement H'' in H by the usual argument:
take the annhilator of $\Psi(-, H')$.

6.1.2 Statement of Deligne semisimplicity theorem

1. Let S be connected.

   Let $\mathscr{C}$ be a full subcategory of the cateogry of continuous families of $\mathbb{Q}$-Hodge structures on S subject to the following conditions.

   - $\mathscr{C}$ is stable under the following operations:
     – taking a direct summand;
     – taking direct sums;
     – taking tensor products.
   - The constant $\mathbb{Q}(n)$, $\forall\, n \in \mathbb{Z}$, are in $\mathscr{C}$.
   - Every element of $\mathscr{C}$ of pure weight n is polarizable.
   - For every $H \in \mathscr{C}$, there is a local system $H'_{\mathbb{Z}}$ on S with

   $$H'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = H_{\mathbb{Q}}. \tag{19}$$

   - For every $H \in \mathscr{C}$, the biggest constant local subsystem T of $\mathcal{H}$ is a constant family of Hodge substructures of H.

2. **Lemma.** *If $\mathscr{C}$ satisifes the first three bullets, then:*

   – $\mathscr{C}$ is an Abelian semisimple subcategory of the category of continuous families of $\mathbb{Q}$-Hodge structures on S.

   – $\mathscr{C}$ is closed under taking duals, wedges and Homs.

3. **Proposition.** *Let me just say this: if $f : X \to Y$ is smooth and proper with Y smooth, then the $R^i f_* \mathbb{Q}$ are polarizable variations og PHS belonging to a suitable category satisfying the requirements made above (five bullets).*

4. **Theorem (Semisimplicity Theorem)**.*Let S be connected, locally connected, locally simply connected and let $s \in S$. Let $\mathscr{C}$ be a category of rational continuous Hodge structures satifying the five bullets above. Then, if $H \in \mathscr{H}$, then $H_{\mathbb{Q}}$ is semisimple (the $\pi_1(S,s)$ representation on $(H_{\mathbb{Q}})_s$ is semisimple.*

6.1.3 Proof of the semisimplicity Theorem

1. We assume we have $H \in \mathscr{C}$ as in the theorem.

2. **Lemma.** *Let $V \subseteq H_{\mathbb{C}}$ be a local subsystem of rank one such that $V^{\otimes n}$ is trivial for some $n \geq 1$. Then $t\,\mathcal{V}$ is horizontal for every $t \in \mathscr{S}$.*

   Proof. By §1.4, we have that it is enough to show that $(t\,\mathcal{V})^{\otimes n}$ is horizontal.

By (17), we have that:

$$(t\,\mathcal{V})^{\otimes n} = t\left(\mathcal{V}^{\otimes n}\right), \tag{20}$$

so that now it is enough to show that $t\,(\mathcal{V}^{\otimes n})$ is horizontal.

We are assuming $\mathcal{V}^{\otimes n}$ is trivial hence generated by a horizontal *global* section v.

We are also assuming that the invariants form a Hodge substructure and this means that $t \star v$ is also invariant and hence is an horizontal section, thus proving $t\,(\mathcal{V}^{\otimes n})$ is horizontal, as desired.

□

3. We prove the theorem by induction on the dimension of $(H_{\mathbb{Q}})_s$.

   Since H splits already according to weights, we may assume H is of some pure weight n.

   Hence it is polarizable by assumption (one of the bullets).

4. Let d the smallest dimension of the non trivial simple local subsystem of $H_{\mathbb{C}}$.

   Take W to be the sum of all the local subsystems of $H_{\mathbb{C}}$ of rank d (each one is simple, the sum is semisimple).

   Then $W \subseteq H_{\mathbb{C}}$ is in fact defined over $\mathbb{Q}$, i.e. of the form $W = W_{\mathbb{Q}} \otimes \mathbb{C}$ for some local subsystem $W_{\mathbb{Q}} \subseteq H_{\mathbb{Q}}$.

   Note that since W is semisimple, so is $W_{\mathbb{Q}}$.

5. Let $H_{\mathbb{Z}}$ be a local system of free $\mathbb{Z}$-modules with $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = H_{\mathbb{Q}}$

   (a local system of f.g. $\mathbb{Z}$-modules exists by assumption (one of bullets); divide by the torsion).

   Set $W_{\mathbb{Z}} := H_{\mathbb{Z}} \cap W_{\mathbb{Q}}$.

   Then $W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = W_{\mathbb{Q}}$ and (let $e :=$ rank of W):

$$\bigwedge^{d} W = \bigwedge_{\mathbb{Z}}^{d} \otimes_{\mathbb{Z}} \mathbb{C}. \tag{21}$$

6. By integrality: **the action of $\pi_1(S, s)$ on $\bigwedge^e W$ is by ± 1.**

   This means we can apply the Lemma above to it.

7. Let $V \subseteq W$ be one of the rank d local subsystems of $H_{\mathbb{C}}$.

   Since W is sum of simples, W is semisimple, so that V has a complement V' in W: $W = V \oplus V'$.

Apply the Lemma to the local subsystem

$$\bigwedge^{d} V \otimes \bigwedge^{e-d} V' \subseteq \bigwedge^{d} H_{\mathbb{C}} \otimes \bigwedge^{e-d} H_{\mathbb{C}} \tag{22}$$

and deduce that, for every $t \in \mathscr{S}$, we have, as in (17):

$$t\left(\bigwedge^{d} V \otimes \bigwedge^{e-d} V'\right) = \bigwedge^{d}(t\,V) \otimes \bigwedge^{e-d}(t\,V') \subseteq \bigwedge^{d} H_{\mathbb{C}} \otimes \bigwedge^{e-d} H_{\mathbb{C}} \tag{23}$$

is horizontal.

By §1.5, we have that both factors in the middle are horizontal.

By §1.6, we have that $t\,\mathcal{V}$ is horizontal, for every $t \in \mathscr{S}$.

But, by the very definition of W, this meand that $t\,\mathcal{V} \subseteq \mathcal{W}$ for every $t \in \mathscr{S}$.

This means that $\mathscr{S}$ preserves $\mathcal{W}$, i.e. that $W_{\mathbb{Q}}$ is a Hodge substructure of H.

8. Take a polarization Ψ of H and split off W using Ψ:

$$H = W \oplus W' \qquad \in \mathscr{C}. \tag{24}$$

We conclude by induction on the dimension: i.e. take W', it must be semsimiple because the dimension went down. $\square$

6.2 Semisimplicity via purity

9. If $f : X \to Y$ is over $\mathbb{C}$, then semsimplicity can be deduced from an analogous result in the context of finite fields.

This circle of ideas is explained in [BBD], §6 and we do not say anyhting more here.

Let us outline the semisimplicity statement in that context.

10. Let $f : X \to Y$ be a map over F an algebraic closure of a finite field.

We assume that f is smooth.

Then $R^i := R^i f_* \overline{\mathbb{Q}}\ell$ is semisimple.

11. **Sketch of proof.** (See [BBB], §5; or Deligne's [Weil2].)
    - Let $f_0 : X_0 \to Y_0$, R^i_0 be the situation over a finite subfield $F_0 = F_q$.
    The main result of Weil2 is that R^i_0 is pure of weight i:
    this means that at every geometric point centered at a point in $Y_0(F_{q^n})$, the action of the n-th iterate of Frobenius has eigenvalues algebraic numbers, all of whose conjugate λ have

$$|\lambda| = (q^n)^{i/2}.$$

- Any subquotient of R_0^i is pure of weight i.
  Take any extension of R_0^i:

$$e_0 : Ext^1(B_0, A_0).$$

  We have the base change map to F:

$$bc\ :\ Ext^1(B_0, A_0) \longrightarrow Ext^1(B, A)$$

  There is a general spectral sequence (of Galois cohomology [BBD], that implies ([BBD], 5.1.2.5) that the base change arrow factors through the Frobenius invariants of the rhs as follows:

$$Ext^1(B_0, A_0) \xrightarrow{epi} Ext^1(B, A)^{Frob} \xrightarrow{mono} Ext^1(B, A)$$

  By the basic properties of weights and Hom ([BBD], 5.1.15.iii), we have that $Ext^1(B, A)$ has weights ≥ 1 (because A, B, being subquotients of R_0^i have weight 0).
  It follows that there are "no" invariants.
  It follows that any extension e_0 splits after passing to F.
- Take $B \subseteq R^i$ be the sum of all simples inside of R^i.
  Then B is semi-simple.
  It is the biggest semi-simple in R^i.
  It follows that it is invariant under Frobenius.
  It follows that it comes from a B_0.
  The corresponding extension

$$0 \longrightarrow B_0 \longrightarrow R_0^i \longrightarrow Q_0 \longrightarrow 0$$

  must split, over F, from the previous point.
  If $Q \neq 0$, then it has a non-zero simple subobject S.
  It follows that we can lift that subobject to $B \oplus S$ and enlarge B, contradiction. $\square$