Final Exam
MAT 534
December 16, 1999

Name:	**ID #:**

Please answer all the questions in the space provided. As usual, you have give a solution, not just an answer. You can use any results we covered in class, in homeworks, or any theorem in Artin's book—if in doubt, ask me. Good luck!

1. Consider the matrix

$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

   (a) Is the bilinear form on $\mathbb{R}^3$, given by $(x, y) = \sum b_{ij} x_i y_j$, positive definite?

   (b) Find an orthogonal matrix C such that CBC^t is diagonal.

Solution: (1) By Sylvester criteria, B is positive definite iff the determinants $|2|, \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}, \det B$ are positive. Explicit calculations shows that they are indeed positive, and so B is positive definite.
(2) By general theory, there exists an orthonormal basis $\{v_1, v_2, v_3\}$ consisting of eigenvectors of B, and if P is a matrix with columns v_1, v_2, v_3, then $P^{-1}BP = P^tBP$ diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ on the diagonal.
To find the eigenvalues, we write $\det(B - \lambda I) = 0$, which gives $\lambda_1 = 2, \lambda_{2,3} = 2 \pm \sqrt{2}$. Finding eigenvectors for each of them, we get $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$. They are indeed orthogonal, but not of unit length. Normalizing them, we get

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \qquad v_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \qquad v_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

and thus

$$C = P^t = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

2. Let V, W be finite-dimensional vector spaces over $\mathbb{C}$.

   (a) Construct an isomorphism $\mathrm{Hom}(V, W) \simeq (\mathrm{Hom}(W, V))^*$.

   (b) Describe explicitly the corresponding pairing $\mathrm{Hom}(V, W) \otimes \mathrm{Hom}(W, V) \to \mathbb{C}$.

Both constructions should be done without using bases in V, W. If you can't do this, partial credit will be given for a construction using a basis, but then you have to show that the result is independent of the choice of basis.

Solution: (1) Using the isomorphism $\mathrm{Hom}(X, Y) = X^* \otimes Y$ (discussed in class), we get

$$(\mathrm{Hom}(W, V))^* \simeq (W^* \otimes V)^* \simeq W^{**} \otimes V^* = W \otimes V^* \simeq \mathrm{Hom}(V, W)$$

(2) It is given by $(f, g) = \mathrm{tr}(fg)$, where $f \in \mathrm{Hom}(W, V), g \in \mathrm{Hom}(V, W)$.

3. Let A be a complex matrix such that $A^3 = A^2$. Describe all possible Jordan normal forms of A. *Solution*: Rewriting $A^3 - A^2 = A^2(A - I) = 0$, we see that the only possible eigenvalues of A are $0, 1$. Moreover, $V = \mathrm{Ker}(A - I) \oplus \mathrm{Ker}(A^2)$ (this was used in proving the generalized eigenspace decomposition). Therefore, the only Jordan blocks that can appear in JNF of A are those for which either $J = I$ or $J^2 = 0$. The only Jordan blocks satisfying $J^2 = 0$ are 1×1 and 2×2 blocks with eigenvalue 0. Therefore, the JNF of A consists of the blocks

$$\boxed{1}, \quad \boxed{0}, \quad \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}}$$

4. Let V be the space of functions on $\mathbb{R}$ of the form $f = e^{\lambda x} p(x)$, where p is a polynomial with complex coefficients in x of degree $\leq n$. Let $T\colon V \to V$ be the operator $\frac{d}{dx}$

   (a) Find Jordan normal form of T.

   (b) Find all values of λ for which there exists an operator Q such that $Q^2 = T$.

   Solution: (1) In the basis $e^{\lambda x} x^k$, the operator T is given by the matrix

$$\begin{bmatrix} \lambda & 1 & 0 & \dots\dots & \\ 0 & \lambda & 2 & 0 & \dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots \\ \dots\dots & 0 & \lambda & n & \\ \dots\dots & \dots & 0 & \lambda & \end{bmatrix}$$

   Therefore, in the basis $e^{\lambda x} \frac{x^n}{n!}$, the matrix of T will be exactly the Jordan block of size $(n+1)$ with λ on the diagonal.

   (2) This is almost the same as one of teh homework problems. Here is teh soution: if $\lambda = 0$, then such a Q does not exist. Indeed, since T is nilpotent, the same must be true for Q. On the other hand, for any nilpotent Q, $\dim \mathrm{Ker}(Q^2) \geq 2$ (suffices to check it for one Jordan block), while $\dim \mathrm{Ker}\, T = 1$.

   If $\lambda \neq 0$, then such Q exists. Indeed, take J to be $(n+1)$ Jordan block with $\sqrt{\lambda}$ on the diagonal. Then explicit calculation (which was in one of the homeworks) shows that the JNF of J^2 is a Jordan block $J_{\lambda, n+1}$; thus, $J^2 = P^{-1} T P$ for some P. Taking $Q = PJP^{-1}$, we get $Q^2 = T$.

5. Let V be a vector space of dimension n over $\mathbb{C}$, and $A\colon V \to V$—a linear operator. Denote by $A^{\otimes 2}, S^2(A), \Lambda^2(A)$ the corresponding operators in the spaces $V^{\otimes 2}, S^2(V), \Lambda^2(V)$ respectively: $A^{\otimes 2}(v \otimes w) = (Av) \otimes (Aw)$, etc. Calculate $\mathrm{tr}\, A^{\otimes 2}, \mathrm{tr}\, S^2(A), \mathrm{tr}\, \Lambda^2(A)$ in terms of $\mathrm{tr}\, A^k$.

Solution: If A is diagonalizable, with eigenbasis $\{v_i\}$ and eigenvalues λ_i, then the basis $\{v_i \otimes v_j\}_{1\le i,j\le n}$ in $V \otimes V$ is an eigenbasis for $A^{\otimes 2}$, with eigenvalues $\lambda_i\lambda_j$. Thus,

$$\operatorname{tr} A^{\otimes 2} = \sum_{i,j} \lambda_i\lambda_j = (\sum \lambda_i)^2 = (\operatorname{tr} A)^2$$

Similarly,

$$\operatorname{tr} \Lambda^2(A) = \sum_{i<j} \lambda_i\lambda_j = \frac{1}{2}((\sum \lambda_i)^2 - \sum \lambda_i^2) = \frac{1}{2}((\operatorname{tr} A)^2 - \operatorname{tr} A^2).$$
$$\operatorname{tr} S^2(A) = \sum_{i\le j} \lambda_i\lambda_j = \frac{1}{2}((\sum \lambda_i)^2 + \sum \lambda_i^2) = \frac{1}{2}((\operatorname{tr} A)^2 + \operatorname{tr} A^2).$$

If A is not diagonalizable, we can choose a basis $\{v_i\}$ in which A has upper triangular form; then $A^{\otimes 2}$ will also have an upper triangular form if we suitably order the basis $v_i \otimes v_j$; thus, the above proof works.

6. Classify all groups of order 44.

   Solution: It follows from Sylow's theorem that there is exactly one subgroup K of order 11 (which implies that K is normal) and either one or 11 subgroups of order 4, all of them conjugate. Let H be one of the subgroups of order 4. Note that $H \cap K = \{e\}$ (all non-identity elments in K have order 11; all non-identity elements in H have order 2 or 4). Thus, if we knew that K and H commute, it would imply that $G = K \times H$. The problem is, we do not know that K and H commute. However, since K is normal, it follows that $hkh^{-1} \in K$; thus, if we choose a generator x of K, so that $x^{11} = e$, then $hxh^{-1} = x^i$ for some i. Let us find possible values of i.

   Since there are exactly two possible groups of order 4: $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$, we have two cases:

   Case 1: $H \simeq \mathbb{Z}_4$. Let y be the generator of $\mathbb{Z}_4$; then $y^4 = e$, and we must have $yxy^{-1} = x^i$. Combining this with $y^4xy^{-4} = x^{i^4} = x$, we see that $i^4 \equiv 1 \mod 11$. Explicit checking shows that the only possible values of i are 1 and 10 (for example: for $i = 5, i^2 \equiv 25 \equiv 4 \mod 11$, so $i^4 \equiv (i^2)^2 \equiv 4^2 \equiv 16 \equiv 5 \mod 11$). Thus, either $yxy^{-1} = x$, which means that H and K commute and $G = H \times K = Z_{11} \times Z_4$, or $yxy^{-1} = x^{10} = x^{-1}$, in which case the group is generated by x, y with the relations $x^{11} = 1, y^4 = 1, yxy^{-1} = x^{-1}$.

   Case 2: $H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. In this case the analysis is similar to the previous one, except that we have two commuting generators y_1, y_2 of order 2, which gives 4 possible cases:

   (i) $y_1xy_1^{-1} = x, \quad y_2xy_2^{-1} = x$

   (ii) $y_1xy_1^{-1} = x^{-1}, \quad y_2xy_2^{-1} = x$

   (iii) $y_1xy_1^{-1} = x, \quad y_2xy_2^{-1} = x^{-1}$

   (iv) $y_1xy_1^{-1} = x^{-1}, \quad y_2xy_2^{-1} = x^{-1}$

   The first case gives $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{11}$; the other 3 cases can be shown to be isomorphic ((ii) and (iii) – by $y_1 \leftrightarrow y_2$, (ii) and (iv)—by $y_2 \mapsto y_1y_2$).

   Yes, this is indeed a complicated problem—I know that, and I didn't really expect you to be able to complete it, but I wanted to see how far you could go. As I promised, I will be rather generous on partial credit for this problem.

7. Define the *commutant* $[G, G]$ of a group G to be the subgroup generated by all elements of the form $xyx^{-1}y^{-1}, x, y \in G$.

   (a) Show that $[G, G]$ is a normal subgroup in G and $G/[G, G]$ is commutative

   (b) Show that for the group T of complex upper-triangular matrices, the commutant $[T, T]$ is the group N of upper-triangular matrices with 1 on the diagonal.

   Solution: (a) Immediate from the definitions.

   (b) Explicit calculation shows that every matrix of the form $xyx^{-1}y^{-1}, x, y \in T$ has ones on the diagonal, so $[T, T] \subset N$. To show equality, consider matrices $x = diag(\lambda_1, \ldots, \lambda_n); y = I + E_{ij}, i < j$. Then: $xyx^{-1}y^{-1} = I + (\frac{\lambda_i}{\lambda_j} - 1)E_{ij}$. Therefore, $[T, T]$ contains all matrices of the form $I + \mu E_{ij}, i < j$. But it is easy to show that these matrices generate N.