MAT 534: Problem Set 10
SOLUTIONS

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In the problems below, V is a finite-dimensional vector space of dimension n, $e_1, \dots, e_n$ is a basis in V. As always, S_d is the group of all permutations of $\{1, \dots, d\}$, acting on $V^{\otimes d}$ by permutation of components, and

$$\mathrm{Sym} = \frac{1}{d!} \sum_{s \in S_d} s, \qquad \mathrm{Alt} = \frac{1}{d!} \sum_{s \in S_d} \mathrm{sgn}(s) s$$

are projectors on the subspaces of symmetric (respectively, antisymmetric) tensors.

6. Let $\xi \in \Lambda^2 V$.

   (a) Prove that it is possible to choose a basis $e_1, \dots, e_n$ in V such that $\xi = e_1 \wedge e_2 + e_3 \wedge e_4 + \dots e_{k-1} \wedge e_k$ for some even $k \le n$. (Hint: we have already proved this before, in different language...)
   Solution: Suffices to note that $\Lambda^2 V$ is the space of skew-symmetric bilinear forms on V^*, and apply the classification theorem for skew-symmetric bilinear forms.

   (b) Show that ξ can be written in the form $v \wedge w$ for some $v, w \in V$ iff $\xi \wedge \xi = 0$.
   Solution: One direction is obvious: if $\xi = v \wedge w$, then $\xi \wedge \xi = v \wedge w \wedge v \wedge w = -v \wedge v \wedge w \wedge w = 0$. Conversely, assume $\xi \wedge \xi = 0$. Write ξ as in part a. Explicit calculation shows that if $k \ge 2$, then $\xi \wedge \xi = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 +$other monomials; thus, $\xi \wedge \xi = 0$ is only possible if $k = 2$, $\xi = e_1 \wedge e_2$.

7. Let $W \subset V^{\otimes d}$ be the subspace spanned by vectors of the form $t - s_i(t)$, where s_i are elementary transpositions. Also, denote by φ the natural surjection $V^{\otimes d} \to V^{\otimes d}/W$

   (a) Prove that $\mathrm{Sym}|_W = 0$.
   Solution: Obvious from $\mathrm{Sym}\, s_i = \mathrm{Sym}$.

   (b) Prove that $S^d V \cap W = \{0\}$ and thus

   $$\varphi|_{S^d V} : S^d V \to V^{\otimes d}/W \tag{1}$$

   is injective.
   Solution: If $t \in S^d V \cap W$, then $\mathrm{Sym}(t) = t = 0$ (by part a).

   (c) Prove that $\varphi(t) = \varphi(s_i t) = \varphi(s(t))$ for any $s \in S_d, t \in V^{\otimes d}$, and thus $\varphi(\mathrm{Sym}(t)) = \varphi(t)$.
   Solution: $\varphi(s_i t) - \varphi(t) = \varphi(s_i(t) - t) = 0$ since $s_i(t) - t \in W$. Applying this repeatedly, we get

   $$\varphi(s_{i_1} \dots s_{i_k}(t)) = \varphi(s_{i_2} \dots s_{i_k}(t)) = \dots = \varphi(t)$$

   (d) Prove that the map (1) is an isomorphism.
   Solution: This map is injective (by part b). It is also surjective: for any $t \in V^{\otimes d}$, the class of t in $V^{\otimes d}/W$ can be written as $\varphi(\mathrm{Sym}(t))$ by part c.

*8 Define operators $\varepsilon_i, \mathrm{i}_j : \Lambda V \to \Lambda V$ by

$$\varepsilon_i w = e_i \wedge w$$
$$\mathrm{i}_j(e_j \wedge w) = w$$
$$\mathrm{i}_j(e_{i_1} \wedge \dots \wedge e_{i_k}) = 0 \quad \text{if none of } i_l = j$$

(a) Show that these conditions uniquely define i_j. Show that i_j satisfies the skew-symmetric Leibniz identity:

$$\mathrm{i}_j(\xi \wedge \eta) = (\mathrm{i}_j \xi) \wedge \eta + (-1)^d \xi \wedge (\mathrm{i}_j \eta)$$

if $\xi \in \Lambda^d V$. Thus, i_j is the skew-symmetric analogue of the operator $\frac{d}{de_j}$ on $SV = \mathbb{C}[e_1, \dots, e_n]$.
Solution: It follows from the definition that

$$\begin{aligned} &\mathrm{i}_{i_k}(e_{i_1} \wedge \dots \wedge e_{i_k} \wedge \dots) \\ &\quad = (-1)^{k-1} \mathrm{i}_{i_k}(e_{i_k} \wedge e_{i_1} \wedge \dots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \dots) \\ &\quad = (-1)^{k-1} e_{i_1} \wedge \dots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \dots \end{aligned}$$

(we assume all i_a are distinct.) Leibniz identity is straightforward from this.

(b) Prove that the operators $\varepsilon_i, \mathrm{i}_j$ satisfy the *Clifford algebra* relations:

$$\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0$$
$$\mathrm{i}_j \mathrm{i}_k + \mathrm{i}_k \mathrm{i}_j = 0$$
$$\mathrm{i}_j \varepsilon_i + \varepsilon_i \mathrm{i}_j = \delta_{ij}$$

(compare with: $x_i \frac{d}{dx_j} - \frac{d}{dx_j} x_i = \delta_{ij}$, where x_i is considered as the operator of multiplication by x_i on $\mathbb{C}[x_1, \dots, x_n]$).
Solution: The first two are straightforward. As for the last one, let us apply the right-hand side to a monomial in $\Lambda^d V$. Such a monomial can always be written in the form $x \wedge w$, where x is one of $1, e_i, e_j, e_i \wedge e_j$ and w does not contain e_i, e_j. For each of these cases the identity is easily checked by direct calculation.

9. Show that the wedge product is associative:

$$(x_1 \wedge \dots \wedge x_p) \wedge (y_1 \wedge \dots \wedge y_q) = x_1 \wedge \dots \wedge x_p \wedge y_1 \wedge \dots \wedge y_q)$$

i.e.

$$\mathrm{Alt}_{p+q}\Big(\mathrm{Alt}_p(x_1 \otimes \dots \otimes x_p) \otimes \mathrm{Alt}_q(y_1 \otimes \dots \otimes y_q)\Big) = \mathrm{Alt}_{p+q}(x_1 \otimes \dots \otimes x_p \otimes y_1 \otimes \dots \otimes y_q)$$

Solution: It follows from $\mathrm{Alt}_{p+q} s = \mathrm{sgn}(s) \mathrm{Alt}_{p+q}$ for every $s \in S_{p+q}$ and definition of $\mathrm{Alt}_p, \mathrm{Alt}_q$ that

$$\mathrm{Alt}_{p+q}(\mathrm{Alt}_p \otimes \mathrm{Alt}_q) = \mathrm{Alt}_{p+q}$$

as operators in $V^{\otimes(p+q)}$.

*10. (a) Show that $\dim \Lambda^{n-1} V = n$
Solution: Follows from general formula: $\dim \Lambda^k V = \binom{n}{k}$.

(b) Construct an isomorphism $\Lambda^{n-1}V = \Lambda^n V \otimes V^*$ (hint: look at Problem 2)

Solution: We have an obvious map $\Lambda^{n-1}V \otimes V \to \Lambda^n V : \xi \otimes v \mapsto \xi \wedge v$. As in problem 2, this gives rise to a map $f\colon \Lambda^{n-1}V \to \Lambda^n V \otimes V^*$. This map can be decribed as follows: $\xi \mapsto \sum(\xi \wedge e_i) \otimes e^i$. One immediately sees that it is isomorphism.

(c) For a linear operator $A\colon V \to V$, consider the corresponding operator $\Lambda^{n-1}A\colon \Lambda^{n-1}V \to \Lambda^{n-1}V$. Write the matrix of $\Lambda^{n-1}A$ in the basis $b_1 = e_2 \wedge_3 \wedge \cdots \wedge e_n$, $b_2 = e_1 \wedge e_3 \wedge \cdots \wedge e_n, \ldots$.

Solution: There are several ways of doing this. Here is one: assume that A is invertible. Then it is rather easy to show that if we identify $\Lambda^{n-1}V = \Lambda^n V \otimes V^*$ as above, then $\Lambda^{n-1}A = \Lambda^n A \otimes (A^*)^{-1} = \det A (A^*)^{-1}$.

On the other hand, it is known that the matrix $C = \det A(A^t)^{-1}$ is the matrix of algebraic complements: $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant of the matrix obtained from A by removing i-th row and j-th column. Thus, the answer (at least for invertible A) is given by the matrix C_{ij} defined above; since the entries of this matrix are polynomials in entries of A, the usual arguments show that the condition $\det A \neq 0$ can be removed.