Math 322 Lecture 1 (Tu/Th) MIDTERM II SPRING 2009 Mark de Cataldo

1. (50 total pts)

(a) (10 pts) State the inverse function theorem for functions of n variables.

If $f: \mathcal{U} \to \mathbb{R}^n$ is continuously differentiable, when $\mathcal{U}$ is open in $\mathbb{R}^n$, and $f(x_0) = y_0$, and $\det((Df)(x_0)) \neq 0 \Rightarrow \exists V, W;\ x_0 \in V \subseteq \mathcal{U}$, $y_0 \in W \subseteq \mathbb{R}^n$, such that $f(\cdot)$ is injective and surjective, from V to W, with a continuously differentiable inverse.

(b) (10 pts) Give the definition of a set of measure zero in R^n.

Z is of measure zero $\iff \forall \varepsilon > 0, \exists \{\mathcal{U}_i\}_{i \in \mathbb{N}}$, $\mathcal{U}_i$: open rectangle, s.t $Z \subseteq \bigcup_{i \in \mathbb{N}} \mathcal{U}_i$ and $\underbrace{\sum_{i \in \mathbb{N}} v(\mathcal{U}_i)}_{} < \varepsilon$.

volume.

(c) (10 pts) Give an explicit example of a nonlinear function $f : R^2 \to R^2$, with $a = (0,0)$ and $f(a) = (0,0)$ that satisfies the assumptions of the inverse function theorem and, for that example, compute the 2×2matrix $(f^{-1})'(0,0)$.

[It's extremely hard to choose when you have got such a tremendously broad range of choice!]

Define $F(x,y) = (x^3+2x, e^y-1) \Rightarrow F(0,0) = (0,0)$.

$$\frac{\partial F}{\partial(x,y)} = \begin{bmatrix} 3x+2 & 0 \\ 0 & e^y \end{bmatrix}$$

$$\det(Df(0,0)) = 2 \Rightarrow (Df)^{-1} = (Df^{-1}) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) (10 pts) Give an example of a set of measure zero but with non zero content. (Provide a short justification for the answer.)

$\mathbb{Z}$, as a subset of $\mathbb{R}$.

countable $\longrightarrow$ of zero measure.

But every set of content zero has to be bounded.

(e) (10 pts) Give an example of a differentiable function $f : R^2 \to R^2$ which is invertible but for which the hypotheses of the inverse function theorem are not met at some point. (Provide a short justification for the answer.)

$F(x, y) = (x^3, y)$

Invertibility is obvious.

$$Df = \begin{bmatrix} 3x & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Rightarrow Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det Df(0,0) = 0 \rightarrow$$

The hypotheses are not satisfied.

2. (50 total pts) Consider the system of equations

$$\sin x + y^2 = u + \cos v^3 - 1, \quad x + \cos y^2 = -e^u.$$

(a) Can you apply the implicit function theorem and deduce that you can express (x, y) implicitly in terms of (u, v), that is

$$(x, y) = g(u, v)$$

in a neighborhood of $(0, 0, 0, 0)$? If yes calculate $g'(0, 0)$.

$F(x, y, u, v) : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$

$(x, y, u, v) \longmapsto (\sin x + y^2 - u - \cos v^3 + 1,\ x + \cos y^2 + e^u)$

$F(x, y, u, v) = 0$ is the system of the equation.

$$Df = \begin{array}{c} \begin{bmatrix} \cos x & 2y & -1 & 3v^2 \sin v^3 \\ 1 & -2y \sin y^2 & e^u & 0 \end{bmatrix} \\ \begin{array}{cccc} x & \quad y & \quad u & \quad\quad v \end{array} \end{array}$$

$$Df(0,0,0,0) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The submatrix corresponding to (x, y) is not invertible

$\Rightarrow$ The assumptions of implicit FT donot hold.

(b) Can you apply the implicit function theorem and deduce that you can express (x, u) implicitly in terms of (y, v), that is

$$(x, u) = g(y, v)$$

in a neighborhood of $(0, 0, 0, 0)$? If yes calculate $g'(0, 0)$.

The submatrix corresponding to (x, u) is invertible: $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} =: A \rightarrow A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$.

$\Rightarrow$

$$g' = \begin{bmatrix} 1/2 & +1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. (50 total pts) Prove that a bounded function $f : A \to R$ on a closed rectangle is integrable if and only if for every $\epsilon > 0$ there is a partition of A into closed subrectangles such that $U(f, P) - L(f, P) < \epsilon$.

If $f(\cdot)$ is integrable, by definition, $\forall \varepsilon > 0, \exists P_1 ; P_2$

$\mathcal{U}(f, P_1) - I < \varepsilon/2 \quad , I - L(f, P_2) < \varepsilon/2$

Let P be the common refinement $\rightarrow$.

$$\begin{cases} \mathcal{U}(f, P) - I < \varepsilon/2 \\ I - L(f, P) < \varepsilon/2 \end{cases} \Rightarrow \mathcal{U}(f,P) - L(f,P) < \varepsilon$$

Conversely, not that always $\sup\limits_P L(f,P) \leq \inf\limits_P \mathcal{U}(f,P)$.

If $\sup\limits_P L(f,P) \neq \inf \mathcal{U}(f,P) \Rightarrow \exists \delta > 0$, s.t

$\forall P: \quad \mathcal{U}(f,P) > L(f,P) + \delta \Rightarrow \mathcal{U}(f,P) - L(f,P) > \delta$

But this contradicts the assumption (that $\forall \varepsilon > 0, \exists P$:

4. (50 pts) Let $f : R^n \to R$ be differentiable with the property that there is a positive integer m such that

$$f(tx) = t^m f(x), \qquad \forall x \in R^n, \qquad \forall t \in R.$$

Prove that

$$\sum_i^n x_i D_i f(x) = m f(x).$$

(Hint: consider $g(t) := f(tx)$ and consider g'.)

Define $\begin{cases} g(t) := f(tx). \\ g : \mathbb{R} \to \mathbb{R} \end{cases} \to$

$$\frac{d}{dt} g = \frac{d}{dt} f(tx) =$$

inner product

By chain-rule: $Df(tx) \cdot \frac{\partial}{\partial t}(tx) = ((D_1 f)(tx), \ldots, (D_n f)(tx)) \cdot \ldots$

$$\ldots (x_1, \ldots, x_n) = \sum_i (D_i f)(tx) \cdot x_i$$

On the other hand:

$$\frac{d}{dt} f(tx) = \frac{d}{dt}\left(t^m f(x)\right) = m\, t^{m-1} f(x)$$

Let $t = 1$

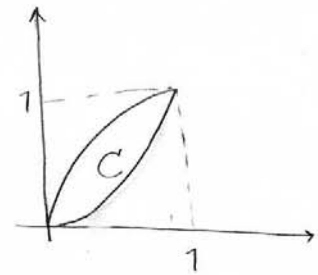
$$\to m f(x) = \sum_i (D_i f)(x) \cdot x_i \quad \blacksquare$$

5. (50 pts) Let C be the bounded region between the two curves $y = x^2$ and $x = y^2$.

(a) What is the definition of $\int_C (x - y)$?

(b) Justify the fact that the integral $\int_C (x - y)$ exists.

(c) Compute $\int_C (x - y)$.



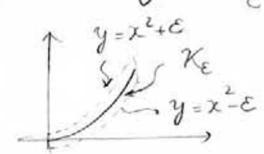
a) You can take any closed rectangle, say $R = [-L, L] \times [L, L]$ which contains C, and then:

$$\int_C (x-y) = \int_R \chi_C \cdot (x-y)$$

b) The boundary of 'C' is of measure zero. Heuristacally, we can, for instance say that the curve $y = x^2$ can be bounded by the region K_ε:

$\Rightarrow \forall \varepsilon$: curve $\subseteq K_\varepsilon$

and K_ε can be made as small as desired.



c) $$\int_C x - y = \int_0^1 \int_{x^2}^{\sqrt{x}} (x-y)\, dy\, dx = \int_0^1 \left(xy - y^2/2\right)\Big|_{x^2}^{\sqrt{x}} dx =$$

$$\int_0^1 \left(x^{3/2} - x^3 + x^4/2 - x/2\right) dx = \left\{\frac{3}{2} x^{5/2} - \frac{1}{4} x^4 + \frac{1}{10} x^5 - \frac{x^2}{4}\right\}\Big|_0^1$$