W7. - Analysis in Several Variables:

Lemma 3-1; (upper sums)

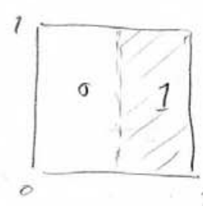
Note that, (following Spivak's Notation), $M_S(f) \geq M_{S_i}(f)$

$$\to M_S(f)\cdot v(S) = M_S(f)[v(S_1) + \dots + v(S_\alpha)] \geq$$

$$M_{S_1}(f)v(S_1) + M_{S_2}(f)\,v(S_2) + \dots + M_{S_\alpha}(f)\cdot v(S_\alpha)$$

$$\Rightarrow \mathcal{U}(f,P) \geq \mathcal{U}(f,P') \quad \blacksquare$$

3-1 - It suffices to introduce a partition for each $\varepsilon > 0$, by Thm 3-3.



let $P_\varepsilon = \{ [0, \tfrac{1}{2}-\varepsilon]\times[0,1], [\overset{+1/2}{-\varepsilon}, \overset{+1/2}{\varepsilon}]\times[0,1], [\tfrac{1}{2}+\varepsilon, 1]\times[0,1]\}$.

It is easy to see that $\mathcal{U}(f,P_\varepsilon) - L(f,P_\varepsilon) = 2\varepsilon$

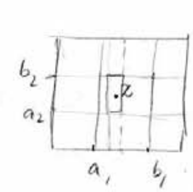
3-2 - We prove it for one point : $\{z\} = \{x \mid f(x) \neq g(x)\}$.

It is easy to see that the same idea works for any finite number of points.

f & g are bounded. Let M be $\max\{\sup|f|, \sup|g|\}$.

f is integrable $\longrightarrow \exists\, P : \quad \mathcal{U}(f,P) - L(f,P) < \varepsilon$.

Now, refine P: let $z \in [a_1,b_1]\times \dots \times [a_n,b_n]$.



Define $P' = (P_1', P_2, P_3, \ldots, P_n)$ where

P_1' refines P_1 : $\underbrace{[a_1, b_1]}_{\in P_1} \longmapsto \underbrace{[a_1, z_1-\delta], [z_1-\delta, z_1+\delta], [z_1+\delta, b_1]}_{\in P_1'}$.

$z_1 = \pi^1(z)$.

and define $\delta = \min\left\{ b_1 - z_1, z_1 - a_1, \frac{\epsilon}{2M(b_2-a_2)(b_3-a_3)\cdots(b_n-a_n)} \right\}$.

$$\rightarrow U(g,P') - L(g,P') \le \underbrace{|L(g,P') - L(f,P')|}_{\text{by construction of } P'} + \underbrace{|L(f,P') - U(f,P')|}_{P' \text{ is finer than } P}$$
$$+ \underbrace{|U(g,P') - U(f,P')|}_{\text{by construction of } P'}$$
$$\le \epsilon + \epsilon + \epsilon = 3\epsilon$$

■

3-3- a) In general, $\sup_{x\in A} f + \sup_{x\in A} g \ge \sup(f+g)$
$\inf_{x\in A} f + \inf_{x\in A} g \le \inf_{x\in A}(f+g)$

Where A is any set.

→ on any subrectangle particularly,

$$\left(\begin{cases} M_S f + M_S g \ge M_S(f+g) \\ m_S f + m_S g \le m_S(f+g) \end{cases}\right.$$
$$\rightarrow \begin{cases} U(f,P) + U(g,P) \ge U(f+g,P) \\ L(f,P) + L(g,P) \le L(g+f,P) \end{cases}$$

By noting that $L(f+g,P) \le U(f+g,P)$, if

$\begin{cases} U(f,P^1) - L(f,P^1) < \epsilon \\ U(g,P^2) - L(g,P^2) < \epsilon \end{cases} \rightarrow$ Define P so that it refines P^1 & P^2.

Then $\mathcal{U}(f+g, P) - L(f+g, P) < 2\varepsilon$.

c) Obviously, $\begin{cases} L(cf, P) = cL(f,P) \\ \mathcal{U}(cf, P) = c\,\mathcal{U}(f,P). \end{cases}$

$\rightarrow$ let P be such that

$\mathcal{U}(f,P) - L(f,P) < \varepsilon/c \Rightarrow c\,\mathcal{U}(f,P) - cL(f,P) =$

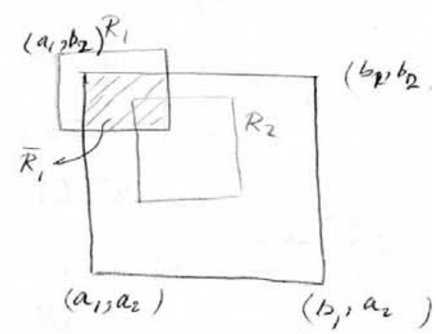
$\mathcal{U}(cf,P) - L(cf,P) < \varepsilon \rightarrow cf$ is inte

$\Rightarrow \int cf = \sup_P L(cf,P) = c \sup_P L(f,P) = c\int f$.

3-5. Evidently, for any subrectangle S: $\begin{cases} m_S(f) \le m_S(g) \\ M_S(f) \le M_S(g) \end{cases}$

$\rightarrow \sup_P L(f,P) \le \sup_P L(g,P)$

3-8; First of all, note that we can just consider rectangles as $\bar{R}_1$, instead of R_1, because $v(R_1) \ge v(\bar{R}_1)$.



Let $\{R_i\}$ be a finite cover of $[a_1,b_1]\times \cdots \times [a_1,b_n]$. We can then divide each member of, R_i, into $\{R_i^1, \dots, R_i^{m_i}\}$, such that the members $\{R_1^1, \dots, R_1^{m_1}, \dots, R_k^1, \dots, R_k^{m_k}\}$ have their intersection at most on their boundary.

It is easy to see that $\sum_{K_i}\sum_i v(R_i^{K_i}) \le \sum_i v(R_i)$

Also, $\sum v(R_i^K) = (b_1-a_1)\times \dots \times (b_n-a_n) > 0$

$\rightarrow$ For every finite cover by rectangles: $\sum v(R_i) > 0$

3-13

13: a) We shall prove that every n-tuple of elements of countable sets can be ordered to a seq. Particularly, every $[a_1,b_1]\times \dots \times [a_n,b_n]$ can be thought of as an 2n-tuple with an extra constraint: $a_i \le b_i$.

We know that a 2-tuple can be ordered in a seq. By induction, let k-tuples be ordered. By applying the same method and the fact that k+1-tuple = (k-tuple, 1-tuple), we get a seq. of k+1-tuples.

~~$\mathbb{R}^n$~~ $\leftarrow$ b) It is easy to see that any open set $\mathcal{U}$ can be covered by elements: $[a_1,b_1]\times \dots \times [a_n,b_n]$, $a_i,b_i \in \mathbb{Q}$, such that $B_\alpha \subseteq \mathcal{U}$. $\Rightarrow$ If $\mathcal{O} = \{\mathcal{U}^\beta\}_\beta$ is an open cover for a set, every $\mathcal{U}^\beta$ can be covered by B_α^β's. But B_α^β's are countable. For every B_α^β (which covers $\bigcup_{\mathcal{O}} \mathcal{U}^\beta$), choose a $\mathcal{U}^\beta$, s.t $B_\alpha^\beta \subseteq \mathcal{U}^\beta$. In (a) we put an order on B_α^β's $\Rightarrow$ we can arrange $\mathcal{U}^\beta$'s by the order on B_α^β's. Note that we now have a countable collection of $\mathcal{U}^\beta$'s.