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Hi;

I don`t know what happenned to the margins! Sorry for some question numbers.

MAT 322 - Analysis in several variables;
HW2 ; (Spivak - Calculus on mnfld's)

1-14; Let $\mathcal{U} = \bigcup_{\alpha} \mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is open (and α is merely a index)

$\rightarrow \forall x \in \mathcal{U} \rightarrow \exists \alpha : x \in \mathcal{U}_\alpha \underset{\text{by openness}}{\Longrightarrow} \exists$ open rectangle $V \subseteq \mathcal{U}_\alpha$ s.t $x \in V$.

$\Rightarrow \mathcal{U}$ is open.

Let V & ~~W~~ $\mathcal{U}$ be open. $\forall x \in \mathcal{U} \cap V$, $\exists \begin{cases} W \text{(open rect.)} \subseteq \mathcal{U} \\ Y \text{(open rect.)} \subseteq V \end{cases}$

$\rightarrow W \cap Y \subseteq \mathcal{U} \cap V$.
It is easy to see that intersection of two open rectangles is an open rectangle $\Rightarrow x \in \underbrace{W \cap Y}_{\text{open rect.}} \subseteq \mathcal{U} \cap V$

$\Rightarrow \mathcal{U} \cap V$ is open.

Take $\mathcal{U}_n = (0, 1 - 1/n) \Rightarrow \bigcup_i \mathcal{U}_i = (0, 1]$ which is not open.

1-16; $A = \{x \in \mathbb{R}^n \mid |x| \le 1\}$

$\text{Int } A = \{x \in \mathbb{R}^n \mid |x| < 1\}$.
$\text{Bd } A = \{x \in \mathbb{R}^n \mid |x| = 1\}$.
$\text{Ext } A = \{x \in \mathbb{R}^n \mid |x| > 1\}$.

$B = \{x \in \mathbb{R}^n \mid |x| = 1\}$

$\text{Int } B = \emptyset$. [By definition, you can check $\text{Int}(\text{Bd } X) = \emptyset$.
$\text{Bd } B = \cancel{\emptyset} B$
$\text{Ext } A = \mathbb{R}^n \setminus B$.

$C = \{x \in \mathbb{R}^n \mid x^i\text{'s are rational}\}$.

$\text{Int } C = \emptyset$ [Every rectangle intersects $\mathbb{R}^n \setminus$
$\text{Bd } C = \mathbb{R}^n$
$\text{Ext } A = \emptyset$.

1-19; A : closed $\Rightarrow \mathbb{R} \setminus A$: open.

Let $\begin{cases} r \in [0,1] \\ r \notin A \end{cases} \Bigg\} \rightarrow \exists \begin{cases} (a,b) \subseteq \mathbb{R} \setminus A \\ r \in (a,b) \end{cases}$

But every interval (a,b) contains at least a rational umber $q \in [0,1] \Rightarrow A$ misses a rational in $[0,1]$ ※ contra

narks;
t script;
~~When can~~
$(a,b) \subseteq [0,1]$, $(1, 0$ are rational$)$. This allows us to cl $q \in [0,1]$.

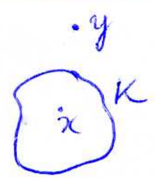
o; Boundedness;

$$\mathbb{R}^n = \bigcup_n \underbrace{\cancel{P}B(0,n)}_{\text{open square with sides equal to "}n\text{"}}.$$

If K is not bounded this cover cannot be reduced.

Closedness;

We prove that K^c (Complement) is open.



Let $y \notin K$. $\rightarrow \forall x \in K,\ \exists \varepsilon_x > 0$ s.t.

$\underbrace{B_{\varepsilon_x}(x)}_{\text{ball with centre at } x,\ \text{radius } \varepsilon_x} \cap B_{\varepsilon_x}(y) = \phi$

Then $\bigcup_{x \in K} B_{\varepsilon_x}(x) \supseteq K \xrightarrow{K\text{-compact}} \bigcup_i B_{\varepsilon_{x_i}}(x_i) \supseteq K$.

and $\bigcap_i B_{\varepsilon_{x_i}}(y)$ is an open set conta

(Finite intersection of balls).

Note that $\left(\bigcup_i B_{\varepsilon_{x_i}}(x)\right) \cap \left(\bigcap_i B_{\varepsilon_{x_i}}(y)\right) = \phi$

$\rightarrow K \cap \left(\bigcap_i B_{\varepsilon_{x_i}}(y)\right) = \phi \Rightarrow K^c$ is open. ▨

(HW2-2)

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2; $\forall x \in \mathcal{U}$, specificly $x \in C$, $\exists$ open rectangle $V_x \subseteq \mathcal{U}$.

For each V_x, construct $\tilde{V}_x$ this way; $\tilde{V}_x$ is a rectang[le] with centre "x", with sides half of that of V_x.

If $\tilde{V}_x = (a_1, b_1) \times \cdots \times (a_n, b_n)$

Define $W_x = [a_1, b_1] \times \cdots \times [a_n, b_n]$. By construction $W_x \subseteq \mathcal{U}$.

and by (1-5) is compact.

$$\rightarrow C \subseteq \bigcup_{x \in C} \tilde{V}_x \underset{\text{Compact}}{\Longrightarrow} C \subseteq \bigcup_{x_i \in C} \tilde{V}_{x_i} \subseteq \overbrace{\bigcup_{x_i \in C} W_{x_i}}^{D}$$

Finite union of comapcts, W_{x_i}, is compact (why?) $\rightarrow D$ is compact. ▨

4; f conti $\Rightarrow f^i$.

Proof; $\rightarrow f^i = \pi^i \circ f$. π^i & f are both conti (prove that π^i is conti!) Composition of two conti. is conti. (prove!) $\rightarrow f^i$ is conti.

If you don't like that proof;

note that $|\pi^i(x)| < |x|$ $^* \Rightarrow$

If for $\delta > 0$, $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

by $*$ $|f^i(x) - f^i(a)| < \epsilon \Rightarrow f^i(\cdot)$ is conti.

Conversely;

$\forall \epsilon > 0$, $\exists \delta_1, \ldots, \delta_n$; s.t $|f^i(x) - f^i(a)| < \epsilon$ if $|x-a| < \delta^i$

Let $\delta = \min_i \{\delta_i\}$.

Then if $|x-a| < \delta$:

$|f(x) - f(a)| \leq \sqrt{n}\,\epsilon$. $\rightarrow f(\cdot)$ is conti.

1-25; By problem 1-10 $\exists M>0 : \forall x; \ |T(x)| < M|x|$

$\rightarrow \forall \epsilon > 0$, if $\delta < \frac{\epsilon}{M} \Rightarrow |x-a|<\delta \rightarrow |T(x)-T(a)|<\epsilon$.

1-28; Spivak already solved it! Instead of $f(y)$ you could cho any function unbound at "x".

To prove unboundedness;

$\forall N > 0$, the set $C = \{y \in A \,|\, |x-y| < 1/N\} \neq \phi$

let $z \in C \rightarrow f(z) > N$.

1-29; $f(A)$ is compact. $B := f(A) \subseteq \mathbb{R}$ $\rightarrow$ bounded. closed

"B" is bounded $\Rightarrow$ has a supremum.

"B" is closed $\rightarrow$ contains its sup $\rightarrow$ has got a max.

In a similar fashion it has a min. or you can look at $-f(\cdot)$.

Note; Every closed set of $\mathbb{R}$ contains its supremum.

If "C" is closed, the set of the set $D = (c,\infty) \cap (\mathbb{R} \setminus C)$, $c \in C$

is open and contains all ~~its~~ upper bounds of C.

$\rightarrow$ If $\sup C \in D \rightarrow \exists (\alpha,\beta) \subseteq D$, $\sup C \in (\alpha,\beta)$,

But then $\alpha < \sup C$. ※.

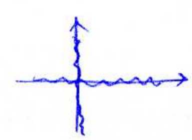
Define $\begin{cases} g : S \to \mathbb{R} \\ g(w) = f(w) \end{cases}$ $\quad S = \{ w \mid |w| = 1 \}$.

$g(w) = g(x, y) = |y| x$. $\quad \begin{cases} g(0,1) = g(1,0) = 0 \\ g(w) = -g(-w). \end{cases}$

$\overset{\cap}{S}$

To prove that it is of the form presented by $|x| g$ it suffices to verify that $f(\lambda x) = \lambda f(x)$ $(\lambda > 0)$. which is obviously satisfied for $f(x, y)$.

In fact $f(\cdot)$ is not differentiable on x- and y-axes. (Try to follow this procedure and prove this statement).



Proof for $(0,0)$;

We assume that $Df|_{(0,0)}$ exists. For any seq. $\{(h,k)\} \to (0,0)$ we should have $\dfrac{|f(0 + (h_n, k_n)) - f(0) - \lambda((h_n, k_n))|}{|(h_n, k_n)|}$

I) $(h_n, k_n) = (0, 1/n) = \frac{1}{n} \hat{e}_2$ — unit standard vector : $(0,1)$

$$\to \frac{|f(0, 1/n) - 0 - \lambda(1/n \, \hat{e}_2)|}{\frac{1}{n}} = \lambda(\hat{e}_2) \quad \text{which is consta}$$

$\Rightarrow \lambda(\hat{e}_2) = 0$. (a)

II) Similarly $(h_n, k_n) = (1/n, 0) \Rightarrow \lambda(\hat{e}_1) = 0$. (b)

$(a, b) \longrightarrow \lambda(\cdot) = 0$.

III) $(h_n,k_n)=(1/n,1/n)=\frac{1}{n}(\hat{e}_1+\hat{e}_2)$.

$$\rightarrow \left|\frac{\sqrt{1/n^2}-\overset{f(0)}{0}-\overset{\lambda(h_n,k_n)}{0}}{\underbrace{\sqrt{1/n^2+1/n^2}}_{|(h_n,k_n)|}}\right| = \frac{1}{\sqrt{2}} \neq 0 \quad ※.$$

-7 ; We claim that the linear transformation : $\begin{cases}\lambda:\mathbb{R}^n\to\mathbb{R}\\ \lambda(x)=0\end{cases}$ satisfies:

$$\frac{|f(x)-\overset{0}{f(0)}-0|}{|x|} \leq \frac{|x|^2}{|x|} = |x| \underset{\text{as } |x|\to 0}{\longrightarrow} 0$$

-8 ; Let for $f=(f^1,f^2)$ we have

$$\frac{|f(x+h)-f(x)-Df(h)|}{\cancel{\ }|h|} \to 0$$

Then in fact ; $\frac{|(\overbrace{f^1(x+h)-f^1(x)-\pi^1\circ Df(h)}^{\text{[first component]}},\ \overbrace{f^2(x+h)-f^2(x)-\pi^2\circ Df(h)}^{\text{2-nd component}})|}{\cancel{\ }|h|} \longrightarrow$

Remember that $2|(a,b)|=2(a^2+b^2)\geq(|a|+|b|)^2$.

$\Longrightarrow$ ~~[illegible]~~ both components tend to zero, if the vector does.

$$\Longrightarrow \frac{|f^i(x+h)-f^i(x)-\pi^i\circ Df(h)|}{|h|} \longrightarrow 0 \qquad i=1,2.$$

$\rightarrow$ f^1, f^2 are differentiable with differentials $\begin{cases}\pi^1\circ Df\\ \pi^2\circ Df\end{cases}$

which are, (by a simple observation about matrices) . $\begin{cases}Df^1\\ Df^2\end{cases}$