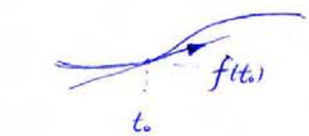
HW13-MAT322;

4-14; $f\circ c : [0,1] \to \mathbb{R}^m$. By definition, Tangent vector to $f\circ c$ is $(f\circ c)_*((e_1)_t) \underset{\text{by } 4\text{-}13}{=} f_* \circ \underbrace{c_*((e_1)_t)}_{v\text{: Tangent to 'c'.}} = f_*(v)$

4-15; Tangent line to the graph can be parameterised by:

$$y = \left(\frac{df}{dt}\Big|_{t=t_0}\right)(x - t_0) + f(t_0) \quad (*)$$



And the end of tangent vector is: $(t_0, f(t_0)) + \left(1, \left(\frac{df}{dt}\right)\Big|_t\right)$

In $(*)$, let $x = t_0 + 1 \Rightarrow y = f(t_0) + \left(\frac{df}{dt}\right)\Big|_{t_0}$

$\square$

4-16; $|c(t)| = 1 \to \langle c(t), c(t)\rangle = 1 \Rightarrow \frac{d}{dt}\langle c(t), c(t)\rangle = 0$

Euclidean inner product.

But $\frac{d}{dt}\langle c(t), c(t)\rangle = 2\langle \underbrace{\frac{d}{dt} c(t)}_{\text{Tangent}}, c(t)\rangle = 0$

that is in a circle, the Tangent is always orthogonal to the radius!

4-17; a) Given a vector-field F, at each 'p': $F(p) \in \mathbb{R}^n_p$, $(F^1, \dots, F^n) \in \mathbb{R}$

Simply, define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(p) = (F^1(p), \dots, F^n(p))$

That's possible because $\mathbb{R}^n_p$, at each point 'p' is just like a copy of $\mathbb{R}^n$.

(b) By definition: $\operatorname{div} f = \sum_i \frac{\partial f}{\partial x_i}$ Recall that $Df = [D_i f_j]_{n\times n}$

$\Rightarrow \operatorname{trace} Df = \sum_i D_i f_i = \operatorname{div} f$.

HW 13 - MAT 322;

4-18; $D_v f(p) = \underbrace{(Df(p))}_{1\times n\text{-matrix}}(v) = [D_1 f(p) \quad \cdots \quad D_n f(p)] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} =$

$$\sum D_i f(p) \cdot v_i = \langle Df(p), v \rangle = \langle v, \underbrace{Df(p)}_{w_p} \rangle$$

Recall that Schwar(t?)z's inequality states;

$D_v f(p) = v \cdot w_p \leq |v||w_p|$ and the equality happens iff $v = \lambda w_p$, $\lambda > 0$.

4-19; a) $df = \sum \frac{\partial f}{\partial x_i} dx^i = \underbrace{\frac{\partial f}{\partial x}}_{F_1} dx + \underbrace{\frac{\partial f}{\partial y}}_{F_2} dy + \underbrace{\frac{\partial f}{\partial z}}_{F_3} dz =$

But

$\nabla f = [\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}]$.

- By definition of exterior derivative:

$$d(\omega_F^1) = \frac{\partial F^1}{\partial x} \cancel{dx \wedge dx}^{\,0} + \frac{\partial F^1}{\partial y} dy \wedge dx + \frac{\partial F^1}{\partial z} dz \wedge dx + .$$

$$\frac{\partial F^2}{\partial x} dx \wedge dy + \frac{\partial F^2}{\partial y} \cancel{dy \wedge dy}^{\,0} + \frac{\partial F^2}{\partial z} dz \wedge dy +$$

$$\frac{\partial F^3}{\partial x} dx \wedge dz + \frac{\partial F^3}{\partial y} dy \wedge dz + \frac{\partial F^3}{\partial z} \cancel{dz \wedge dz}^{\,0} =$$

Note that $dx \wedge dy = -dy \wedge dx$ and ... .

Therefore $:= \left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F^1}{\partial z} - \frac{\partial F^3}{\partial x}\right) dz \wedge dx$

$$+ \left(\frac{\partial F^3}{\partial y} - \frac{\partial F^2}{\partial z}\right) dy \wedge dz. = \omega^2_{\underbrace{\nabla \times F}_{curl}}$$

HW13 – MAT322;

4.19-a – Cont'd;

$$d(\omega_F^2) = \frac{\partial F^1}{\partial x}\, dx \wedge dy \wedge dz + \frac{\partial F^1}{\partial y}\, \cancelto{0}{dy \wedge dy} \wedge dz + \frac{\partial F^1}{\partial z}\, \cancelto{0}{dz \wedge dy \wedge dz}$$

$$+ \frac{\partial F^2}{\partial x}\, \cancelto{0}{dx \wedge dz \wedge dx} + \frac{\partial F^2}{\partial y}\, dy \wedge dz \wedge dx + \frac{\partial F^2}{\partial z}\, \cancelto{0}{dz \wedge dz \wedge dx}$$

$$+ \frac{\partial F^3}{\partial x}\, \cancelto{0}{dx \wedge dx} \wedge dy + \frac{\partial F^3}{\partial y}\, \cancelto{0}{dy \wedge dx \wedge dy} + \frac{\partial F^3}{\partial z}\, dz \wedge dx \wedge dy$$

$$= \left(\frac{\partial F^1}{\partial x} + (-1)^2 \frac{\partial F^2}{\partial y} + (-1)^2 \frac{\partial F^3}{\partial z}\right) dx \wedge dy \wedge dz$$

$$= \underbrace{(\nabla \cdot F)}_{div}\, dx \wedge dy \wedge dz$$

b) $$d(\omega^1_{\vec{\nabla} f}) \underset{by (a)}{=} d(df) = \omega^2_{\nabla \times (\vec{\nabla} f)} = 0$$

$$\Rightarrow \nabla \times \vec{\nabla} f = 0. \quad \text{Note that } \omega_F^2 = 0 \longleftrightarrow F = 0.$$

$$d(\omega_F^1) = \omega^2_{\nabla \times F}$$

$$d(d(\omega_F^1)) = 0 = d(\omega^2_{\nabla \times F}) = \nabla \cdot (\nabla \times F)\, dx \wedge dy \wedge dz$$

$$\Rightarrow \nabla \cdot (\nabla \times F) = 0$$

c) $\nabla \times F = 0 \Rightarrow \omega^2_{\nabla \times F} = 0 \Rightarrow d(\omega_F^1) = 0 \Rightarrow$ By Poincaré's lemma, $\exists\, \alpha$, s.t $\omega_F^1 = d\alpha$, That is:

$F^1 = \frac{\partial \alpha}{\partial x}$, $F^2 = \frac{\partial \alpha}{\partial y}$, $F^3 = \frac{\partial \alpha}{\partial z}$, so $\nabla \alpha = F$.

HW13 - MAT322;

4-19-c - Cont'd; $\nabla . F = 0 \to d(\omega_F^2) = 0 \Rightarrow$ By Poincaré's lemma;

$\exists \beta;\quad d\beta = \omega_F^2 \qquad$ Let $\beta = \beta^1 dx + \beta^2 dy + \beta^3 dz$

$$\to d\beta = \left(\frac{\partial \beta^1}{\partial y} - \frac{\partial \beta^2}{\partial x}\right) dx \wedge dy + \left(\frac{\partial \beta^1}{\partial z} - \frac{\partial \beta^3}{\partial x}\right) dz \wedge dz +$$

$$\left(\frac{\partial \beta^2}{\partial z} - \frac{\partial \beta^3}{\partial y}\right) dz \wedge dy = \omega^2_{\nabla \times G}$$

where $G = (\beta^1, \beta^2, \beta^3)$.

4-20; Let β be defined on $f(\mathcal{U})$, and closed: $d\beta = 0$.

$\to$ We can define $f^*(\beta)$, which is a form on $\mathcal{U}$.

and:

$d(f^*(\beta)) = f^*(d\beta) = f^*(0) = 0 \quad \Rightarrow$ By assumption,

closedness of $f^*(\beta)$ gives exactness: $\to \exists \theta:\ d\theta = f^*(\beta)$

$$\Rightarrow (f^{-1})^*(d\theta) = (f^{-1})^* f^*(\beta) = \beta = d\big(\underbrace{(f^{-1})^*\theta}_{\omega}\big) = d\omega. \qquad \blacksquare$$