3.3.3.

(a) $\{(5,0)+s(-3,1) \mid s \in R\}$

(b) $\{(1,1,1)+s(-1,1,1) \mid s \in R\}$

(c) $\{(2,1,1)+s(-1,1,1) \mid s \in R\}$

(d) $\{(2,0,-1)+s(0,1,1) \mid s \in R\}$

(e) $\{(1,0,0,0)+r(-2,-1,0,0)+s(3,0,1,0)+t(-1,0,0,0) \mid r,s,t \in R\}$.

(f) $\{(1,0)\}$.

(g) $\{(0,0,0,1)+r(-3,1,1,0)+s(1,-1,0,1) \mid r,s \in R\}$.

3.3.4. (a) (1) $A^{-1}=\begin{pmatrix}-5 & 3\\ 2 & -1\end{pmatrix}$ (2) $\begin{pmatrix}x_1\\x_2\end{pmatrix}=A^{-1}\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}-11\\5\end{pmatrix}$

(b) (1) $A^{-1}=\begin{pmatrix}\frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9}\\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9}\end{pmatrix}$ (2) $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=A^{-1}\begin{pmatrix}5\\1\\4\end{pmatrix}=\begin{pmatrix}3\\0\\-2\end{pmatrix}$.

3.3.6. It is equivalent to solve $\begin{cases}a+b=1\\ 2a-c=11\end{cases}$. The answer is $\left\{\begin{pmatrix}\frac{11}{2}\\-\frac{9}{2}\\0\end{pmatrix}+t\begin{pmatrix}1\\-1\\2\end{pmatrix} \,\middle|\, t\in R\right\}$

3.3.7. (a) $\mathrm{rank}(A)=2$, $\mathrm{rank}(A \vdots b)=3$.

(b) $$(A \vdots b)=\left[\begin{array}{ccc:c}1 & 1 & -1 & 1\\ 2 & 1 & 3 & 2\end{array}\right]\xrightarrow{r_2-2r_1}\left[\begin{array}{ccc:c}1 & 1 & -1 & 1\\ 0 & -1 & 5 & 0\end{array}\right]\longrightarrow\left[\begin{array}{ccc:c}1 & 1 & -1 & 1\\ & 1 & -5 & 0\end{array}\right]$$

$$\xrightarrow{r_1-r_2}\left[\begin{array}{ccc:c}1 & 0 & 4 & 1\\ & 1 & -5 & 0\end{array}\right].$$

$\mathrm{rank}(A)=\mathrm{rank}(A|b)=2$

(c) $\mathrm{rank}(A)=\mathrm{rank}(A|b)=3$.

(d) $\mathrm{rank}(A)=\mathrm{rank}(A|b)=4$

(e) $\mathrm{rank}(A)=2$, $\mathrm{rank}(A|b)=3$.

3.3.10. Let $Ax = b$ be this system of linear equations.

Then $A$ is a $m \times n$ matrix,

And the condition translates to : $\operatorname{rank}(A) = m$.

By Thm 3.5, if we write $A = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \in M_{m\times 1}$

$m = \operatorname{rank}(A) =$ the dimension of the subspace generated by its columns $\alpha_1, \cdots \alpha_n$

$\leq$ the dimension ——— " ——— $\alpha_1, \cdots, \alpha_n$ and an additional column $b$.

$= \operatorname{rank}(A \mid b)$ --- ①

By Corollary 2(b) to Thm 3.6.

$\operatorname{rank}(A \mid b) =$ the dimension of the subspace generated by its rows

$\leq$ the number of its rows $= m$. --- ②

By ① and ②, $m = \operatorname{rank}(A) \leq \operatorname{rank}(A \mid b) \leq m$

So, $\operatorname{rank}(A) = \operatorname{rank}(A \mid b)$

By Thm 3.11, we know $Ax = b$ has solution.

# HW9

## 3.4

2 (a) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right] \to$

$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & -2 & 3 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \to$

$\to \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

$\begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = -1 \end{cases}$ Solution: $\left\{ \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} \right\}$.

(c) $\begin{cases} x_1 = 2 \\ x_2 = 3 \\ x_3 = -2 \\ x_4 = -1 \end{cases}$ Solution: $\left\{ \begin{pmatrix} 2 \\ 3 \\ -2 \\ -1 \end{pmatrix} \right\}$.

(e) $\left[\begin{array}{cccc|c} 1 & -4 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ which gives the linear system:

$\begin{cases} x_1 - 4x_2 - x_4 = 4 \\ x_3 - 2x_4 = 1 \end{cases} \to$

$\to \begin{cases} x_1 = 4t_1 + t_2 + 4 \\ x_2 = t_1 \\ x_3 = 2t_2 + 1 \\ x_4 = t_2 \end{cases}$ and the solution is:

$$\left\{ t_1 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} : t_1, t_2 \in R \right\}.$$

(g) $\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 23 & -23 \\ 0 & 0 & 1 & 0 & -6 & 7 \\ 0 & 0 & 0 & 1 & -9 & 9 \end{array}\right]$. As above, one gets solution:

$$\left\{ t_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -23 \\ 0 \\ 6 \\ 9 \\ 1 \end{pmatrix} + \begin{pmatrix} -23 \\ 0 \\ 7 \\ 9 \\ 0 \end{pmatrix} : t_1, t_2 \in R \right\}$$

(i) $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution: $\left\{ t_1 \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ -4 \\ 0 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} : t_1, t_2 \in R \right\}.$

3 (a) If $(A'|b')$ is rref, then $A'$ is rref too. So, by thm. 3.16a:

- $rk(A')$ = number of nonzero rows in $A'$
- $rk(A'|b')$ = " " " " " $(A'|b')$.

Thus $rk(A') \neq rk(A'|b') \iff$

$\iff$ there is a zero row in $A'$ which is nonzero in the augmented matrix $(A'|b') \iff$

$\iff (A'|b')$ contains a row whose only nonzero entry is in the last column.

(b) By thm. 3.11, $Ax = b$ is consistent $\iff$

$\iff rk(A) = rk(A|b)$.

Since $rk(A) = rk(A')$ and $rk(A|b) = rk(A'|b')$,
$Ax=b$ is consistent $\iff$
$\iff rk(A') = rk(A'|b') \iff$ (by part (a))
$\iff (A'|b')$ has no row whose only nonzero entry is in the last column.

(4) (a) rref is
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 4/3 \\ 0 & 1 & 0 & 1/2 & 1/3 \\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right]$$
$rk(A') = 3 = rk(A'|b') \Rightarrow$ system is consistent.

Solution is: $\left\{ t\begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4/3 \\ 1/3 \\ 0 \\ 0 \end{pmatrix} : t \in R \right\}$.

Basis for $K_H$ is: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix} \right\}$.

(b) rref is
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$
$rk(A') = 2 = rk(A'|b') \Rightarrow$ system is consistent.

Solution is: $\left\{ t_1\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} : t_1, t_2 \in R \right\}$.

Basis for $K_H$ is: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$.

(c) rref is:
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$
$rk(A') = 2 \neq 3 = rk(A'|b') \Rightarrow$ system not consistent.

⑥ Let $a_1, \ldots, a_6$ be the columns of A.
The rref form of A tells us that:

$a_2 = -3a_1$

$a_4 = 4a_1 + 3a_3$

$a_5 = 5a_1 + 2a_3 - a_6$

Thus: $A = \begin{bmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & -4 & 0 & 2 & 5 \end{bmatrix}$

⑧ $\begin{bmatrix} 2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\ -3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\ 4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\ -5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\ 2 & -6 & 1 & 6 & -3 & 12 & -2 & 7 \end{bmatrix}$

rref is: $\begin{bmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Thus $\{u_1, u_3, u_5, u_7\}$ is a basis.