Math 310 MIDTERM I SPRING 2017

Name and ID# : ______________________

Problem	Points Possible	Points
1	30	
2	40	
3	30	
4	40	
5	30	
6	30	
Total	200	

INSTRUCTIONS:

1. Show your work. Use the correct notation. Use only the methods developed so far in the course (example: do not use methods from later chapters that you may have learned elsewhere). Correct answers without sufficient work, or not using the method required will receive minimal or no credit. If you use a theorem from the book, you need to tell us which one you are using: give its name (example: "dimension theorem"), or its statement (example: "any two bases have the same number of elements").
2. Provide clearly written answers in the space provided. You can use the flip sides of the pages and page 8 as scrap paper. **Do not tear off any page. You must return all 8 pages.**
3. No books. No notes. Cell phones off and in backpack. No devices. No calculators.

1. (30 pts) Let V be the vector space of real functions $f : R \to R$, where addition and multiplication by scalars are defined in the usual way:

$$\forall f, g \in V, \forall t \in R: \ (f+g)(t) := f(t) + g(t), \qquad \forall f \in V, \forall c, t \in R: \ (cf)(t) := c \cdot f(t).$$

A real function $f \in V$ is said to be periodic, with period a fixed real number α, if $f(t+\alpha) = f(t)$, $\forall t \in R$.

Prove that the subset $W \subseteq V$ of periodic functions with period α is a subspace of V.

a) f_0, the zero function $\in W$: $f_0(t+\alpha) = 0 = f_0(t)$, $\forall t \in R$.

b) $f, g \in W \Rightarrow f+g \in W$: $(f+g)(t+\alpha) = f(t+\alpha) + g(t+\alpha)$
$= f(t) + g(t) =: (f+g)(t)$, $\forall t \in R$.

c) $c \in R, f \in W \Rightarrow cf \in W$: $(cf)(t+\alpha) = cf(t+\alpha) = c\, f(t)$
$= (cf)(t)$.

2. (40 total pts; each subproblem has the indicated value)

(a) (20 pts) Prove that the set $\{(1,1,1),(1,1,2),(2,0,0)\}$ is a basis for R^3.

We need to show that $\begin{cases} a+b+2c=0 \\ a+b=0 \\ a+2b=0 \end{cases}$ has a unique solution $(a,b,c)=(0,0,0)$.

$\begin{cases} a+b+2c=0 \\ 0\ \ 0-2c=0-0 \\ 0\ \ b\ \ \ \ =0-0 \end{cases}$ has solution $a=0$, $c=0$, $b=0$.

The 3 vectors are linearly indep. so that, since $\dim R^3=3$, they form a basis for R^3.

(b) (20 pts) Express $(2,0,-1)$ as a linear combination of the elements of the basis in part (a).

We need to solve $\begin{cases} a+b+2c=2 \\ a+b=0 \\ a+2b=-1 \end{cases}$

$\begin{cases} a+b+2c=2 \\ -2c=-2 \\ b=-1 \end{cases}$ $\begin{cases} a=+1 \\ c=1 \\ b=-1 \end{cases}$

$1(1,1,1)-1(1,1,2)+1(2,0,0)=(2,0,-1)$

3. (30pts) A real matrix $A \in M_{n\times n}(R)$ is symmetric if $A = A^t$, and it is anti-symmetric if $A = -A^t$. Prove that every matrix $M \in M_{n\times n}(R)$ can be written as the sum $M = B + C$ with B symmetric and C anti-symmetric.

(Hint: consider $M - M^t$.) (You may use the usual properties of transposition.)

$C := (M - M^t)$ is anti-symmetric: $(M-M^t)^t = M^t - (M^t)^t = M^t - M = -(M-M^t)$

$\frac{1}{2}(M-M^t)$ is anti-symmetric: $\left[\frac{1}{2}(M-M^t)\right]^t = \left(\frac{1}{2}M - \frac{1}{2}M^t\right)^t = -\frac{1}{2}(M-M^t)$

$B := (M+M^t)$ is symmetric: $(M+M^t)^t = M^t + M = M + M^t$

$\frac{1}{2}(M+M^t)$ is symmetric; $\left[\frac{1}{2}(M+M^t)\right]^t = \left(\frac{1}{2}M + \frac{1}{2}M^t\right)^t = \frac{1}{2}(M+M^t)$

We have $M = B + C = \frac{1}{2}(M+M^t) + \frac{1}{2}(M-M^t) = \frac{1}{2}2M = M.$

4. (40 pts) Let $T : F^3 \to F^2$ be the linear transformation $T(a,b,c) = (a+b, b+c)$.

(a) (20 pts) Find bases for $N(T)$ and for $R(T)$, and determine nullity(T) and rank(T).

$N(T) = \{ ae_1 + be_2 + ce_3 \mid a = -b,\ c = -b \}$. So $N(T)$ has $(-1,1,-1)$ as basis.

nullity(T) = dim $N(T) = 1$

By the dimension theorem, rank$(T) = 3 - 1 = 2$.

Since dim $F^2 = 2$, $R(T) = F^2$ and $\{(1,0),(0,1)\}$ is a basis.

(b) (20 pts) Let (e_1, e_2, e_3) be the standard basis in R^3, let (ϵ_1, ϵ_2) be the standard basis in R^2. Let $\beta := \{e_3, e_1, e_2\}$ and $\gamma := \{\epsilon_2, \epsilon_1\}$. Determine $[T]_\beta^\gamma$.

$T(e_3) = (0,1) = 1 \cdot \varepsilon_2 + 0 \cdot \varepsilon_1$

$T(e_1) = (1,0) = 0\,\varepsilon_2 + 1\,\varepsilon_1$

$T(e_2) = (1,1) = 1\,\varepsilon_2 + 1\,\varepsilon_1$

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

5. (30 pts) Let $P_2(\mathbb{R})$ be the vector space of polynomials of degree at most 2 over a field $\mathbb{R}$.

(a) (10 pts) Is there a surjective linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}^4$? Explain.

$\dim P_2(\mathbb{R}) = 3$ $\quad \dim F^4 = 4$

By the dimension theorem $\operatorname{rank}(T) \leq 3$.

Then $R(T)$ cannot have dimension 4, so $R(T) \neq \mathbb{R}^4$

and $\forall T$ fails to be surjective.

(b) (10 pts) Give an example of a linear transformation $T : P_2(\mathbb{R}) \to F^4$ such that $1, 1 + x^2 \in N(T)$ and $(1, 1, 1, 0) \in R(T)$.

To give T, it is enough to assign $T(\beta)$, where β is a basis.

$\beta := \{1, 1 + x^2, x\}$ form a basis

So it is enough to define $T(1) = 0$, $T(1+x^2) = 0$, $T(x) = (1,1,1,0)$.

(c) (10 pts) Is T as in (b) unique? Explain.

No

$\beta' = \{1, 1+x^2, 2x\}$ is a basis, define $T'(1) = 0$, $T'(1+x^2) = 0$
$T'(2x) = (1,1,1,0)$

Then $T \neq T'$, because $T'(x) = (1/2, 1/2, 1/2, 0) \neq T(x) = (1,1,1,0)$.

6. (30 pts) Let $T : V \to V$ be a linear transformation such that $T \circ T = T_0$ (the zero linear transformation) and such that $V \neq \{0\}$ is not the zero vector space.

Prove that T is not 1-1.

Since $V \neq \{0\}$, $\exists\ 0 \neq v \in V$.

Either $T(v) = 0$, and $v \in N(T)$,

or $T(v) \neq 0$, but $T(T(v)) = 0$,

In either case, we have $N(T) \neq \{0\}$.

Since T is 1-1 $\iff$ $N(T) = \{0\}$,

T is not 1-1.

SCRAP PAPER