1. (30 pts) Let V be a finite dimensional vector space over a field F, let β be an ordered basis of V. Given any vector $v \in V$, we have the vector $[v]_\beta \in F^n$ (the coordinates of v with respect to β). Prove that the assignment $v \mapsto [v]_\beta$ defines an isomorphism $\phi : V \to F^n$.

Let $\beta = \{v_1, \dots, v_n\}$. For $v, w \in V$, we can write $v = \sum_{i=1}^n a_i v_i$, $w = \sum_{i=1}^n b_i v_i$ for some $a_1, \dots, a_n, b_1, \dots, b_n \in F$ (since β is a basis).

Then, for $c \in F$,

$$cv + w = c\left(\sum_{i=1}^n a_i v_i\right) + \sum_{i=1}^n b_i v_i = \sum_{i=1}^n (ca_i + b_i) v_i.$$

Thus:

$$\phi(cv+w) = [cv+w]_\beta = (ca_1+b_1, \dots, ca_n+b_n) =$$
$$= c\cdot(a_1, \dots, a_n) + (b_1, \dots, b_n) =$$
$$= c\cdot[v]_\beta + [w]_\beta = c\cdot\phi(v) + \phi(w).$$

Therefore ϕ is linear.

Since $\dim V = n = \dim F^n$, to prove ϕ is an isomorphism it is enough to prove it is 1-1 (by the dimension theorem).

Let $v \in V$, $v = \sum_{i=1}^n a_i v_i$, $\phi(v) = 0$. Then $(a_1, \dots, a_n) = [v]_\beta = 0 \in F^n \implies a_i = 0 \ \forall i = 1, \dots, n$ $\implies v = 0$. Thus ϕ is 1-1.

2. (30pts) Let $P_2(R)$ be the space of polynomials of degree at most two over R. Let $\beta := (1, 1+x, 1+x+x^2)$ and $\beta' := (1-x, 1+x^2, -x+x^2)$. Determine the change of coordinate matrix Q from β'-coordinates to β-coordinates.

$$1-x = a\cdot 1 + b\cdot(1+x) + c\cdot(1+x+x^2) =$$
$$= a+b+c + (b+c)x + cx^2 \Longrightarrow$$
$$\Longrightarrow \begin{cases} a+b+c=1 \Longrightarrow a = 1-b = 1+1 = 2 \\ b+c=-1 \Longrightarrow b=-1 \\ c=0 \end{cases}$$

$$1+x^2 = a\cdot 1 + b(1+x) + c\cdot(1+x+x^2) \Longrightarrow$$
$$\Longrightarrow \begin{cases} a+b+c=1 \Longrightarrow a = 1-b-c = 1+1-1 = 1 \\ b+c=0 \Longrightarrow b=-1 \\ c=1 \end{cases}$$

$$-x+x^2 = a\cdot 1 + b(1+x) + c(1+x+x^2) \Longrightarrow$$
$$\Longrightarrow \begin{cases} a+b+c=0 \Longrightarrow a = -b-c = 2-1 = 1 \\ b+c=-1 \Longrightarrow b=-2 \\ c=1 \end{cases}$$

Thus $$Q = \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

3. (40pts)

(a) (20pts) Let $V = P_1(R)$ and consider the two linear functionals $f_1, f_2 \in V^*$ given by $f_1(p(x)) = \int_0^1 p(t)dt$, $f_2(p(x)) = p(0) - p'(0)$. They form an ordered basis $\{f_1, f_2\}$ for V^* (do not prove this fact). Find an ordered basis $\beta = \{p_1(x), p_2(x)\}$ whose dual basis β^* equals $\{f_1, f_2\}$.

let $p_1(x) = a + bx$, $p_2(x) = c + dx$.

$$\begin{cases} f_1(p_1(x)) = 1 \\ f_2(p_1(x)) = 0 \end{cases} \Rightarrow \begin{cases} \int_0^1 (a+bt)dt = \left(at + b\frac{t^2}{2}\right)\Big|_0^1 = a + \frac{1}{2}b = 1 \\ p_1(0) - p_1'(0) = a - b = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{3}{2}a = 1 \Rightarrow a = \frac{2}{3} \\ a = b \Rightarrow b = \frac{2}{3}. \end{cases}$$

$$\begin{cases} f_1(p_2(x)) = 0 \\ f_2(p_2(x)) = 1 \end{cases} \Rightarrow \begin{cases} c + \frac{1}{2}d = 0 \\ c - d = 1 \end{cases} \Rightarrow \begin{cases} \frac{3}{2}d + 1 = 0 \Rightarrow d = -\frac{2}{3} \\ c = d + 1 \Rightarrow c = \frac{1}{3} \end{cases}$$

Thus: $\beta = \left\{\frac{2}{3} + \frac{2}{3}x, \frac{1}{3} - \frac{2}{3}x\right\}$.

(b) (20pts) Let V be a vector space of dimension n, and let $\{f_1, f_2, \ldots, f_n\}$ be an ordered basis for V^*. Prove that there exists an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ of V such that its dual basis β^* equals $\{f_1, f_2, \ldots, f_n\}$. (Hint: consider the double dual V^{**}).

$\exists \psi : V \longrightarrow V^{**}$ isomorphism, where $\forall f \in V^*$,
$x \longmapsto \hat{x}$ $\qquad \hat{x}(f) = f(x)$.

Therefore, we can write the dual basis associated to β^* as $\beta^{**} = \{\hat{v}_1, \ldots, \hat{v}_n\}$ for some $v_1, \ldots, v_n \in V$.

By construction, $\hat{v}_i(f_j) = \delta_{ij}$ $\forall i,j = 1, \ldots, n$.

Thus $f_j(v_i) = \hat{v}_i(f_j) = \delta_{ij} = \delta_{ji}$ $\forall i,j = 1, \ldots, n$.

$\{v_1, \ldots, v_n\} = \psi^{-1}(\beta^{**})$ is a basis for V since ψ^{-1} is isom., and $\{v_1, \ldots, v_n\}^* = \{f_1, \ldots, f_n\}$ since $f_j(v_i) = \delta_{ji}$.

4. (30 pts) Express the invertible matrix

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$$

as an explicit product[1] of elementary matrices.

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \xrightarrow[\cdot \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}]{} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot]{} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

which is an elementary matrix.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Longrightarrow$$

$$\Longrightarrow \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

[1]The final answer should be a product of matrices: if, for example, a factor is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$, you need to write it explicitly as $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

5. (30pts) Let $T : R^3 \to R^3$ be defined by $T(a,b,c) := (a+b, b-2c, a+2c)$. Determine whether $v := (2,1,1) \in R(T)$.

$v \in R(T) \iff T(x,y,z) = v$ for some $(x,y,z) \in \mathbb{R}^3$

$$\iff \begin{cases} x+y=2 \\ y-2z=1 \\ x+2z=1 \end{cases} \text{ has a solution.}$$

This can be rewritten as $Ax=b$,

with $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

$Ax=b$ has a solution $\iff rk(A|b) = rk(A)$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 2 & 1 \end{array}\right) \xrightarrow[R_3-R_1]{} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \end{array}\right) \longrightarrow$$

$$\xrightarrow[R_3+R_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus:

$rk(A) = 2 = rk(A|b)$.

So $v \in R(T)$.

6. (40pts) Let $V \subseteq R^6$ be the subspace of solutions to the system of linear equations

$$x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 0, \quad 2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 0.$$

Let $S := \{(0,-1,0,1,1,0),(1,0,1,1,1,0)\} \subseteq R^6$. Then S is linearly independent and contained in V (do not show this). Complete S to a basis for V.

$$\begin{pmatrix} 2 & -1 & 0 & 2 & -3 & 1 \\ 2 & -1 & -1 & 3 & -4 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & -1 & 2 & 2 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & 3 \\ 0 & 1 & -1 & -1 & 2 & 2 \end{pmatrix}$$

$$\begin{cases} x_1 = x_3 - x_4 + x_5 - 3x_6 \\ x_2 = x_3 + x_4 - 2x_5 - 2x_6 \end{cases}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} = t_1 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + t_2 \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + t_3 \underbrace{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_3} + t_4 \underbrace{\begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_4}$$

$\{v_1, v_2, v_3, v_4\}$ is a basis for V.

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -3 \\ -1 & 0 & 1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & -3 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & -3 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The marked columns are linearly independent, therefore the required basis is:

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$