1. (30 pts) A real function f defined on the real line R is said to be an odd function if $f(t) = -f(-t)$, $\forall t \in R$. Prove that the set O of odd functions with the operations of addition and scalar multiplication defined as follows:

$$\forall f, g \in O, \forall t \in R : (f+g)(t) := f(t) + g(t), \qquad \forall f \in O, \forall c, t \in R : (cf)(t) := cf(t),$$

is a vector space over the field R.

VS1 : $(f+g)(t) = f(t) + g(t) = g(t) + f(t) = (g+f)(t)$

VS2 : $((f+g)+h)(t) = (f+g)(t) + h(t) = f(t) + g(t) + h(t)$
$= \cancel{f(t)}(g+h)(t) = (f+(g+h))(t)$

VS3 $0(t) = 0$ is the zero vector, ~~and 0(t) ... 0(t)~~ $(f+0)(t) = f(t)$.

VS4 if $f \in O$, then $-f \in O$. and $f + (-f) = 0$

VS5 ~~1(t) = 1 satisfies 1(t) = 1~~
$1 \in R$. $(1\cdot f)(t) = 1 f(t) = f(t)$ So $1 \cdot f = f$.

VS6 $a, b \in R$ $((ab)f)(t) = (ab) f(t) = a(bf(t)) = a((bf)(t))$

VS7 $a, b \in R$, $f \in O$

$((a+b)f)(t) = (a+b)(f(t)) = af(t) + bf(t) = (af)(t) + (bf)(t)$
$= (af + bf)(t)$

- Closed under addition: if $f, g \in O$, we must show $f+g \in O$. In fact
$-(f+g)(-t) = -(f(-t) + g(-t)) = -f(-t) + (-g(-t)) = f(t) + g(t) = (f+g)(t)$.

- closed under scalar multiplication, if $a \in F$, $f \in O$, we must show $af \in O$. In fact
$-(af)(-t) = \cancel{af(-t)}\ -a f(-t) = a(-f(-t)) = af(t) = (af)(t)$.

2. (40 total pts; each subproblem has the indicated value)

(a) (5 pts) Give the definition of a linearly dependent set S in a vector space V over a field F.

- S is ~~called~~ linearly dependent if there exists $v_1, \cdots, v_n \in S$ and $a_1, \cdots a_n \in F$ such that $a_1 v_1 + \cdots + a_n v_n = 0$ and $a_1, \cdots a_n$ are not all zeros.

(b) (15 pts) Give the definition of a linearly independent S set in a vector space V over a field F and state without proof an" if and only if" condition on a subset T of V that ensures that T is linearly independent.

- S is linearly independent if S is not linearly dependent.
- S is linearly indep iff $\forall v_1, \cdots, v_n \in V$, $\forall a_1, \cdots, a_n \in F$ with $a_1 v_1 + \cdots + a_n v_n = 0$, we must have $a_i = 0 \ \forall i$.

(c) (20 pts) Determine whether the following set of vectors in R^3 is linearly dependent

$$\{(-2,2,2), (1,2,-1), (-3,-3,3)\}.$$

Let $a, b, c \in R$ s.t. $a(-2,2,2) + b(1,2,-1) + c(-3,-3,3) = 0$

then $\begin{cases} -2a + b - 3c = 0 \\ 2a + 2b - 3c = 0 \\ 2a - b + 3c = 0 \end{cases} \Rightarrow \begin{cases} b = -4a \\ c = -2a \end{cases}$ So put $a = 1$

we get $(-2,2,2) - 4(1,2,-1) - 2(-3,-3,3) = 0$

~~So they are~~ So, they are linearly dependent.

3. (30pts; each subproblem has the indicated value)

(a) (10pts) Give the definitions of an :

i) $n \times n$ symmetric matrix over a field F;

ii) $n \times n$ anti-symmetric matrix over a field F.

i) $\{A \in M_{n\times n}(F) \mid A^t = A\}$ If $A = (a_{ij})$, then this means $a_{ij} = a_{ji}$

ii) $\{A \in M_{n\times n}(F) \mid -A^t = A\}$ If $A = (a_{ij})$, then this means $-a_{ij} = a_{ji}$

(b) (20 pts) Find a basis of the vector space consisting of all 4×4 symmetric matrices over a field F, and justify your answer.

Let $S = \{ E_{11} \quad E_{22} \quad E_{33} \quad E_{44}$

$E_{12} + E_{21}, \quad E_{13} + E_{31}, \quad E_{23} + E_{32}$

$E_{14} + E_{41}, \quad E_{24} + E_{42}, \quad E_{34} + E_{43} \}$

- S is linearly indepedent, because $\{E_{ij} \mid i, j = 1, \cdots 4\}$ ~~are~~ is linearly independent.

- S span V, where V denote the vector space of all 4×4 symmetric matrices

Put $A \in V$, then $A = \begin{bmatrix} a_1 & b & c & d \\ b & a_2 & e & f \\ c & e & a_3 & g \\ d & f & g & a_4 \end{bmatrix} = a_1 E_{11} + a_2 E_{22} + a_3 E_{33} + a_4 E_{44}$

$+ b(E_{12} + E_{21}) + c(E_{13} + E_{31}) + d(E_{14} + E_{41}) + e(E_{23} + E_{32}) + f(E_{24} + E_{42})$

$+ g(E_{34} + E_{43})$

4. (30 pts) Determine the values of $a \in R$ such that the following four vectors in R^4 are linearly dependent

$$(1, a, 3a + a^2, 2),\ (0, 1, 0, 4a^3),\ (0, 0, 1, -1),\ (0, 0, a^2 - 1, 1).$$

Let $k_1, k_2, k_3, k_4 \in R$ s.t.

$$k_1(1, a, 3a+a^2, 2) + k_2(0, 1, 0, 4a^3) + k_3(0,0,1,-1) + k_4(0,0,a^2-1,1) = 0$$

i.e.

$$\begin{cases} k_1 = 0 & ① \\ k_1 a + k_2 = 0 & ② \\ (3a+a^2)k_1 + k_3 + (a^2-1)k_4 = 0 & ③ \\ 2k_1 + 4a^3 k_2 - k_3 + k_4 = 0 & ④ \end{cases}$$

By ①, ② $k_1 = k_2 = 0$, then ~~by~~ ③ ④ become

$$\begin{cases} k_3 + (a^2-1)k_4 = 0. \\ -k_3 + k_4 = 0. \end{cases} \Rightarrow \begin{cases} k_3 = k_4 \\ a^2 k_3 = 0. \end{cases}$$

If $a = 0$, then we can choose $k_3 = k_4 = 1 \neq 0$, so they are linearly dependent

If $a \neq 0$, then $k_3 = k_4 = 0$ so they are linearly independent

5. (a) $N(T) = \left\{ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \;\middle|\; \begin{array}{l} a_{12} + 3a_{21} = 0 \\ a_{22} - a_{21} + a_{23} = 0 \\ a_{11} = 0 \end{array} \right\}$

Let $a_{21} = t$, $a_{22} = s$, $a_{13} = r$, then $a_{12} = -3t$, $a_{23} = a_{21} - a_{22} = t - s$

So $A = \begin{bmatrix} 0 & -3t & r \\ t & s & t-s \end{bmatrix} = t\begin{bmatrix} 0 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} + r\begin{bmatrix} 0 & 0 & 1 \\ & & 0 \end{bmatrix}$

$=: tG_1 + sG_2 + rG_3.$

So $N(T) = \operatorname{Span}\{G_1, G_2, G_3\}$. Clearly, G_1, G_2, G_3 are linearly independent.

So $\operatorname{nullity}(T) = \dim N(T) = 3$.

(b) First $R(T) \subset \left\{ \begin{bmatrix} t & 0 \\ s & r \end{bmatrix} : t, s, r \in R \right\}$. Conversely, we observe

that $T\left(\begin{bmatrix} r & t & 0 \\ 0 & s & 0 \end{bmatrix} \right) = \begin{bmatrix} t & 0 \\ s & r \end{bmatrix}$, thus, $R(T) \supset \left\{ \begin{bmatrix} t & 0 \\ s & r \end{bmatrix} : t, s, r \in R \right\}$.

So $R(T) = \left\{ \begin{bmatrix} t & 0 \\ s & r \end{bmatrix} : t, s, r \in R \right\}$

So $\operatorname{rank}(T) = \dim R(T) = 3$.

(b) T is not one-one, since $N(T) \neq \{0\}$.

T is not onto, since $\operatorname{rank}(T) = 3 < \dim M_{2\times 2}(F) = 4$.

6. (a). Suppose T is such a linear map.

Then $(1,1) = T(2,0,4) = T\big((-2)(-1,0,-2)\big)$

$= -2T(-1,0,-2) = -2\,(3,1) = (-6,-2) \neq (1,1)$

a contradiction.

So such T does not exist.

(b). Define $T(x,y) = x$, $U(x,y) = 2x$, then $T \neq U$

But $N(T) = N(U) = \{(0,y) \mid y \in R\} \neq 0$

$R(T) = R(U) = \{(x,0) \mid x \in R\} \neq 0$.