Use the blue booklet for the actual answers. Use the two yellow ones as scrap paper. Write name and ID number on all booklets before starting

You may use any tautology, axiom, law of logic and theorem throughout the test.

8 Questions (Answer $(a), (b), \cdots, (h)$ with <u>brief</u> justification(s).)

(a) Is $(P \wedge \sim P) \Rightarrow \sim (\sim Q \wedge Q)$ a tautology?

(b) Given that $[P \Rightarrow (Q \Rightarrow P)]$ is a tautology, explain why $[P \Rightarrow (Q \Rightarrow P)] \Rightarrow Q$ is not a tautology.

(c) For integers m and n, let mRn denote the relation "2 divides mn."
If mRn and nRl, does it follow that mRl?

(d) Let A, B, C be three distinct points on a line l. Assume $|AB| + |BC| > |AC|$. Can B lie inside $\overline{AC}$?

(e) Let A, B and C be three non-empty sets.
Does $A \times B = A \times C$ imply $B = C$?

(f) Assume $A \cap B = A \cap C$. Conclude that $A \cap (B \cup C) = A \cap B$.

(g) Is it possible to have a $1-1$ function from a set with 5 elements to a set with 4 elements?

(h) Is there an onto function between the set of even natural numbers and the set of natural numbers?

4 Questions (Midterms I and II)

Problem 1 Use induction to prove that 3 divides $n^3 - n$ for all natural numbers n.

Problem 2 Is $[Q \Rightarrow (P \wedge Q)] \Rightarrow P$ a tautology?
If true, prove it. If false, find a counterexample.

Problem 3 Write the following statement in symbolic form :

$P(x)$: *For any real number x, there is a real number bigger than x^2.*

For which values of x does $P(x)$ hold true?

Problem 4 Let A, B, C be sets. Assume $C \subseteq B$ and $B \cap A = (B - C) \cap A$. Deduce that $C \cap A = \emptyset$.

4 Questions (Last part of the semester)

Problem 1 Let $l_1, l_2, \ldots, l_s$ be distinct and mutually parallel lines. Let $m_1, \ldots, m_t$ be distinct and mutually parallel lines. Assume l_1 and m_t are transverse.
Find the total number of intersection points of these $(s+t)$ lines.

Problem 2 Let $\angle BAC = \pi/2$, $\angle DAC = \pi/2$, $D \neq B$ and A, B, D be on the same line l. Assume $|DB| + |BA| = |DA|$. Which of the following is true :

(i) $D - A - B$ $\qquad$ (ii) $A - D - B$ $\qquad$ (iii) $A - B - D$?

Problem 3 (a) Define the mid-point of a segment $\overline{AB}$.
(b) Prove that the mid-point of $\overline{AB}$ exists and is unique.

Problem 4 Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be given by :

$$f(x) = -x, \; g(x) = \tfrac{1}{x^2}.$$

Answer each of the following three questions separately. Is the function $f \circ g$:
4.1 injective?
4.2 surjective?
4.3 bijective?

Section I

(a) Since $(P \wedge \sim P)$ is a contradiction (and hence always false), it would imply any statement and $\sim(\sim Q \wedge Q)$, in particular. Hence the given implication is a tautology.

(b) Suppose on the contrary, i.e., $[P \Rightarrow (Q \Rightarrow P)] \Rightarrow Q$ is a tautology. Since $[P \Rightarrow (Q \Rightarrow P)]$ is a tautology, combining the two tautologies we conclude Q is a tautology. But that gives a contradiction if Q is false! Hence our assumption was wrong and $[P \Rightarrow (Q \Rightarrow P)] \Rightarrow Q$ isn't a tautology.

(c) 2 divides mn (or mRn) is equivalent to mn being an even integer. Similarly, nl is an even integer. From this it is NOT possible to conclude that ml is even (equivalently, mRl) for if m and l are odd integers and n is an even integer, ml is NOT even!

(d) If B lay inside $\overline{AC}$, then choose a co-ordinate system f on the line s.t. $f(P) = |AP|$. Then

$$|AC| = f(C) - f(A) = f(C) - f(B) + f(B) - f(A) = |BC| + |AB|$$

gives a contradiction. Thus B does NOT lie inside $\overline{AC}$.

Section II

Problem 1 When $n = 1$, $P(n) = n^3 - n$ equals 0 and 3 divides it. Assume $P(n)$ to be true for n. Then a simple computaion shows that

$$P(n+1) = (n+1)^3 - (n+1) = n^3 - n + 3(n^2 + n) = P(n) + 3(n^2 + n).$$

Since 3 divides the RHS of the above equality, it divides the LHS which is $P(n+1)$ and we're done.

Problem 2 This is NOT a tautology. Writing down the truth table would prove that. One could also define :

$$P : \textit{The streets are wet} \; ; \; Q : \textit{It has rained.}$$

When Q is FALSE, $[Q \Rightarrow (P \wedge Q)]$ is still TRUE but P may not necessarily be TRUE!

Problem 3 $\forall x \in \mathbb{R} \, \exists y \in \mathbb{R} \, (y > x^2)$.
$P(x)$ is true for all real numbers as for any x, one can choose $y = x^2 + 1$.

Problem 4 If possible, assume the contrary, i.e., $C \cap A \neq \phi$, and pick $x \in C \cap A$. Since $C \subseteq B$, $x \in B$. Since $x \in A$, it follows that $x \in B \cap A = (B - C) \cap A$, i.e., $x \in (B - C)$, a contradiction.

QUESTIONS e, f, g, h

(e) Yes. If $B \neq C$, then either ① $\exists b \in B, b \notin C$
or ② $\exists c \in C, c \notin B$.

In either case, get $\rightarrow\leftarrow$: ① $(a,b) \in A \times B$ but $\notin A \times C$.
② $(a,c) \in A \times C$ but $\notin A \times B$.

(f) $A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$, but $A \cap B = B \cap C$,

so $= A \cap B$.

(g) No: codomain should have ≥ 5 elements.

(h) Yes
$$\begin{array}{ccc} \{2,4,6,8,\ldots\} & & 2m \\ \downarrow & & \downarrow \\ \{1,2,3,4,\ldots\} & & m \end{array}$$
is onto (bijective).

Problems from last part of semester

(1) There are $s \cdot t$ intersection points.
The l lines do not meet each other; same for the m lines.
Every l line meets every m line only once and all those points are distinct.

(2) (iii) $A-B-D$ that is B is in the middle.
In the other two cases: $|DB| = |DA| + |AB| > |DA| \rightarrow\!\leftarrow$
$|AB| = |AD| + |DB| > |AD| = |DA| \rightarrow\!\leftarrow$

(3) a) M ~~is~~ point in $\overline{AB}$ such that $|AM| = |MB|$
b) By ruler axiom: take $f: l \to \mathbb{R}$ $f(A) = 0$, $f(B) > 0$.
Take M to be the point going to $f(B)/2$.
Then M is a midpoint.
If M' is another then $f(M') = f(B)/2$, but f is bijective, so $M = M'$.

(4) $f \circ g: \mathbb{R} - 0 \to \mathbb{R}$ $\quad x \to -1/x^2$
1) Not inj : $1 \to -1$ $\quad -1 \to -1$
2) Not surj : nobody goes to 0 or any $\# < 0$.
3) Not bij : it is not inj, not surj.