MAT 142 FALL 2003
MIDTERM II

!!! WRITE YOUR NAME, SUNY ID N. AND SECTION BELOW !!!

NAME :

SUNY ID N. :

SECTION :

CHECK THAT THERE ARE 6 PROBLEMS. THEY DO NOT HAVE EQUAL VALUE. SHOW YOUR WORK!!!

1	40pts	
2	40pts	
3	40pts	
4	40pts	
5	50pts	
6	40pts	
Total	250	

1. [**40 points**] Evaluate $\int \sin(\ln x)dx$. (Hint: Let $u = \ln x$.)

Solution: As per the hint, let $u = \ln x$; then $x = e^u$, and $dx = e^u du$. So we rewrite the integral as $\int e^u \sin u du$. Integrating by parts once, we have that $\int e^u \sin u du = -e^u \cos u + \int e^u \cos u du$. Integrating by parts once more, we have that $\int e^u \sin u du = -e^u \cos u + e^u \sin u - \int e^u \sin u du$. Adding $\int e^u \sin u du$ to both sides of the previous equation and dividing by 2, we have that $\int e^u \sin u du = \frac{e^u}{2}(\sin u - \cos u) + C$, or $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$.

2. [**40 points**] Prove that

$$\frac{d}{dx}(sinh^{-1}(x)) \;=\; \frac{1}{\sqrt{1+x^2}}.$$

3. [**40 points**] Evaluate

$$\int_1^2 4z^3 \ln z dz.$$

Solution: Integrating by parts, letting $u = \ln z$ and $dv = 4z^3 dz$, we have that $\int_1^2 4z^3 \ln z dz = 16 \ln 2 - \int_1^2 z^3 dz = 16 \ln 2 - \frac{15}{4}$.

4. [**40 points**] Evaluate the following integral

$$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt$$

Solution: There are one linear factor and one quadratic factor. Set

$$\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1}.$$

We get

$$3t^2+t+4 = A(t^2+1) + (Bt+C)t$$

or

$$3t^2+t+4 = (A+B)t^2 + Ct + A$$

so $A = 4, B = -1, C = 1$. We get

$$\frac{3t^2+t+4}{t^3+t} = \frac{4}{t} + \frac{-t+1}{t^2+1} = \frac{4}{t} - \frac{t}{t^2+1} + \frac{1}{t^2+1}$$

Integrating each summand we get

$$\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}\,dt = 4\ln t|_1^{\sqrt{3}} - \frac{1}{2}\ln(t^2+1)|_1^{\sqrt{3}} + tan^{-1}t|_1^{\sqrt{3}} =$$

$$\ln\frac{9\sqrt{2}}{2} + \frac{\pi}{12}.$$

5. [**50 points**] Evaluate

$$\int \frac{x^2\,dx}{\sqrt{4-x^2}}.$$

Solution. Set $x = 2\sin t$. Then $dx = 2\cos t dt$, $-\pi/2 < t < \pi/2$.
$4 - x^2 = 4 - 4\sin^2 t = 4(1-\sin^2 t) = 4\cos^2 t$.
We have

$$\int \frac{x^2\,dx}{\sqrt{4-x^2}} = \int \frac{4\sin^2 t\; 2\cos t\, dt}{2\cos t} =$$

$$\int 4\sin^2 t\, dt = \int 2 - 2\cos 2t\, dt = 2t - \sin 2t + C = 2\sin^{-1}\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2} + C.$$

6. [**50 points**] Determine the limit:

$$\lim_{x\to 0}\left(\frac{1}{\sin x^2}-\frac{1}{x^2}\right)$$

Solution. It is a form of type $\infty-\infty$. Using the substitution $u=x^2$, the limit is the same as

$$\lim_{u\to 0^+}\left(\frac{1}{\sin u}-\frac{1}{u}\right).$$

We have $\frac{1}{\sin u}-\frac{1}{u}=\frac{u-sinu}{u\sin u}$. We apply L'Hopital:

$$\lim_{u\to 0}\left(\frac{1}{\sin u}-\frac{1}{u}\right)=\lim_{u\to 0}\left(\frac{u-sinu}{u\sin u}\right)=\lim_{u\to 0}\left(\frac{1-\cos u}{\sin u+u\cos u}\right)=$$

applying L'Hopital again

$$\lim_{u\to 0}\left(\frac{\sin u}{-u\sin u+2\cos u}\right)=\frac{0}{2}=0.$$

The wanted limit is 0.