MAT 142 FALL 2003
MIDTERM I

!!! WRITE YOUR NAME, SUNY ID N. AND SECTION BELOW !!!

NAME :

SUNY ID N. :

SECTION :

CHECK THAT THERE ARE 6 PROBLEMS. THEY DO NOT HAVE EQUAL VALUE. SHOW YOUR WORK!!!

1	40pts	
2	40pts	
3	40pts	
4	40pts	
5	50pts	
6	40pts	
Total	250	

1. [**40 points**] Find the center of mass of the triangular plate bounded by the graphs of $y = |x|$ and $y = 1$. The density is constant.

Solution: We may assume that the density of the plate is 1 everywhere (if not use δ and find that the answer is the same). Since the region has the line $x = 0$ as its axis of symmetry, the x-coordinate of its center of mass is 0, so we need only find the y-coordinate, i.e. the moment of the region about the x-axis divided by the mass of the plate. The moment of the region about the x-axis is $(\int_0^1 2y^2 dy)$, or $\frac{2}{3}$ (we consider horizontal strips). The mass of the plate is $(\int_0^1 2y dy)$, or 1. The center of mass of the plate is $(0,\frac{2}{3})$.

2. [**40 points**] Solve the initial value problem

$$\frac{dy}{dx} = y \sin x, \qquad y(0) = 1.$$

Solution: Separating variables, we obtain $\frac{dy}{y} = \sin x dx$. Integrating both sides gives $\ln |y| = -\cos x + C$; exponentiating both sides gives $y = ke^{-\cos x}$ where k is a nonnegative constant. Due to the initial condition, we have that $1 = ke^{-1}$, or $k = e$. Therefore the solution is $y = e^{1-\cos x}$.

3. [**40 points**] If a semicircular plate whose radius is 4 feet is submerged in water so that it is perpendicular to the surface, it points down and its diameter is 6 feet below the surface, set up, but do not evaluate, an integral which gives the force exerted by the water on one side of the plate. (The weight-density of water is 62.4 pounds per cubic foot.)

Solution: If we work in a coordinate system in which the surface of the water is the line y=0, the integral representing the force exerted by the water on one side of the plate is
$\int_{-10}^{-6}(62.4)(-y)(2\sqrt{(16-(y+6)^2})dy.$

4. [**40 points**] Let $f(x)$ be a differentiable function on the interval $(0, \infty)$. Assume that
1) $f(1) = 0$,
2) $f'(1) = 1$,
3) $f(ab) = f(a) + f(b)$ for every $a, b > 0$.

a) Prove that $f(\frac{1}{x}) = -f(x)$.

Solution: $0 = f(1) = f(x\frac{1}{x}) = f(x) + f(\frac{1}{x})$.
So we have $f(\frac{1}{x}) = -f(x)$.

b) Prove that $f(x) = \ln x$. You can use that $f(\frac{1}{x}) = -f(x)$ even if you did not prove it. (Hint: consider $f'(1)$)

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \overset{3)}{=} \lim_{h\to 0} \frac{f((x+h)\frac{1}{x})}{h} =$$

$$\lim_{h\to 0} \frac{f(1+\frac{h}{x})}{h} \overset{1)}{=} \lim_{h\to 0} \frac{f(1+\frac{h}{x}) - f(1))}{h} =$$

$$\lim_{h\to 0} \frac{f(1+\frac{h}{x}) - f(1)}{\frac{h}{x}} \frac{1}{x} \overset{2)}{=} 1 \cdot \frac{1}{x} = \frac{1}{x}.$$

Since $f'(x) = \frac{1}{x}$, $f(x) = \ln x + C$.
By 1), $f(x) = \ln x$.

5. [**50 points**] Evaluate the following integral and derivative.

a) $\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos e^{x^2} dx$

Solution. Set $u = e^{x^2}$.
$= \int_1^{\pi} \cos u \, du = -\sin 1$.

b) If $y = (\ln(\sin t))^{\pi}$, find $\frac{dy}{dt}$.

Solution. $\pi \frac{1}{t} \cos t (\ln(\sin t))^{\pi - 1}$

6. [**40 points**] Solve the initial value problem

$$y' + xy = x, \qquad y(0) = -6.$$

$v(x) = e^{x^2/2}$
$y = e^{-x^2/2} \int e^{x^2/2} x dx = 1 + Ce^{-x^2/2}$
Using the initial condition, we get $C = -7$.
$y = 1 - 7e^{-x^2/2}$.