Let

$$y = f(x) = Ax^2 + Bx + C$$

be the equation of a parabola.
Let a be a number and h be a positive number.
Assume

$$f(a-h) = y_0, \quad f(a) = y_1, \quad f(a+h) = y_2.$$

Find an expression for

$$I := \int_{a-h}^{a+h} f(x)\,dx$$

involving only y_0, y_1 and y_2 and h.
(Hint: it makes no difference if you assume $a = 0$ (why?))

Solution. Let us assume that $a = 0$. The answer will not depend on this assumption, since since we can shift the graph to the left (if $a > 0$) or to the right (if $a < 0$) without changing the values of h, y_0, y_1 and y_2. This will change A, B and C, but not the answer. So we can also use the same letters A, B, and C. We have

$$I := \int_{-h}^{h} (Ax^2 + Bx + C)\,dx = A/3x^3 + B/2x^2 + Cx|_{-h}^{h} = 2/3Ah^3 + 2Ch.$$

Since $f(0) = y_1$, by plugging in $f(x)$, we get $C = y_1$ and $I = 2/3Ah^3 + 2y_1h$.
We have

$$f(-h) = Ah^2 - Bh + y_1 = y_0 \qquad\qquad f(h) = Ah^2 + Bh + y_1 = y_2$$

and adding the second to the first we have

$$2Ah^2 = y_0 + y_2 - 2y_1.$$

So

$$I = 1/3(2Ah^2)h + 2y_1h = 1/3(y_0 + 4y_1 + y_2)h.$$