Minimal Log Discrepancy of Isolated Singularities and Reeb Orbits

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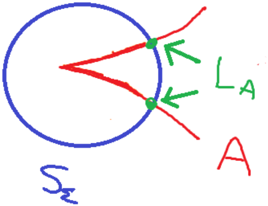
- $A \subset \mathbb{C}^N$ affine variety of dimension n with an isolated singularity at 0.
- $L_A = A \cap S_\epsilon$ where $S_\epsilon = \{z \in \mathbb{C}^N \| |z| = \epsilon\}$.
- Here L_A is a real $2n - 1$ dimensional C^∞ manifold called the **link** of A at 0 for ϵ small enough.
- A C^∞ manifold diffeomorphic to L_A is said to be **Milnor fillable by** A.



- *Examples:*

$$L_{\mathbb{C}^n} = S^{2n-1}$$

$$L_{\{x^2+y^2+z^2=0\}} = \mathbb{RP}^3.$$

- A is called **differentiably smooth** if L_A is diffeomorphic to $L_{\mathbb{C}^n} = S^{2n-1}$.
- *Question:* Which singularities are differentiably smooth?

- **Theorem** (Mumford)
  Let A be a normal surface singularity with link diffeomorphic to S^3. Then A is smooth at 0.
- This is false in higher dimensions (Brieskorn):

$$L_{\{x^2+y^2+z^2+w^3=0\}} = S^5$$

- Need more structure on the link.

Introduction to Contact Geometry

- Let C be a real $2n-1$ dimensional C^∞ manifold and let $\xi \subset TC$ be a hyperplane distribution. For simplicity, assume $\xi = \ker(\alpha)$ for some 1-form α on M.
- (Frobenious Integrability Theorem): ξ is the tangent space to a foliation iff $d\alpha|_\xi = 0$.

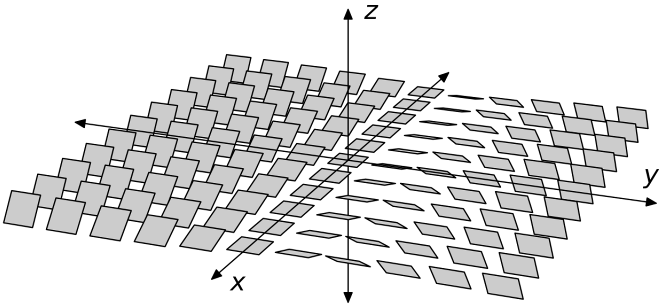
- *Definition:* ξ is a **Contact structure** if $d\alpha|_\xi$ is a non-degenerate 2-form at every point.
- A contact structure is the opposite of a Foliation!
- Equivalently: ξ is a **Contact structure** iff $\alpha \wedge (d\alpha)^{n-1} \neq 0$ at every point.

- We will call (C, ξ) a **contact manifold**.
- Any 1-form α satisfying $\mathrm{Ker}(\alpha) = \xi$ is a **contact form associated to** ξ.
- Two contact manifolds are **contactomorphic** if there is a diffeomorphism preserving the respective hyperplane distributions.
- (*Gray's stability theorem*). If I have a smooth family of contact structures on a compact manifold, then they are all contactomorphic.

Example:

$$(\mathbb{R}^{2n-1}, \ker(dz - \sum_{j=1}^{n-1} y_j dx_j))$$

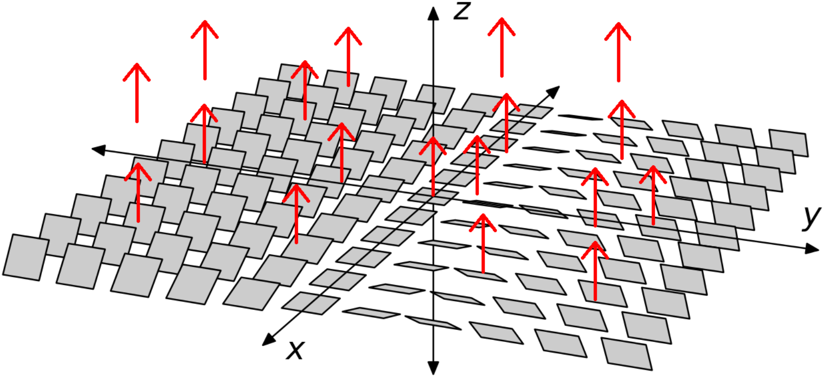
where $(x_1, y_1, \cdots, x_{n-1}, y_{n-1}, z)$ are the natural coordinates.



- **The Reeb vector field of** α is the unique vector field R on C satisfying $i_R d\alpha = 0$, $i_R \alpha = 1$.
- Intuition: Think of C as the level set of a Hamiltonian, and R is the Hamiltonian flow inside that level set. I.e. some dynamical system in some fixed energy level.
- R is uniquely determined by α, but R is not an invariant of ξ. If I replace α with $f\alpha$ for some $f : C \to \mathbb{R} \setminus \{0\}$, the associated Reeb vector field changes a lot.
- A **periodic Reeb orbit of period** L is a map $\mathbb{R}/L\mathbb{Z} \to C$ tangent to R.

Example: Reeb vector field of

$$dz - \sum_{j=1}^{n-1} y_j dx_j \quad \text{is} \quad \frac{\partial}{\partial z}.$$



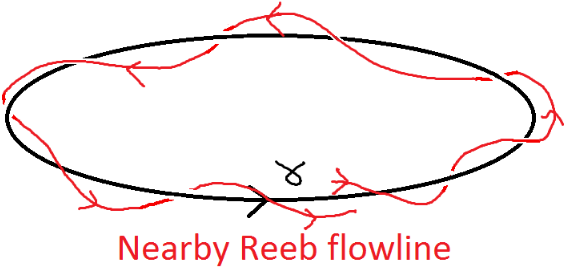
- Let $A \subset \mathbb{C}^N$ have an isolated singularity at 0 with link $L_A = A \cap S_\epsilon$ as before. Let $i : T(A \setminus \{0\}) \to T(A \setminus \{0\})$ be complex multiplication.
- *Define:* $\xi_A := TL_A \cap iTL_A$.
- **Lemma** (Varchenko): For all $\epsilon > 0$ small enough, (L_A, ξ_A) is a contact manifold and is an invariant of the germ of A at 0 up to contactomorphism.

- **Conjecture** (Seidel) If A is normal and (L_A, ξ_A) is contactomorphic to $(L_{\mathbb{C}^n}, \xi_{\mathbb{C}^n})$ then A is smooth at 0.
- Seidel observed that this is true for hypersurface singularities using work by Eliashberg,Gromov,McDuff.

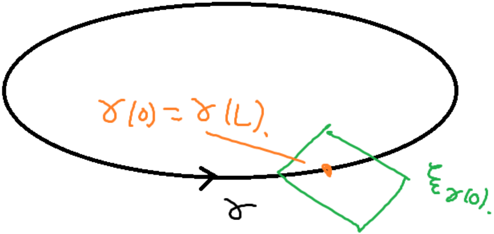
Definition of the Conley-Zehnder index

- Let (C, ξ) be a general contact manifold with $\xi = \ker(\alpha)$.
- Choose a complex structure J on the bundle ξ compatible with the symplectic form $d\alpha|_\xi$. We define $c_1(\xi) := c_1(\xi, J)$.
- We will assume $H^1(C; \mathbb{Q}) = 0$, $c_1(\xi) = 0$.

- These topological conditions tell us that for each periodic Reeb orbit γ, we get an index: $\mathrm{CZ}(\gamma) \in \mathbb{Q}$ called the **Conley-Zehnder index**.
- Intuition: $\mathrm{CZ}(\gamma)$ describes how many times the Reeb flow 'wraps' around γ.

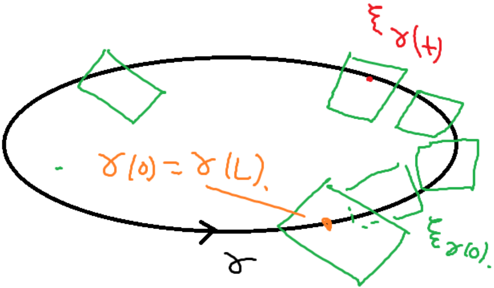


- Let $\phi_t : C \to C$ be the Flow of the Reeb vector field R of α.
- This flow preserves ξ (i.e. $D\phi_t(\xi) = \xi$).
- The **linearized return map** of $\gamma : \mathbb{R}/L\mathbb{Z} \to C$ is the natural map $D\phi_L|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(L)} = \xi_{\gamma(0)}$.



- For simplicity, we will define $\mathrm{CZ}(\gamma)$ under the following conditions:
  1. $D\phi_t|_\xi$ is J holomorphic for some compatible almost complex structure J on ξ.
  2. $D\phi_L|_{\xi_{\gamma(0)}} = \mathrm{id}$.
  3. $c_1(\xi) = 0$.
- Choose a trivialization of the complex vector bundle $\gamma^*\xi$ with complex structure J.

- Using this trivialization and the above properties, the map $t \to (\phi_t|_{(\xi)_{\gamma(0)}})$ is viewed as a map from $Q : \mathbb{R}/L\mathbb{Z} \to U(n-1)$. We define $\mathrm{CZ}(\gamma)$ to be twice the degree of the map $\det(Q) : \mathbb{R}/L\mathbb{Z} \to U(1)$.

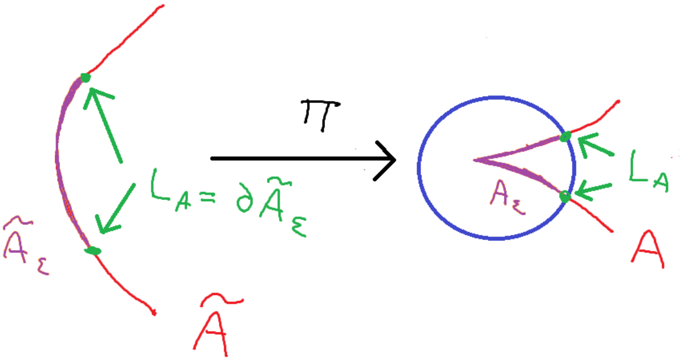


- Define
  $\mathrm{ISFT}(\gamma) := \mathrm{CZ}(\gamma) - \frac{1}{2}\dim\ker(D\phi_L|_{\xi|_{\gamma(0)}} - \mathrm{id}) + (n-3)$.
- For any α such that $\ker(\alpha) = \xi$, define the **minimal index of** α as $\mathrm{mi}(\alpha) := \inf(\mathrm{ISFT}(\gamma))$.
- Define the **highest minimal index** $\mathrm{hmi}(C, \xi) := \sup_\alpha \mathrm{mi}(\alpha)$ where the supremum is taken over all α such that $\ker(\alpha) = \xi$.

Minimal discrepancy

- Recall: A is an isolated singularity and L_A is its link.
- Assume $c_1(TA|_{L_A})$ is torsion. Fact: $c_1(TA|_{L_A}) = c_1(\xi_A)$. Such a singularity is called **numerically $\mathbb{Q}$-Gorenstein**.
- Fix some resolution $\pi : \widetilde{A} \twoheadrightarrow A$ so that $\pi^{-1}(0)$ has smooth normal crossing exceptional divisors $E_1, \cdots, E_l$.

- Define: $B_\epsilon := \{|z| \leq \epsilon\}$, $A_\epsilon := B_\epsilon \cap A$ and $\widetilde{A}_\epsilon := \pi^{-1}(A_\epsilon)$.
  Note: $\partial \widetilde{A}_\epsilon = \partial A_\epsilon = L_A$.



$$
\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \| \\
 & & \exists c_1(\widetilde{A}_\epsilon, L_A; \mathbb{Q}) & \longrightarrow & c_1(\widetilde{A}_\epsilon; \mathbb{Q}) & \longrightarrow & c_1(L_A; \mathbb{Q}) \\
 & & \pitchfork & & \pitchfork & & \pitchfork \\
H^1(L_A; \mathbb{Q}) & \xrightarrow{0} & H^2(\widetilde{A}_\epsilon, L_A; \mathbb{Q}) & \longrightarrow & H^2(\widetilde{A}_\epsilon; \mathbb{Q}) & \longrightarrow & H^2(L_A; \mathbb{Q}) \\
 & & \| & & & & \\
 & & H_{2n-2}(\widetilde{A}; \mathbb{Q}) & & & &
\end{array}
$$

$H_{2n-2}(\widetilde{A}; \mathbb{Q})$ ← freely generated by $[E_j]$

So $c_1(\widetilde{A}_\epsilon, L_A; \mathbb{Q}) = \sum_i a_i [E_i]$ for unique $a_i \in \mathbb{Q}$.

- Define a_j to be the **discrepancy of** E_j.
- Define **Minimal discrepancy** to be

$$\mathrm{md}(A) = \begin{cases} \min(a_j) & \text{if} \quad \min(a_j) \geq -1 \\ 0 & \text{otherwise.} \end{cases}$$

- Minimal discrepancy measures how singular A is at 0.
- **Examples:**
  1. $\mathrm{md}(\mathbb{C}^n) = n - 1$.
  2. $\mathrm{md}(\{x^2 + y^2 + z^2 + w^3 = 0\}) = 1$.
  3. $\mathrm{md}(\{x^7 + y^{11} + z^{13} + w^{17} = 0\}) = -\infty$.

- **Theorem**: If A is numerically $\mathbb{Q}$-Gorenstein (i.e. $c_1(\xi_A)$ is torsion) and $H^1(L_A;\mathbb{Q}) = 0$ then:

$$\mathrm{hmi}(L_A, \xi_A) = \begin{cases} 2\mathrm{md}(A) & \text{if} \quad \mathrm{md}(A) \geq 0 \\ < 0 & \text{otherwise.} \end{cases}$$

- **Shokurov's Conjecture** (Combined with work from: Boucksom, de Fernex, Favre, Urbinati): If A is numerically $\mathbb{Q}$-Gorenstein with $\mathrm{md}(A) = n - 1$ then A is smooth at 0.
- **Corollary.** Suppose that Shokurov's Conjecture is true. If A is normal and $(L_A, \xi_A) \underset{\text{cont.}}{\cong} (L_{\mathbb{C}^n}, \xi_{\mathbb{C}^n})$ then A is smooth at 0.
- (Markushevich, Reid, Kawamata), Shokurov's conjecture is true in dimension ≤ 3.
- **Corollary.** For all $n \leq 3$, if A is normal and $(L_A, \xi_A) \underset{\text{cont.}}{\cong} (L_{\mathbb{C}^n}, \xi_{\mathbb{C}^n})$ then A is smooth at 0.

Proof

- **Easier part:** Find some contact form α_A associated to ξ_A so that:

$$\mathrm{mi}(\alpha_A) = 2\mathrm{md}(A)$$

This gives us a lower bound form $\mathrm{hmi}(\xi)$.

- **Hard part:** For *every* compatible contact form, find a Reeb orbit γ so that:

$$\mathrm{ISFT}(\gamma) \leq \begin{cases} 2\mathrm{md}(A) & \text{if} \quad \mathrm{md}(A) \geq 0 \\ < 0 & \text{otherwise.} \end{cases}$$

This gives us a upper bound form $\mathrm{hmi}(\xi)$.

Proof in the case of cone singularities.

- Assume A is the cone over a smooth projective $X \subset \mathbb{CP}^{N-1}$. E.g. $X = \mathbb{CP}^{n-1}$, $A = \mathbb{C}^n$.
- $\widetilde{A} = \mathrm{Bl}_0 A$ and let $\pi : \widetilde{A} \twoheadrightarrow A$ be the blowdown map.
- We also have the $\mathcal{O}(-1)$ bundle $P : \widetilde{A} \to X$. We identify X with the zero section of P.

Easier Part:

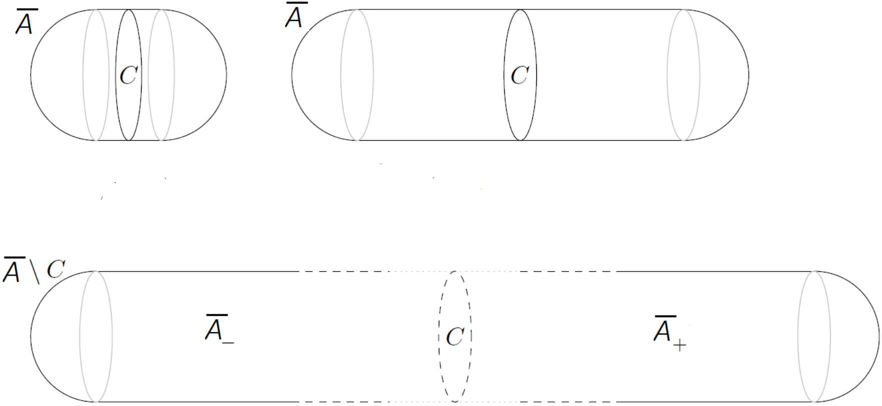
- $A \subset \mathbb{C}^N$. Define $\alpha_A := \sum_j x_j dy_j - y_j dx_j|_{L_A}$ where $z_j = x_j + iy_j$.
- $P : \widetilde{A} \twoheadrightarrow X$ is a Hermitian line bundle $\mathcal{O}_X(-1)$ with Hermitain form coming from the standard symplectic form on $\mathbb{C}^N$.
- The Reeb flow uniformly rotates the fibers of P. I.e. $\phi_t(z) = e^{it}(z)$ (up to a time reparameterization).

- So through each point p in L_A there are Reeb orbits of period $2k\pi$ wrapping k times around X.
- The ISFT index of such an orbit is $2k(a_1 + 1) - 2$ where a_1 is the discrepancy of $X \subset \widetilde{A}$.
- Hence $\mathrm{mi}(\alpha_A) = 2a_1 = 2\mathrm{md}(A)$.

Sketch of Proof of Hard Part

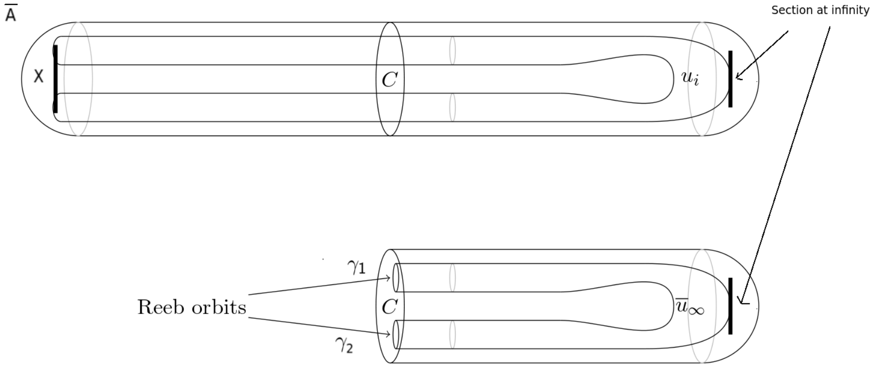
- We now start with any contact form α associated to ξ_A. We wish to find an orbit γ with the right bound on its index.
- Compactify $\widetilde{A}$ to $\overline{A} = \mathbb{P}(\mathbb{C} \oplus \mathcal{O}_X(-1))$ and let $\overline{\pi} : \overline{A} \twoheadrightarrow X$ be the natural map.
- Let $[F] \in H_2(\overline{A})$ be the class of the fiber of $\overline{\pi}$, then $\mathrm{GW}_{[F],0}([\mathrm{pt}]) \neq 0$. Hence for any compatible almost complex structure J on $\overline{A}$, there is a J-holomorphic curve: $u_J : \mathbb{P}^1 \to \overline{A}$ representing $[F]$.

- We now deform the symplectic form on $\overline{A}$ through symplectic forms to a new symplectic form ω so that we have an embedding $\iota : C \hookrightarrow \overline{A}$ so that $\iota^* \omega = d\alpha$.
- We now choose a family J_i's compatible with ω which 'stretch' along C.
- The associated u_{J_i}'s 'break' and their ends converge to Reeb orbits $\gamma_1, \cdots, \gamma_k$.
- Simple Example: X is a point, so $\overline{A} = \mathbb{CP}^1$. Our degeneration is $\{x^2 + y^2 = t\} \subset \mathbb{CP}^2$ as $t \to 0$. The complex structure here stretches along the equator $\mathbb{RP}^1$.



$\overline{A}$
C
$\overline{A}$
C
$\overline{A} \setminus C$
$\overline{A}_-$
C
$\overline{A}_+$

Schematic Picture



- The space of such broken maps $\overline{u}_\infty$ converging to $\gamma_1, \cdots, \gamma_k$ has dimension given by a formula involving the discrepancy and Reeb orbits.
- This gives us an inequality:

$$2a_1 - \sum_j \mathrm{ISFT}(\gamma_j) \geq 0$$

proving the hard part of the theorem.

Further directions

- What other parts of the resolution can we recover? E.g. Information from the dual graph? other invariants such as Log Canonical Threshold?
- Some of the holomorphic curves involved look like arcs. What is the relationship between these curves and the (short) arc space?
- Secretly our proof is showing that a group called Contact Homology has lowest non-zero degree equal to $\mathrm{md}(A)$ or is < 0 depending on the sign of $\mathrm{md}(A)$. What is the relationship between this group and the singularity?