Complex cobordism, Hamiltonian loops and global Kuranishi charts.

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(Work is joint with Mohammed Abouzaid and Ivan Smith.)

Main Results

- Let P be a closed symplectic manifold.
- Let $\pi : P \longrightarrow \mathbb{CP}^1$ be a smooth submersion, whose fibers are symplectic submanifolds and let (X, ω) be a fiber.
- **Theorem 1:** (Abouzaid, M., Smith). We have an additive isomorphism

$$H^*(P; \mathbb{Z}) \cong H^*(X; \mathbb{Z}) \otimes H^*(\mathbb{C}P^1; \mathbb{Z}).$$

- **Theorem 2:** (Abouzaid, M., Smith). More generally, we have an additive isomorphism

$$H^*(P; \mathbb{E}) \cong H^*(X; \mathbb{E}) \otimes_{H^*(pt;\mathbb{E})} H^*(\mathbb{C}P^1; \mathbb{E})$$

for any complex oriented cohomology theory $\mathbb{E}$ (such as complex cobordism MU).

Main Results

- Lalonde, McDuff and Polterovich proved that such a splitting holds with $\mathbb{Z}$ coefficients for monotone symplectic manifolds.
- McDuff proved this splitting with $\mathbb{Q}$ coefficients in general.
- Both theorems are new in the case where $\pi : P \longrightarrow \mathbb{CP}^1$ is a morphism of smooth projective varieties (which was proven with $\mathbb{Q}$-coefficients by Deligne).

Examples

- Theorem 1 holds for Hamiltonian fibrations over $\mathbb{C}P^1$, but it does not hold for all symplectic fibrations over $\mathbb{C}P^1$ (such as the Hopf surface $S^1 \times S^3 \to S^2$). One needs a symplectic form on the total space P.
- Also Theorem 2 does not hold for all generalized cohomology theories.
- For example the Hirzebruch surface $F_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$. But $H^*(F_1, KO)$ is not isomorphic to $H^*(\mathbb{CP}^1) \otimes_{H^*(pt;KO)} H^*(\mathbb{CP}^1)$ where KO is real K-theory (Bahri and Bendersky).

Alternative Description of P.

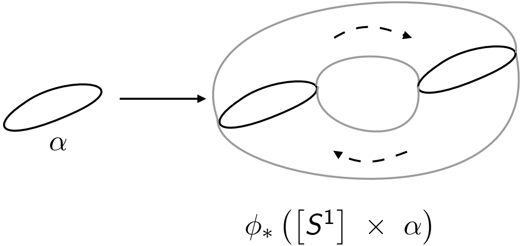
- The fibration π can be described in a different way by the *clutching construction*.
- Take a loop $\phi : S^1 \longrightarrow \mathrm{Ham}(X, \omega)$ of Hamiltonian symplectomorphisms of a closed symplectic manifold (X, ω).
- Define
$$P = P_\phi := (\mathbb{D} \times X)_{(1)} \sqcup (\mathbb{D} \times X)_{(2)} / \sim$$
where $\mathbb{D} \subset \mathbb{C}$ is the closed disk and $\sim$ identifies $(z, x) \in (\partial\mathbb{D} \times X)_{(1)}$ with $(z, \phi(z)(x)) \in (\partial\mathbb{D} \times X)_{(2)}$.
- $\pi : P \longrightarrow \mathbb{CP}^1$ is the natural projection map to $\mathbb{CP}^1 = \mathbb{D}_{(1)} \sqcup \mathbb{D}_{(2)} / \sim$ where $\sim$ identifies $\partial\mathbb{D}_{(1)}$ with $\partial\mathbb{D}_{(2)}$ via the identity map.

Hamiltonian Loops

- For any loop $\phi : S^1 \longrightarrow \mathrm{Ham}(X, \omega)$, define the *sweepout map*

$$\delta_\phi : H_*(X; \mathbb{Z}) \longrightarrow H_{*+1}(X; \mathbb{Z})$$

to be the map sending a cycle α to $\phi_*([S^1] \times \alpha)$.



- **Corollary** (Abouzaid, M., Smith). The sweepout map vanishes.

Proof of Corollary

Recall that we have a Serre spectral sequence computing $H_*(P_\phi)$ with E^2 page:

$$\begin{tikzcd}
H_{p+1}(X) & 0 & H_{p+1}(X) \\
H_p(X) & 0 & H_p(X) \arrow[ull, "\delta_\phi"'] \\
H_{p-1}(X) & 0 & H_{p-1}(X) \arrow[ull, "\delta_\phi"']
\end{tikzcd}$$

By Theorem 1, we know that this spectral sequence degenerates and so $\delta_\phi = 0$. A similar argument using Theorem 2 shows that the sweepout map vanishes for any complex oriented homology theory.

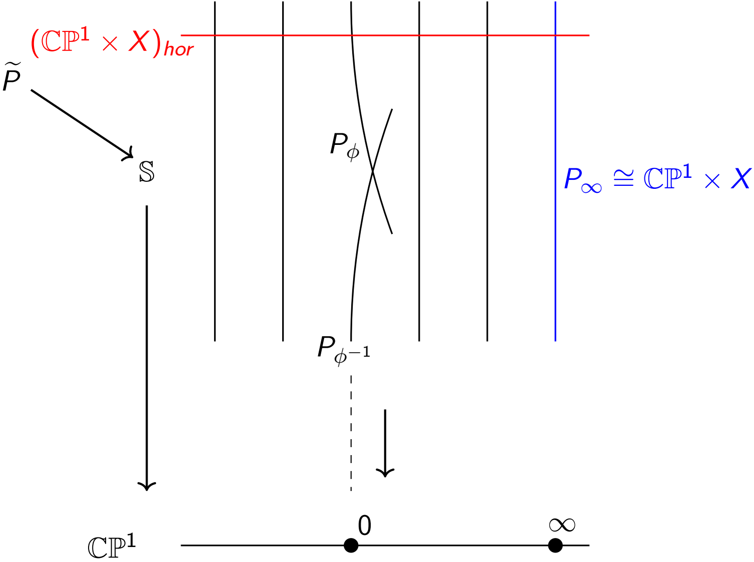
Main Argument

- We will first give an outline of the proof of Theorem 1 under the assumption that our moduli spaces are smooth closed manifolds and where all evaluation maps are submersions.
- It is sufficient for us to construct a section $s : H^*(P_\phi) \longrightarrow H^*(X)$ of the natural fiber restriction map $H^*(P_\phi) \longrightarrow H^*(X)$.
- This section will then give us our isomorphism:

$$H^*(X) \otimes H^*(\mathbb{CP}^1) \xrightarrow{\cong} H^*(X) \oplus H^*(P_\phi, P_\phi - X) \xrightarrow{s \oplus q^*} H^*(P_\phi)$$

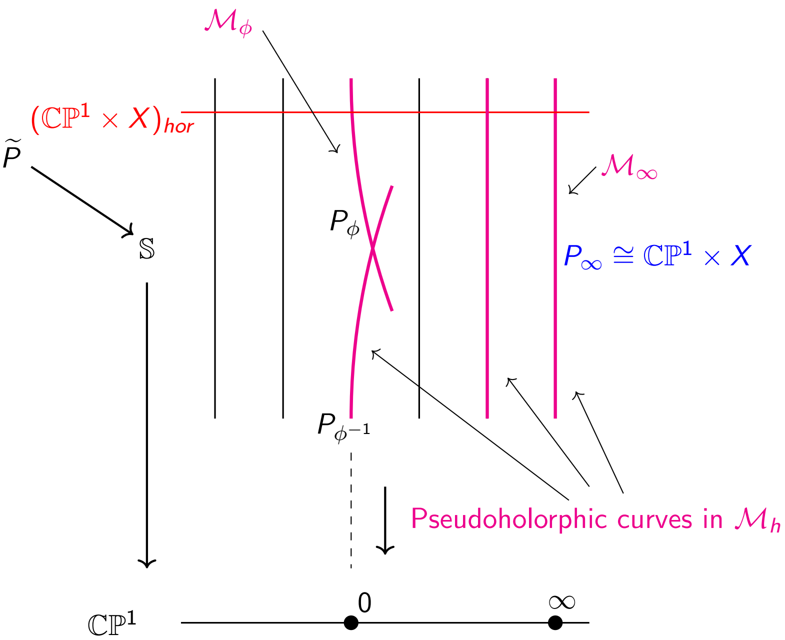
where q is the natural quotient map.

- Let $\mathbb{S} = Bl_{(0,0)}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ be the blowup at $(0,0)$.
- Now construct a Hamiltonian fibration $\pi_{\mathbb{S}} : \widetilde{P} \longrightarrow \mathbb{S}$ with fiber (X, ω) so that:
  1. The restriction of $\pi_{\mathbb{S}}$ to the exceptional divisor of $\mathbb{S}$ is the fibration $\pi : P_\phi \longrightarrow \mathbb{CP}^1$.
  2. The restriction of $\pi_{\mathbb{S}}$ to $\mathbb{CP}^1 \times \{\infty\} \subset \mathbb{S}$ is the trivial fibration $P_\infty := \mathbb{CP}^1 \times X$.
  3. Also $\widetilde{P}$ restricted to the proper transform of $\{0\} \times \mathbb{CP}^1$ is the trivial fibration $\mathbb{CP}^1 \times X$ which we will write as $(\mathbb{CP}^1 \times X)_{hor}$.



$(\mathbb{CP}^1 \times X)_{hor}$
$\widetilde{P}$
$\mathbb{S}$
P_ϕ
$P_\infty \cong \mathbb{CP}^1 \times X$
$P_{\phi^{-1}}$
$\mathbb{CP}^1$
0
∞

- Choose an almost complex structure making $\pi_{\mathbb{S}}$ holomorphic and which is a product near P_∞ and $(\mathbb{CP}^1 \times X)_{hor}$.
- We let $\mathcal{M}_h$ be the moduli space of genus 0 pseudo-holomorphic maps to $\widetilde{P}$ with two marked points representing $[\mathbb{CP}^1 \times pt] \in H_2(P_\infty) \subset H_2(\widetilde{P})$ so that one marked point maps to $(\mathbb{CP}^1 \times X)_{hor}$ and the other is free.
- Let $\mathrm{ev} : \mathcal{M}_h \longrightarrow \widetilde{P} \times (\mathbb{CP}^1 \times X)_{hor}$ be the evaluation map.
- Define $\mathcal{M}_\bullet := \mathrm{ev}^{-1}(P_\bullet \times (\mathbb{CP}^1 \times X)_{hor})$ for $\bullet = \phi$ or ∞ (in other words, the restriction of $\mathcal{M}_h$ to $P_\phi \cup P_{\phi^{-1}}$ and P_∞ respectively).



$\mathcal{M}_\phi$
$(\mathbb{CP}^1 \times X)_{hor}$
$\widetilde{P}$
$\mathcal{M}_\infty$
P_ϕ
$\mathbb{S}$
$P_\infty \cong \mathbb{CP}^1 \times X$
$P_{\phi^{-1}}$
Pseudoholorphic curves in $\mathcal{M}_h$
0
∞
$\mathbb{CP}^1$

- We then have pushpull maps:

$$\Psi_\bullet : H^*((\mathbb{CP}^1 \times X)_{hor}) \xrightarrow{\mathsf{ev}^*} H^*(\mathcal{M}_\bullet) \xrightarrow{-\cap[\mathcal{M}_\bullet]}$$

$$H_{\dim(P_\bullet)-*}(\mathcal{M}_\bullet) \xrightarrow{\mathsf{ev}_*} H_{\dim(P_\bullet)-*}(P_\bullet) \cong H^*(P_\bullet)$$

for $\bullet = \phi$ or ∞.

- Similarly we have a pushpull map

$$\Psi_h : H^*((\mathbb{CP}^1 \times X)_{hor}) \longrightarrow H^*(\widetilde{P})$$

associated to $\mathcal{M}_h$.

Our map s is constructed, and shown to be a section, by staring at the following commutative diagram:

$$
\begin{tikzcd}
 & & H^*(P_\infty) \arrow[dr, "res"] & \\
H^*(X) \arrow[r, "pr^*"'] \arrow[rrr, bend left, "id"] \arrow[rr, bend right, "s"'] & H^*((S^2 \times X)_{hor}) \arrow[ur, "\Psi_\infty = (pr_X)_*"] \arrow[r, "\Psi_h"] \arrow[dr, "\Psi_\phi"'] & H^*(\widetilde{P}) \arrow[u, "res"] \arrow[r, "res"'] \arrow[d, "res"] & H^*(X) \\
 & & H^*(P_\phi) \arrow[ur, "res"'] &
\end{tikzcd}
$$

where $pr_X : (S^2 \times X)_{hor}$ is the projection map to $0 \times X \subset P_\infty$.

QED

Problems

- Now, we have a problem, which is that the (virtual) fundamental class $[\mathcal{M}_\bullet]$, $\bullet = \phi$, ∞, h is usually defined over $\mathbb{Q}$ (since our moduli spaces are usually not nice smooth manifolds).
- How do we deal with $\mathbb{Z}$?
- First of all, it is sufficient for us to prove our theorem over $\mathbb{Z}/p^k\mathbb{Z}$ for every prime power p^k.
- The key idea (of Abouzaid and Blumberg) is to use *Morava K-theory* which 'approximates' $\mathbb{Z}/p^k\mathbb{Z}$-cohomology and which also gives our moduli spaces a virtual fundamental class.

- In order to construct such a fundamental class, we need an appropriate topological description of our moduli space.
- **Definition**: A *global Kuranishi chart* is a tuple $(G, \mathcal{T}, E, s)$ where
    1. G is a compact Lie group,
    2. $\mathcal{T}$ is a manifold (called the *thickening*) admitting a G-action with finite stabilizers,
    3. E is a G-vector bundle and
    4. s is a G-equivariant section whose zero locus is compact.
- **Definition**: Such a chart *describes* a metric space M if M is homeomorphic to $s^{-1}(0)/G$ (here, M will be our moduli space).
- **Theorem:** (we will explain this later). There are global Kuranishi charts as above describing our moduli spaces with $T\mathcal{T}$ and E complex G-vector bundles.

- Given a global Kuranishi chart $(G, \mathcal{T}, E, s)$ describing M, how do we put a fundamental class on $M \cong s^{-1}(0)/G$?
- This will be a map $\mathrm{vfc} : H^*(M; \mathbb{K}) \longrightarrow \mathbb{K}_* := H^*(pt; \mathbb{K})$ where $H^*(-, \mathbb{K})$ is an appropriate generalized cohomology theory.
- Define $H^*(A|B; \mathbb{K}) := H^*(A, A - B; \mathbb{K})$.
- We will write $H^*(A) = H^*(A; \mathbb{K})$ to avoid clutter.

Sketch of vfc construction.

- ASSUMPTION:
  1. G-equivariant Thom isomorphism holds:

     $$H^*_G(E|\mathcal{T}) \xrightarrow{Thom} H^{*+e}_G(\mathcal{T})$$

     where $e = \dim(E)$.
  2. G-equivariant Poincaré duality holds

     $$H^*_G(\mathcal{T}) \xrightarrow{PD} H^G_{d-k-*}(\mathcal{T}, \partial\mathcal{T}),$$

     $d = \dim(\mathcal{T})$, $k = \dim(G)$.
- These hold over $\mathbb{Q}$.
- But not over $\mathbb{Z}$ since G might have non-trivial stabilizers.
- They do for Morava K-theories when $T\mathcal{T}$ and E are complex.

Sketch of vfc construction continued...

- For each G-equivariant relatively compact open neighborhood U of $s^{-1}(0)$, we have a map

$$\mathrm{vfc}_U : H_G^{*-\mathrm{vdim}}(U) \xrightarrow{PD} H_{-e-*}(U, \partial U) \tag{1}$$

$$\xrightarrow{s_*} H_{-e-*}(E|\mathcal{T}) \xrightarrow{Thom} H_{-*}(\mathcal{T}) \longrightarrow \mathbb{K}_{-*}. \tag{2}$$

  where $\mathrm{vdim} := d - k - e$.

- The virtual fundamental class is:

$$\mathrm{vfc}_{\mathbb{K}} : H^{*-\mathrm{vdim}}(M) \longrightarrow \varinjlim_{U} H_G^{*-\mathrm{vdim}}(U) \xrightarrow{\varinjlim_U \mathrm{vfc}_U} \mathbb{K}_{-*}.$$

- Actually it is quite handy to work with vfc_U sometimes.

What is Morava K-theory?

Proposition: For any prime power p^k and any $n \in \mathbb{N}$, there is a generalized cohomology theory $H^*(-, K_{p^k}(n))$ called *Morava K-theory* satisfying the following properties:

1. The coefficient ring is $K_{p^k}(n)_* := H_*(pt, K_{p^k}(n)) = (\mathbb{Z}/p^k\mathbb{Z})[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$.
2. Any stably complex vector bundle is $K_{p^k}(n)$-oriented and so the G-equivariant Thom isomorphism theorem holds.
3. (Cheng): G-equivairant Poincaré duality holds for manifolds admitting a G-equivariant stable almost complex structure.

- As a result, we can construct virtual fundamental classes in Morava K-theory.
- For any CW complex Y, the *Atiyah-Hirzebruch spectral sequence* (AHSS) tells us that there is a spectral sequence converging to $H^*(Y;\mathbb{K})$ whose E_2-page is $H^p(Y;H^q(pt;\mathbb{K}))$, for any generalized cohomology theory $H^*(-,\mathbb{K})$.
- Now if our CW complex Y is finite dimensional and the parameter n is large, then AHSS for $H^*(Y;K_{p^k}(n))$ must degenerate for degree reasons. Therefore $H^*(Y;K_{p^k}(n)) \cong H^*(Y;\mathbb{Z}/p^k\mathbb{Z})[v_n, v_n^{-1}]$.
- Using all these facts, we can prove our splitting theorem $H^*(P;\mathbb{Z}/p^k\mathbb{Z}) \cong H^*(Y;\mathbb{Z}/p^k\mathbb{Z}) \otimes H^*(\mathbb{CP}^1;\mathbb{Z}/p^k\mathbb{Z})$ for all prime powers p^k and hence over $\mathbb{Z}$ too.

Moduli Spaces of Curves

- How do we construct a global Kuranishi chart for the moduli space of genus 0 curves?
- Let (M, ω) be a closed symplectic manifold and J an ω-tame almost complex structure and $\beta \in H_2(M)$.
- Recall $\mathcal{M}_{(0,0)}(J, \beta)$ is the space of J-holomorphic maps $\Sigma \longrightarrow M$ where Σ is a genus zero nodal curve representing β up to equivalence:

$$\begin{tikzcd}
\Sigma \arrow[r] \arrow[d, "\cong"'] & M \\
\Sigma' \arrow[ur] &
\end{tikzcd}$$

- **First Problem:** The domain Σ isn't fixed. In order to do analysis, one really should identify this domain with something 'standard'.
- One typical way of doing this is adding marked points to Σ, until the domain becomes stable. This then identifies Σ with an element of $\overline{\mathcal{M}}_{0,n}$.
- Another way, suggested by Siebert, is to choose a basis of holomorphic sections of an ample line bundle on Σ. These sections then identify Σ with a curve mapping to projective space. We will use this approach.

Framed Curves

- Fix a Hermitian line bundle $L \longrightarrow M$ whose curvature is $-2i\pi\Omega$ where Ω is a symplectic form taming J.
- **Definition:** A *framed curve* is a triple (u, Σ, F) where $u : \Sigma \longrightarrow M$ is a smooth map from a nodal curve to M representing β and $F = (f_0, \cdots, f_d)$ is an orthonormal basis of $H^0(u^* L)$. We also require Ω to have positive degree on each irreducible component of Σ.
- Given any such framed curve, there is a natural degree d map

$$\phi_F : \Sigma \longrightarrow \mathbb{CP}^d, \quad \phi_F(\sigma) = [f_0(\sigma), \cdots, f_d(\sigma)].$$

Framed Curves

- Therefore, the domains Σ of framed curves (u, Σ, F) are identified with the fibers of the universal curve $\mathcal{C}$ over the automorphism free locus $\mathcal{F} \subset \mathcal{M}_{(0,0)}(\mathbb{CP}^d, d)$.
- As a result, a framed curve is equivalent to a smooth map $u : \mathcal{C}|_x \longrightarrow X$ from a fiber $\mathcal{C}|_x$ over $x \in \mathcal{F}$.
- There is a natural *Gromov topology* on the space of framed curves coming from the Hausdorff distance metric on the set of graphs of such curves, viewed as subsets of $M \times \mathcal{C}$.

- **Second problem:** The linearized Cauchy-Riemann equation is not surjective.
- To solve this, we need to find a natural vector space to surject onto the cokernel.
- Natural candidates are holomorphic sections of pullbacks of vector bundles over $X \times \mathcal{C}$.

- Choose large integer $k \gg 1$. And let $\mathcal{L}$ be an ample line bundle on our universal curve $\mathcal{C}$.
- For each framed curve (u, Σ, F) let $\iota_F : \Sigma \hookrightarrow \mathcal{C}$ be the natural domain inclusion map.
- **Definition**. We will define the *thickened moduli space* $\mathcal{T}$ to be the moduli space of tuples (u, Σ, F, η) where
  1. (u, Σ, F) is a framed curve.
  2. and $\eta \in H^0(\overline{\mathrm{Hom}}(\iota_F^* T\mathcal{C}, u^* TX) \otimes \iota_F^* \mathcal{L}^k) \otimes \overline{H^0(\iota_F^* \mathcal{L}^k)}$.

  satisfying
  $$\overline{\partial}_J u + \langle \eta \rangle \circ d\iota_F = 0$$
  where $\langle , \rangle : H^0(\overline{\mathrm{Hom}}(\iota_F^* T\mathcal{C}, u^* TX) \otimes \iota_F^* \mathcal{L}^k) \otimes \overline{H^0(\iota_F^* \mathcal{L}^k)} \longrightarrow C^\infty(\iota_F^* T\mathcal{C}, u^* TX)$ is the natural pairing.

- There is a bundle E over $\mathcal{T}$ whose fiber over (u, Σ, F, η) is $H^0(\overline{\mathrm{Hom}}(\iota_F^* T\mathcal{C}, TX) \otimes \iota_F^* \mathcal{L}^k) \otimes \overline{H^0(\iota_F^* \mathcal{L}^k)}$.
- This bundle has a canonical section s sending (u, Σ, F, η) to η.
- There is also a natural $U(d+1)$ action on $\mathcal{T}$ given by changing the framing F.
- $(U(d+1), \mathcal{T}, E, s)$ is our global Kuranishi chart.

- Hömanders theorem can be used to show that $\langle \eta \rangle$ can approximate any dirac delta section of $\overline{\mathrm{Hom}}(\iota_F^* T\mathcal{C}, u^* TX)$.
- This then can be used to show that the fibers of E surject onto the cokernel of the linearized Cauchy-Riemann operator.
- This ensures that $\mathcal{T}$ is a manifold.
- A 'Gromov trick' allows us to describe our moduli space as the space of holomorphic curves in a bundle over $X \times \mathcal{C}$ (i.e. we can get rid of obstruction bundles).
- G-equivariant smoothing theory of Lashof can be used to make $\mathcal{T}$ smooth after 'stabilizing' our global Kuranishi chart.

Splitting of Complex Oriented Cohomology.

- We will now work in the category of spectra.
- Our generalized cohomology theory is equal to $H^*(X; \mathbb{E}) = \pi_*(F(X, \mathbb{E}))$ where $F(X, \mathbb{E})$ is the space of maps from X to a ring spectrum $\mathbb{E}$. Homology is $H_*(X; \mathbb{E}) := \pi_*(X \wedge \mathbb{E})$.

Sweepout map.

- Let
$$\Phi : S^1 \times X \longrightarrow X, \quad \Phi(t,x) = \phi(t)(x)$$
be our loop of Hamiltonian diffeomorphisms and let
$$pr_X : S^1 \times X \longrightarrow X, \quad pr_X(t,x) = x$$
be the projection map.
- Then since they agree on $1 \times X$, we have an induced map of spectra:
$$\eta_\phi = (\Phi - pr_X) : S^1 \wedge X_+ \wedge \mathbb{S} \longrightarrow X_+ \wedge \mathbb{S}$$
called the *stable sweepout map* where $\mathbb{S}$ is the sphere spectrum.
- This induces the usual sweepout map on homology after smashing with $\mathbb{E}$.

Sweepout map continued.

- **Lemma:** There is a bijection between null homotopies of the map:

$$\eta_\phi \wedge \mathbb{E} : S^1 \wedge X_+ \wedge \mathbb{S} \longrightarrow X_+ \wedge \mathbb{E}$$

  and $\mathbb{E}$-module homotopies

$$(P_\phi)_+ \wedge \mathbb{E} \cong (X_+ \wedge \mathbb{E}) \vee (\Sigma^2 X_+ \wedge \mathbb{E}).$$

- It is therefore enough to show that the stable sweepout map vanishes in order to prove Theorem 2:
- Recall: **Theorem 2**:

$$H^*(P; \mathbb{E}) \cong H^*(X; \mathbb{E}) \otimes_{H^*(pt;\mathbb{E})} H^*(S^2; \mathbb{E})$$

  for any complex oriented cohomology theory $\mathbb{E}$ (such as complex cobordism).

Vanishing of stable sweepout map

- **Lemma:** If $\eta_\phi \wedge MU$ vanishes, then so does $\eta_\phi \wedge \mathbb{E}$ for any complex oriented cohomology theory $\mathbb{E}$.
- *Proof:* We have a map $\iota : MU \longrightarrow \mathbb{E}$ and so $\eta_\phi \wedge \mathbb{E} = (\mathrm{id} \wedge \iota) \circ (\eta_\phi \wedge MU)$.
- **Lemma:** $\eta_\phi \wedge MU$ vanishes iff $\eta_\phi \wedge MU_{(p)}$ vanishes for each prime p.

- **Fact:** There is a p-local spectrum BP so that:
  1. the p-localization $MU_{(p)}$ is a finite wedge sum of copies of shifts of BP and
  2. the natural map $BP \longrightarrow \prod_{n=1}^{\infty} L_{K_p(n)}BP$ into a product of localizations is the inclusion of a wedge summand.
- Therefore it is sufficient to show that $\eta_\phi \wedge L_{K_p(n)}BP$ vanishes.
- This follows from the fact that $\eta_\phi \wedge K_p(n)$ vanishes (using our moduli spaces of curves $\mathcal{M}_h$, $\mathcal{M}_\phi$, $\mathcal{M}_\infty$ as above).
- QED for Theorem 2.