Symplectic geometry of Stein manifolds and affine varieties

Mark McLean
Introduction

Hamiltonian Dynamics
Symplectic manifold definition

Stein manifolds
Cotangent bundles
Smooth affine varieties
Stein manifold definition

Main theorems
Questions concerning Stein manifolds
Cotangent bundles and smooth affine varieties
Symplectic and algebraic invariants
Suppose we have some physical system with position coordinates $q_1, \cdots, q_n$ and momentum coordinates $p_1, \cdots, p_n$.

We have some function $H(q_1, \cdots, q_n, p_1, \cdots, p_n)$ called the Hamiltonian which tells us the energy of the system at each state.

The system given by a path $(q_1(t), \cdots, q_n(t), p_1(t), \cdots, p_n(t))$ satisfies:

\[
\frac{\partial H}{\partial q_1} = -\frac{dp_1}{dt} \\
\vdots \\
\frac{\partial H}{\partial q_n} = -\frac{dp_n}{dt} \\
\frac{\partial H}{\partial p_1} = \frac{dq_1}{dt} \\
\vdots \\
\frac{\partial H}{\partial p_n} = \frac{dq_n}{dt}
\]
Suppose we have some physical system with position coordinates \( q_1, \cdots, q_n \) and momentum coordinates \( p_1, \cdots, p_n \).

We have some function \( H(q_1, \cdots, q_n, p_1, \cdots, p_n) \) called the Hamiltonian which tells us the energy of the system at each state.

The system given by a path \((q_1(t), \cdots, q_n(t), p_1(t), \cdots, p_n(t))\) satisfies:

\[
\begin{pmatrix}
    H_{q_1} \\
    \vdots \\
    H_{q_n} \\
    H_{p_1} \\
    \vdots \\
    H_{p_n}
\end{pmatrix} = -
\begin{pmatrix}
    0 & 1 & 0 \\
    0 & \ddots & \vdots \\
    -1 & 0 & 1 \\
    \vdots & \ddots & 0 \\
    0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
    \frac{dq_1}{dt} \\
    \vdots \\
    \frac{dq_n}{dt} \\
    \frac{dp_1}{dt} \\
    \vdots \\
    \frac{dp_n}{dt}
\end{pmatrix}
\]
Example: A single bead on a wire

\[ q(t) = \text{horizontal component of position at time } t \]
\[ p(t) = \text{horizontal component of momentum at time } t \]

Click to Animate http://math.mit.edu/~mclean/beadonwire/manybeadsonawire.html
- A symplectic manifold is a manifold with a 2-form $\omega$ which locally looks like:

$$\sum_{j=1}^{n} dp_j \wedge dq_j = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & \ddots \\
0 & \ddots & 0 \\
0 & 0 & -1
\end{pmatrix}$$

- (Darboux) Equivalently it is a manifold with a closed non-degenerate 2-form.
To construct a cotangent bundle $T^*M$ of some manifold $M$ you think of a bead constrained to that manifold.

If $M$ is the circle $S^1$ then our phase space is a cylinder $T^*S^1$:
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Definition

A smooth affine variety is a complex submanifold of $\mathbb{C}^N$ given by the zero locus of some polynomial equations. This has a symplectic form given by restricting the standard one $\sum_j dp_j \wedge dq_j$ on $\mathbb{C}^N = \mathbb{R}^{2N}$.

Examples

1. $\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^2 + z_2^2 + z_3^2 = 1 \} \cong_{\text{symp}} T^*(S^2)$.

2. $\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid \frac{z_1^2 + z_2^2}{z_3^2 + z_4^2} = 1 \} \cong_{\text{symp}} T^*(T^2)$

3. $\left\{ \frac{(z_1 z_3 + 1)^2 - (z_2 z_3 + 1)^3}{z_3} = 0 \right\}$

This smooth affine variety is not symplectomorphic to any cotangent bundle. It is in fact contractible.
Smooth affine varieties and cotangent bundles are both examples of Stein manifolds.

A *Stein manifold* is a properly embedded complex submanifold of $\mathbb{C}^N$. This has a symplectic form given by restricting the standard one $\sum_{j=1}^n dp_j \wedge dq_j$ on $\mathbb{C}^N = \mathbb{R}^{2N}$. 
Some questions about Stein manifolds and affine varieties

- What are good ways of describing Stein manifolds symplectically?
- What is the relationship between algebraic/analytic properties of the affine variety and the symplectic structure?
- Dynamical questions.
  - e.g. how many 1-periodic orbits does a Hamiltonian system on this affine variety have?
- Mirror Symmetry questions.
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- Mirror Symmetry questions.
Theorem
(Weinstein). Every Stein manifold has an explicit handle decomposition. Each handle has an explicit symplectic structure on it.

1d example.

2d example.
Theorem

(Eliashberg) If there is a diffeomorphism between Stein manifolds $A$ and $B$ (satisfying an additional topological condition) then $\mathbb{C} \times A$ is symplectomorphic to $\mathbb{C} \times B$.

Theorem

(Seidel-Smith) There are at least two non-symplectomorphic smooth affine varieties diffeomorphic to $\mathbb{R}^{2n}$ for each $n \geq 3$. 
Theorem
(Eliashberg) If there is a diffeomorphism between affine varieties $A$ and $B$ (satisfying an additional topological condition) then $\mathbb{C} \times A$ is symplectomorphic to $\mathbb{C} \times B$.

Theorem
(Seidel-Smith, M) There are infinitely many pairwise non-symplectomorphic smooth affine varieties diffeomorphic to $\mathbb{R}^{2n}$ for each $n \geq 3$. 
Theorem

(M). There is no algorithm telling us in general if two Weinstein handle presentations diffeomorphic to $\mathbb{R}^{2N}$ ($N > 7$) are symplectomorphic or not.

- There is an earlier result by Seidel where $\mathbb{R}^{2N}$ is replaced by a more complicated manifold.
Theorem

(M) Most cotangent bundles are not symplectomorphic to smooth affine varieties.

Here ‘most’ means that these cotangent bundles $T^*Q$ have complicated topology.

- If $\pi_1(Q)$ grows exponentially.
- If $\pi_1(Q) = 0$ and the sum of the Betti numbers is greater than $2^{\dim Q}$.
The previous theorem is similar to a result by Kulkarni.

Let $Q$ be a compact manifold described by the zero locus of some polynomial equations with real variables:

$$
\left\{ \begin{array}{c}
(x_1, \cdots, x_N) \in \mathbb{R}^N \\
p_1(x_1, \cdots, x_N) = 0 \\
\vdots \\
p_k(x_1, \cdots, x_N) = 0
\end{array} \right.
$$

Now replace the real coordinates $x_i$ with complex ones $z_i$:

$$
Q(\mathbb{C}) := \left\{ \begin{array}{c}
(z_1, \cdots, z_N) \in \mathbb{C}^N \\
p_1(z_1, \cdots, z_N) = 0 \\
\vdots \\
p_k(z_1, \cdots, z_N) = 0
\end{array} \right.
$$

Theorem

(Kulkarni) Suppose that the inclusion $Q \hookrightarrow Q(\mathbb{C})$ is a homotopy equivalence. Then $Q$ has nonnegative Euler characteristic.
The main tool used to prove this is called the growth rate of symplectic cohomology. This assigns to any Stein manifold $M$ a number $\Gamma(M) := \{-\infty\} \cup [0, \infty]$.

**Theorem (M)** For smooth affine varieties $A$, $\Gamma(A) < \dim_{\mathbb{C}} A$.

**Theorem**
(Abbondandolo-Schwartz, Salamon-Weber, Viterbo)
$\Gamma(T^*Q) = \infty$ for sufficiently complicated compact manifolds $Q$. 
For any Hamiltonian $H$, we get: $HF^*(H)$.

Chain complex freely generated by fixed points of the time 1 Hamiltonian flow of $H$.

The differential is given by a matrix (with respect to the basis of fixed points). Each entry is a count of solutions to:

$$\partial_s u + J\partial_t u = JX_H$$
For a Stein manifold $M \subset \mathbb{C}^N$, we choose a Hamiltonian $H = r^2$ where $r$ is the distance from the origin in $\mathbb{C}^N$.

The growth rate $\Gamma(M)$ (roughly) is the rate at which the rank of $HF^*(\lambda H)$ grows as $\lambda$ tends to infinity.

One can use other Hamiltonians $H$ (satisfying certain properties) and still get the same invariant $\Gamma(M)$. 
Bounding growth rate of affine varieties

For an affine variety $A$, we can find a nice Hamiltonian $H$ so that the number of fixed points of $\lambda H$ grows like a polynomial of degree $\dim_{\mathbb{C}} A$.

This bounds $HF^*(\lambda H)$ and hence $\Gamma(A) \leq \dim_{\mathbb{C}}(A)$. 
- $X =$ compactification of $A$.
- $D = X \setminus A =$ smooth normal crossing.
- **log Kodaira dimension** $\kappa(A) =$ rate at which $\rho_m := \text{rank}(H^0(m(K_X + D)))$ grows.
Theorem

(M)

▶ Log Kodaira dimension is a symplectic invariant for acyclic smooth affine varieties of dimension 2.

▶ Partial results in dimension 3.

▶ (Work in progress): If $A, B$ are symplectomorphic affine varieties and $\kappa(A) < 1 + \text{technical conditions}$ then $\kappa(B) < 1$. 

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Theorem

\((M)\) Suppose \(A\) is uniruled \((i.e.\ there\ is\ a\ rational\ curve\ passing\ through\ every\ point)\) and \(B\) is symplectomorphic to \(A\) then \(B\) is also uniruled.
We use Gromov-Witten invariants to prove these results:

- Embed $A$ as an open subset of a projective variety $X$.
- Count holomorphic maps $\mathbb{P}^1 \to X$.
- Relate log Kodaira dimension to these counts of curves.
What other algebraic structures are remembered by the symplectic structure? (rational connectedness?). What about log general type affine varieties?

Relationship between dynamical properties and algebraic properties.

What can symplectic/contact topology say about singularities?