Local Uniruledness Does Not Imply Global Uniruledness in Symplectic Topology

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A smooth projective variety is **rationally connected** if there exists a rational holomorphic curve passing through any two points. In [Kol96], it was conjectured that the property of being rationally connected is a symplectic invariant. One possible approach to such a conjecture would involve showing that if a symplectic manifold with a compatible almost complex structure is locally ruled by genus 0 holomorphic curves representing some homology class, then it is globally ruled by such curves. This leads us to the following question posed by Zinger.

Question. Let (X, ω) be a connected symplectic manifold and let J be an ω-compatible (or ω-tame) almost complex structure. If there is a non-empty open subset $W \subset X$ so that $(W, J|_W)$ is uniruled, then doesn't it follow that (X, J) is uniruled?

See below for a definition of uniruledness. In this note, we show that the answer to the above question is no by producing an explicit example. The construction of this example and the proof follow from the key observation that the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ over $\mathbb{P}^1$ is trivial as a symplectic vector bundle.

Another related but important question which will not be answered in this paper is the following.

Question. Suppose that (X, ω) is a compact symplectic manifold and J_1, J_2 are ω-compatible (or ω-tame) almost complex structures on X. If (X, J_1) is uniruled, doesn't it follow that (X, J_2) is uniruled?

Definition. An almost complex manifold (X, J) is **uniruled** if for every $x \in X$, there exists a non-constant J-holomorphic map $u : \mathbb{P}^1 \to X$ satisfying $x \in u(\mathbb{P}^1)$. Now suppose that we have a non-trivial homology class $\beta \in H_2(X; \mathbb{Z})$. We say that (X, J) is **β-uniruled** if for every $x \in X$, there exists a J-holomorphic map $u : \mathbb{P}^1 \to X$ representing β whose image contains x.

Let $\mathbb{T} = \mathbb{C}/\mathbb{Z}^2$ be the standard complex torus and let X be the one-point blowup of $\mathbb{T}^3 \equiv \mathbb{T} \times \mathbb{T} \times \mathbb{T}$ at the point $(0, 0, 0)$. Let ω be a Kähler form on X.

Theorem. There is an almost complex structure J on X compatible with ω, a non-empty open set $W \subset X$, a point $y \in X$, and a non-trivial homology class $\beta \in H_2(W; \mathbb{Z})$ so that

1. $(W, J|_W)$ is β-uniruled and
2. $y \notin u(\mathbb{P}^1)$ for all non-constant J-holomorphic maps $u : \mathbb{P}^1 \to X$.

In order to prove this theorem we need two preliminary lemmas. We let $J_X : TX \to TX$ be the standard complex structure on X and let $E \subset X$ be the exceptional divisor of the blowdown map $\mathrm{Bl} : X \to \mathbb{T}^3$.

Lemma 1. *There is an open set $U \subsetneq X$ containing E so that every non-constant J-holomorphic map $u : \mathbb{P}^1 \to X$ has image in U for every almost complex structure J on X satisfying $J|_{X-K} = J_X|_{X-K}$ for some compact $K \subset U$.*

Proof. For every $r \in \mathbb{R}^+$, define

$$D_r \equiv \{z \in \mathbb{T} = \mathbb{C}/\mathbb{Z}^2 \ : \ |z| < r\} \subset \mathbb{T}, \quad D_r^c \equiv \mathbb{T} - D_r.$$

We take $U \equiv \mathrm{Bl}^{-1}((D_{\frac{1}{2}})^3)$ where $(D_{\frac{1}{2}})^3 \subset \mathbb{T}^3$. For $i = 1, 2, 3$, let

$$\pi_i : \mathbb{T}^3 \twoheadrightarrow \mathbb{T}, \quad \pi_i(z_1, z_2, z_3) = z_i$$

be the i-th component projection map.

Let J be any almost complex structure on X satisfying $J|_{X-K} = J_X|_{X-K}$ for some compact $K \subset U$ and let $u : \mathbb{P}^1 \to X$ be any non-constant J-holomorphic map. For each $i = 1, 2, 3$, let $u_i \equiv \pi_i \circ \mathrm{Bl} \circ u$. Since $K \subset U$ is compact and $u_i|_{u^{-1}(X-K)}$ is J_X-holomorphic, there exists $r \in (0, \frac{1}{2})$ such that $K \subset \pi_i^{-1}(D_r)$ and ∂D_r^c is contained in the set of regular values of u_i for every $i = 1, 2, 3$. Define $\Sigma_i \equiv u_i^{-1}(D_r^c)$ and

$$u_i^{\geq r} : \Sigma_i \to D_r^c, \quad u_i^{\geq r}(v) = u_i(v), \quad \forall v \in \Sigma_i, \quad \forall i \in \{1, 2, 3\}.$$

Since $\partial \Sigma_i = u_i^{-1}(\partial D_i^c)$ is contained in the set of regular values of u_i, $u_i^{\geq r}$ is a holomorphic map between compact Riemann surfaces with boundary sending $\partial \Sigma_i$ to ∂D_r^c.

Suppose that the image of u is not contained in U. Then there is some $i \in \{1, 2, 3\}$ so that the image of u_i is not contained in $D_{\frac{1}{2}} \subset \mathbb{T}$ and hence $\Sigma_i \neq \emptyset$. Thus, there is a connected component Σ of Σ_i so that $v \equiv u_i^{\geq r}|_\Sigma : \Sigma \to D_r^c$ is a non-constant holomorphic map. Since v is a non-constant proper holomorphic map of degree $d \geq 1$, we get that

$$\chi(\Sigma) \leq d\chi(D_r^c) = -d \tag{1}$$

by the Riemann-Hurwicz formula. Let $B \in \mathbb{N}$ be the number of boundary components of Σ. Since Σ is connected and of genus 0,

$$\chi(\Sigma) = 2 - B. \tag{2}$$

Since $v|_{\partial\Sigma} : \partial\Sigma \to \partial D_r^c$ is an orientation-preserving covering map of degree d as ∂D_r^c is contained in the set of regular values of u_i, $B \leq d$. Hence by Equations (1) and (2), $2 - d \leq -d$ which gives us a contradiction. Therefore $u(\mathbb{P}^1) \subset U$. □

Lemma 2. *For every compact set $K \subset X$ whose interior contains E, there is an almost complex structure J compatible with ω, a non-empty open set $W \subset K$ and an element $\beta \in H_2(W; \mathbb{Z})$ so that $J|_{X-K} = J_X|_{X-K}$ and $(W, J|_W)$ is β-uniruled.*

Proof. Let $L \subset E$ be a complex line in the exceptional divisor $E \cong \mathbb{P}^2$. Let $\omega_{\mathbb{C}^2}$ be the standard symplectic form on $\mathbb{C}^2$ and define $\omega_L \equiv \omega|_L$. Since the normal bundle of L is isomorphic as a complex vector bundle to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$, it has trivial first Chern class. Hence by [MS98, Theorem 2.69], it is trivial as a symplectic vector bundle. Therefore, by the Symplectic Neighborhood Theorem [MS98, Theorem 3.30], there is a neighborhood $\mathcal{N}(L)$ of L in X, a neighborhood $\mathcal{N}$ of $L \times \{0\}$ inside $L \times \mathbb{C}^2$, and a symplectomorphism

$$\Phi : (\mathcal{N}, (\omega_L + \omega_{\mathbb{C}^2})|_{\mathcal{N}}) \to (\mathcal{N}(L), \omega|_{\mathcal{N}(L)}).$$

Let $J_{\mathbb{C}^2}$ be the standard complex structure on $\mathbb{C}^2$ and let J_L be equal to the induced complex structure on the J_X-holomorphic submanifold $L \subset X$. For $\epsilon > 0$ small enough,

$$L \times B_\epsilon \subset \Phi^{-1}(K \cap \mathcal{N}(L)),$$

where $B_\epsilon \subset \mathbb{C}^2$ is the ball of radius $\epsilon > 0$. Define $W \equiv \Phi(L \times B_\epsilon)$. By Proposition , there is an almost complex structure J such that

$$J|_W = \Phi_*(J_L \oplus J_{\mathbb{C}^2})|_W, \quad J|_{X-K} = J_X|_{X-K}.$$

Choose a J_L-holomorphic isomorphism $\phi : \mathbb{P}^1 \to L$. For each $b \in B_\epsilon$, define

$$u_b : \mathbb{P}^1 \to X, \quad u_b(x) = \Phi(\phi(x), b), \quad \forall x \in \mathbb{P}^1.$$

Since $W \subset \Phi(L \times B_\epsilon)$ and $J|_W = \Phi_*(J_L \oplus J_{\mathbb{C}^2})|_W$, u_b is a J_L-holomorphic map.

Since for every w in W there is a unique point $(l, b) \in L \times B_\epsilon$ so that $\Phi(l, b) = b$, the images of $(u_b)_{b \in B_\epsilon}$ cover W. Define $\beta \equiv [L] \in H_2(W; \mathbb{Z})$. Then each of the J-holomorphic maps $(u_b)_{b \in B_\epsilon}$ also represent β. □

Proof of Theorem . We let U be the open set from Lemma 1. Hence there exists a point $y \in X - U$ and a compact set $K \subset U$ whose interior contains E. By Lemma 2, there is a non-empty open set $W \subset K$, an almost complex structure J compatible with ω, and an element $\beta \in H_2(X; \mathbb{Z})$ so that so that $J|_{X-K} = J_X|_{X-K}$ and $(W, J|_W)$ is $\beta|_W$-uniruled. By Lemma 1, we get that $y \notin u(\mathbb{P}^1)$ for all J-holomorphic maps $u : \mathbb{P}^1 \to X$. □

Appendix: Extending Compatible Almost Complex Structures.

Proposition. Let (X, ω) be a symplectic manifold and $U_1, W_1, U_2, W_2 \subset X$ be open subsets such that

$$\overline{U}_1 \subset W_1, \qquad \overline{U}_2 \subset W_2, \qquad W_1 \cap W_2 = \emptyset\,.$$

If J_1 and J_2 are almost complex structures on W_1 and W_2, respectively, compatible with $\omega|_{W_1}$ and $\omega|_{W_2}$, then there exists an almost complex structure J on X compatible with ω such that $J|_{U_1} = J_1$ and $J|_{U_2} = J_2$.

The above Proposition will follow immediately either from Lemma 3 or from Lemma 4 below. Lemma 3 uses general results by Palais and Steenrod, whereas Lemma 4 gives us a more direct proof of the above Proposition.

For a finite-dimensional symplectic vector space (V, Ω), we denote by $\mathcal{J}(V, \Omega)$ the manifold of linear complex structures compatible with Ω. For a symplectic vector bundle (E, ω) over a topological space X, we denote by $\mathbb{J}(E, \omega) \longrightarrow X$ the fiber bundle with fiber $\mathcal{J}(E_x, \omega_x)$ over $x \in X$. If (E, ω) is a smooth symplectic vector bundle, then $\mathbb{J}(E, \omega) \longrightarrow X$ is a smooth fiber bundle.

Because the fibers of $\mathbb{J}(E, \omega) \longrightarrow X$ are contractible by [MS98, Proposition 2.50(iii)], we have that the proposition above follows immediately from the following lemma.

Lemma 3. *Let $\pi : Q \to B$ be a fiber bundle whose fiber is a contractible metrizable manifold and whose base is a metrizable topological space. Let $U, U' \subset B$ be open sets so that $\overline{U'} \subset U$ and let $\sigma_U : U \to Q$ be a section of $\pi|_U$. Then there is a section σ of π so that $\sigma|_{U'} = \sigma_U|_{U'}$. If $\pi : Q \to B$ is a smooth fiber bundle and σ_U is smooth section then the section σ can be chosen to be smooth.*

Proof. Since B is a normal topological space, there is an open set U'' in B so that $\overline{U'} \subset U'' \subset \overline{U''} \subset U$. By [Pal66, Theorem 9], there is a continuous section s of π so that $s|_{U''} = \sigma_U|_{U''}$. If $\pi : Q \to B$ is smooth then by the Steenrod approximation theorem [Ste99, Section 6.7, Main Theorem], there is a smooth section σ of π so that $\sigma|_{\overline{U'}} = s|_{\overline{U'}}$. Therefore $\sigma|_{U'} = \sigma_U|_{U'}$. □

Lemma 4. *Let (E,ω) be a symplectic vector bundle over a paracompact topological space X and $U, W \subset X$ be open subsets such that $\overline{U} \subset W$. If J_W is a section of $\mathbb{J}(E,\omega)|_W$, then there exists a section J of $\mathbb{J}(E,\omega)$ such that $J|_U = J_W|_U$. If (E,ω) is a smooth symplectic vector bundle and J_W is a smooth section of $\mathbb{J}(E,\omega)|_W$, then the section J can be chosen to be smooth.*

Proof. For a finite-dimensional vector space V, we denote by $\mathcal{M}(V)$ the manifold of positive-definite inner-products on V. For a finite-dimensional symplectic vector space (V,Ω), we denote by $\mathrm{Sp}(V,\Omega) \subset \mathrm{GL}(V)$ the subgroup of linear automorphisms Φ preserving the symplectic form Ω and define

$$s_{V,\Omega} \colon \mathcal{J}(V,\Omega) \longrightarrow \mathcal{M}(V), \qquad \big\{s_{V,\Omega}(J)\big\}(v,w) = \Omega(v, Jw) \quad \forall\, v,w \in V,\ J \in \mathcal{J}(V,\Omega).$$

We note that

$$s_{V,\Omega}\big(\Phi^* J\big) = \Phi^*\big(s_{V,\Omega}(J)\big) \qquad \forall\, J \in \mathcal{J}(V,\Omega),\ \Phi \in \mathrm{Sp}(V,\Omega). \tag{3}$$

By [MS98, Proposition 2.50], there exists a continuous map

$$r_{V,\Omega} \colon \mathcal{M}(V) \longrightarrow \mathcal{J}(V,\Omega) \qquad \text{s.t.}$$
$$r_{V,\Omega} \circ s_{V,\Omega} = \mathrm{id}_{\mathcal{J}(V,\Omega)}, \qquad r_{V,\Omega}\big(\Phi^* g\big) = \Phi^*\big(r_{V,\Omega}(g)\big) \quad \forall\, g \in \mathcal{M}(V),\ \Phi \in \mathrm{Sp}(V,\Omega). \tag{4}$$

By [MS11, Exercise 2.52], the map $r_{V,\Omega}$ is in fact smooth.

For a real vector bundle E over X, we denote by $\mathbb{M}(E) \longrightarrow X$ the fiber bundle with fiber $\mathcal{M}(E_x)$ over $x \in X$. If E is a smooth vector bundle, then $\mathbb{M}(E) \longrightarrow X$ is a smooth fiber bundle. By (3) and the second identity in (4), the maps $s_{V,\Omega}$ and $r_{V,\Omega}$ induce continuous bundle maps

$$s_{E,\omega} \colon \mathbb{J}(E,\omega) \longrightarrow \mathbb{M}(E) \qquad \text{and} \qquad r_{E,\omega} \colon \mathbb{M}(E) \longrightarrow \mathbb{J}(E,\omega).$$

If (E,ω) is a smooth symplectic vector bundle, then these maps are smooth. By the first identity in (4),

$$r_{E,\omega} \circ s_{E,\omega} = \mathrm{id}_{\mathbb{J}(E,\omega)}. \tag{5}$$

Choose any section g_X of $\mathbb{M}(E)$ and a continuous function

$$\rho \colon X \longrightarrow [0,1] \qquad \text{s.t.} \qquad \rho(x) = \begin{cases} 1, & \text{if } x \in U; \\ 0, & \text{if } x \notin W. \end{cases}$$

Let $g_W = s_{E|_W,\omega|_W}(J_W)$ and define

$$g \colon X \longrightarrow \mathbb{M}(E), \qquad g|_x = \begin{cases} \rho(x) g_W|_x + (1-\rho(x)) g_X|_x, & \text{if } x \in W; \\ g_X|_x, & \text{if } x \notin W; \end{cases}$$
$$J \colon X \longrightarrow \mathbb{J}(E,\omega), \qquad J(x) = r_{E,\omega}(g|_x).$$

By (5), $J|_U = J_W|_U$. □

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