1. Cohomology of Grassmannian.

We will first compute the cohomology ring in the case when $n = 1$ (this is in the homework)

Lemma 1.1. We have the following graded algebra isomorphism

$$H^*(Gr_1(\mathbb{R}^\infty), \mathbb{Z}/2\mathbb{Z}) = H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[a]$$

where $a \in H^2(\mathbb{RP}^\infty, \mathbb{Z}/2\mathbb{Z})$ has degree 2.

Proof. $\mathbb{RP}^k$ is constructed as a CW complex by attaching a $k-1$ to $\mathbb{RP}^{k-1}$ via the double covering map $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$. The cellular cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients is then the vector space. Hence we just need to compute the ring structure. This follows from the following commutative diagram where $i + j = n$:

$$\begin{array}{ccc}
H^i(\mathbb{RP}^n; Z/2Z) \times H^j(\mathbb{RP}^n; Z/2Z) & \xrightarrow{\quad\cup\quad} & H^n(\mathbb{RP}^n; Z/2Z) \\
\uparrow \cong & & \uparrow \cong \\
H^i(\mathbb{RP}^n; \mathbb{RP}^n - \mathbb{RP}^j; Z/2Z) \times H^j(\mathbb{RP}^n; \mathbb{RP}^n - \mathbb{RP}^i; Z/2Z) & \xrightarrow{\cup} & H^n(\mathbb{RP}^n; \mathbb{RP}^n - \mathbb{RP}^0; Z/2Z) \\
\downarrow \cong & & \downarrow \cong \\
H^i(\mathbb{R}^n; \mathbb{R}^n - \mathbb{R}^j; Z/2Z) \times H^j(R^n; R^n - R^i; Z/2Z) & \xrightarrow{\quad\cup\quad} & H^n(\mathbb{R}^n; \mathbb{R}^n - 0; Z/2Z) \\
\downarrow \cong & & \downarrow \cong \\
H^i(\mathbb{R}^i; \mathbb{R}^i - 0; Z/2Z) \times H^j(\mathbb{R}^j; \mathbb{R}^j - 0; Z/2Z) & \xrightarrow{\quad\times\quad} & H^n(R^n; R^n - 0; Z/2Z)
\end{array}$$

□

Corollary 1.2. $H^*((\mathbb{RP}^\infty)^n) \cong (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n]$.

Note that $(\mathbb{RP}^\infty)^n$ classifies vector bundles of the form $\oplus_{i=1}^n \gamma_i$ where γ_i is a line bundle up to isomorphism which *preserve* the direct sum decomposition and the ordering of the line bundles $\gamma_1, \cdots, \gamma_n$.

Theorem 1.3 (Leray Hirsch Theorem). Let $\pi : E \longrightarrow B$ be a fiber bundle (all our spaces are CW complexes). Let $\iota : F \longrightarrow E$ be the natural inclusion map of the fiber and suppose that there is a linear map

$$s : H^*(F; \Lambda) \longrightarrow H^*(E; \Lambda)$$

satisfying $\iota^* \circ s = id_{H^*(F)}$. Then the natural linear map

$$H^*(F; \Lambda) \otimes H^*(B; \Lambda) \longrightarrow H^*(E; \Lambda), \quad \alpha \otimes \beta \longrightarrow s(\alpha) \cup \pi^*\beta$$

is an isomorphism.

In particular the natural map

$$\pi^* : H^*(B; \Lambda) \longrightarrow H^*(E; \Lambda)$$

is injective.

Later on we will also need a proof of a relative version of the Leray-Hirsch theorem.

Theorem 1.4 (Relative Leray Hirsch Theorem). Let $\pi : E \longrightarrow B$ be a fiber bundle (all our spaces are CW complexes) and let $E_0 \subset E$ be a subbundle. Let $\iota : F \longrightarrow E$ be the natural inclusion map of the fiber and let $F_0 \subset F$ be the fiber of E_0. Suppose that there is a linear map

$$s : H^*(F, F_0; \Lambda) \longrightarrow H^*(E, E_0; \Lambda)$$

satisfying $\iota^* \circ s = id_{H^*(F,F_0)}$. Then the natural linear map

$$H^*(F, F_0; \Lambda) \otimes H^*(B; \Lambda) \longrightarrow H^*(E, E_0; \Lambda), \quad \alpha \otimes \beta \longrightarrow s(\alpha) \cup \pi^* \beta$$

is an isomorphism.

We will only prove the Leray-Hirsch theorem as the relative version of this theorem has exactly the same proof.

Proof. This argument would be straightforward if we knew about spectral sequences, but we don't. As a result we will do this a different (but directly related) way. Now B is a direct limit of compact sets $K_0 \subset K_1 \subset \cdots$. Therefore is sufficient for us to show that

$$H^*(F; \Lambda) \otimes H^*(K_i; \Lambda) \longrightarrow H^*(E|_{K_i}; \Lambda), \quad \alpha \otimes \beta \longrightarrow s(\alpha) \cup \pi^* \beta$$

is an isomorphism for all i.

So from now on we will assume that B is compact. Let $U_1, \cdots, U_m$ be open subsets of B so that $E|_{U_i}$ is trivial. Define $U_{<i} \equiv \cup_{j<i} U_i$. Suppose (by induction) we have shown that the map

$$F_{<i} : H^*(F; \Lambda) \otimes H^*(U_{<i}; \Lambda) \longrightarrow H^*(E|_{U_{<i}}; \Lambda), \quad F_{<i}(\alpha \otimes \beta) \equiv s(\alpha) \cup \pi^* \beta$$

is an isomorphism for some i. We now wish to show that the corresponding map $F_{<i+1}$ is an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
H^*(F; \Lambda) \otimes H^*(U_{<i} \cap U_{i+1}; \Lambda) & \xrightarrow{\alpha \otimes \beta \to (s(\alpha)|_{U_{<i} \cap U_{i+1}}) \cup \beta} & H^*(E|_{U_{<i} \cap U_{i+1}}; \Lambda) \\
\uparrow & & \uparrow \\
H^*(F; \Lambda) \otimes H^*(U_{<i}; \Lambda) \oplus H^*(F; \Lambda) \otimes H^*(U_{i+1}; \Lambda) & \xrightarrow{\alpha \otimes \beta \oplus \alpha' \otimes \beta' \to (s(\alpha)|_{U_{<i}}) \cup \beta \oplus (s(\alpha')|_{U_{<i}}) \cup \beta'} & H^*(E|_{U_{<i}}; \Lambda) \oplus H^*(E|_{U_{i+1}}; \Lambda) \\
\uparrow & & \uparrow \\
H^*(F; \Lambda) \otimes H^*(U_{<i+1}; \Lambda) & \xrightarrow{\alpha \otimes \beta \to (s(\alpha)|_{U_{<i+1}}) \cup \beta} & H^*(E|_{U_{<i+1}}; \Lambda) \\
\uparrow & & \uparrow
\end{array}$$

The vertical arrows form a Mayor-Vietoris long exact sequence. Also the horizontal arrows are isomorphisms at the top and the bottom for all i. Hence by the five lemma we get our isomorphism. $\square$

We have the following corollary of the Leray-Hirsch theorem:

Theorem 1.5. Thom Isomorphism Theorem over $\mathbb{Z}/2$ Let $\pi : E \longrightarrow B$ be a rank n vector bundle and define $E_0 \equiv E - B$ where $B \subset E$ is the zero section. Then there is a class $\alpha \in H^n(E, E_0; \mathbb{Z}/2\mathbb{Z})$ so that the map

$$H^*(B; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{*+n}(E, E_0; \mathbb{Z}/2\mathbb{Z}), \quad \beta \longrightarrow \beta \cup \alpha$$

is an isomorphism.

This theorem is true over any coefficient field if we assumed that E is an oriented vector bundle.

Definition 1.6. The **unoriented Euler class** of a vector bundle $\pi : E \longrightarrow B$ as above is a class $e(E; \mathbb{Z}/2\mathbb{Z}) \in H^n(E; \mathbb{Z}/2\mathbb{Z})$ given by the image of the class α under the composition $H^n(E, E_0; \Lambda) \longrightarrow H^n(E; \Lambda) \longrightarrow H^n(B; \Lambda)$.

Note that if E is an oriented vector bundle then we can define the Euler class $e(E;\Lambda)$ over any coefficient ring λ. Usually when people talk about the Euler class, they are talking about $e(E;\mathbb{Z})$ (we will call this the **Euler class**) and we will write $e(E)$.

Proof. We will only prove our theorem when the coefficient field is $\mathbb{Z}/2\mathbb{Z}$. The proof is exactly the same if we have oriented vector bundles and another coefficient ring.

Our fiber is $F = \mathbb{R}^n$ and the fiber of $\pi|_{E_0}$ is $F_0 = \mathbb{R}^n - 0$. By the relative Leray-Hirsch theorem it is sufficient to show that there is a class $\alpha \in H^n(E, E_0; \Lambda)$ whose restriction to $H^*(\mathbb{R}^n, 0; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is the unit $1 \in \mathbb{Z}/2\mathbb{Z}$. We assume that B is connected.

Let $(U_i)_{i\in\mathbb{N}}$ be an open cover by relatively compact sets where $E|_{U_i}$ is trivial. Define $U_{<i} \equiv \cup_{j<i} U_j$. We'll suppose that $U_{<i}$ and U_i is connected for all $i \in \mathbb{N}$ and that $F = \mathbb{R}^n \subset E|_{U_0}$. Suppose (by induction) there is a class $\alpha_i \in H^n(E|_{U_{<i}}; E_0|_{U_{<i}}; \mathbb{Z}/2\mathbb{Z})$ whose restriction to $H^n(\mathbb{R}^n; \mathbb{R}^n - 0; \mathbb{Z}/2\mathbb{Z})$ is 1. Consider the Mayor-Vietoris sequence:

$$\longrightarrow H^n(E|_{U_{<i+1}}; E_0|_{U_{<i+1}}; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{a}$$

$$H^n(E|_{U_{<i}}; E_0|_{U_{<i}}; \mathbb{Z}/2\mathbb{Z}) \oplus H^n(E|_{U_{i+1}}; E_0|_{U_{i+1}}; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{b} H^n(E|_{U_{<i}\cap U_{i+1}}; E_0|_{U_{<i}\cap U_{i+1}}; \mathbb{Z}/2\mathbb{Z}).$$

Since

$$H^*(E|_{U_{i+1}}; E_0|_{U_{i+1}}; \mathbb{Z}/2\mathbb{Z}) = H^*(U_{i+1}; \mathbb{Z}/2\mathbb{Z}) \otimes H^*(\mathbb{R}^n, 0; \mathbb{Z}/2\mathbb{Z}) \cong H^{*-n}(U_{i+1}; \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}$$

we get a class $\alpha' \in H^*(E|_{U_{i+1}}; E_0|_{U_{i+1}}; \mathbb{Z}/2\mathbb{Z})$ mapping to 1 under the above isomorphism and hence whose restriction to $H^*(\mathbb{R}^n, 0; \mathbb{Z}/2\mathbb{Z})$ is 1. Also since $E|_{U_{<i}\cap U_{i+1}}$ is trivial, we get get using similar reasoning that the images of α_n and α' in $H^*(E|_{U_{<i}\cap U_{i+1}})$ are equal. Hence $b(\alpha_i \oplus \alpha') = 0$. Hence there is a class $\alpha_{i+1} \in H^n(E|_{U_{<i+1}}; E_0|_{U_{<i+1}}; \mathbb{Z}/2\mathbb{Z})$ so that $a(\alpha_i \oplus \alpha')$. This class maps to $1 \in H^*(\mathbb{R}^n, 0; \mathbb{Z}/2\mathbb{Z})$. □

The Euler class satisfies the following properties:

(1) (**Functoriality**) If $\pi : E \longrightarrow B$ is isomorphic to f^*E' for some other bundle $\pi' : E' \longrightarrow B'$ and function $f : B \longrightarrow B'$ then $e(E;\Lambda) = f^*(e(E';\Lambda))$.
(2) (**Whitney Sum Formula**) If $\pi : E \longrightarrow B$ and $\pi' : E' \longrightarrow B$ are two vector bundles over the same base then $e(E \oplus E';\Lambda) = e(E;\Lambda) \cup e(E';\Lambda)$.
(3) (**Normalization**) If E admits a nowhere zero section then $e(E;\Lambda) = 0$.
(4) (**Orientation**) If E is an oriented vector bundle and $\overline{E}$ is the same bundle with the opposite orientation then $e(E) = -e(\overline{E})$.

The following is a geometric interpretation of the Euler class when the base is a compact manifold. We need a definition first.

Definition 1.7. Let M_1, M_2 be submanifolds of a manifold X. Then M_1 is **transverse** to M_2 if for every point $x \in M_1 \cap M_2$, we have that

$$\text{codim}(T_xM_1 \cap T_xM_2 \subset T_xX) = \text{codim}(T_xM_1 \subset T_xX) + \text{codim}(T_xM_2 \subset T_xX).$$

Let $\pi : E \longrightarrow B$ be a smooth vector bundle over a smooth compact manifold B. A smooth section $s : B \longrightarrow E$ is **transverse to** 0 if the submanifold $s(B) \subset E$ is transverse to the zero section $B \subset E$.

Note that if M_1 intersects M_2 transversely then $M_1 \cap M_2$ is a manifold. Also if X, M_1 and M_2 are oriented (in other words TX,TM_1 and TM_2 are oriented) then $M_1 \cap M_2$ has an orientation defined as follows: Let $N(M_1 \cap M_2)$, NM_1 and NM_2 be the normal bundles of $M_1 \cap M_2$, M_1 and M_2 inside X. Then we have isomorphisms

$$TM_1 \oplus NM_1 \cong TX|_{M_1}, \quad TM_2 \oplus NM_2 \cong TX|_{M_2},$$

$$T(M_1 \cap M_2) \oplus N(M_1 \cap M_2) \cong TX|_{M_1 \cap M_2},$$

$$N(M_1) \oplus N(M_2) \cong NM_1|_{M_1 \cap M_2} \oplus NM_1|_{M_1 \cap M_2}.$$

The first two isomorphisms give us an orientation on NM_1 and NM_2 and the last one gives us an orientation on $N(M_1 \cap M_2)$. The third isomorphism then gives us an orientation on $T(M_1 \cap M_2)$ called the **intersection orientation**.

Also note that any compact manifold M (whether oriented or not) has a fundamental class $[M] \in H^n(M; \mathbb{Z}/2\mathbb{Z})$ over $\mathbb{Z}/2\mathbb{Z}$.

It turns out that a 'generic' section is transverse ('generic' will be defined precisely later in the course). I won't prove this for the moment (maybe later).

Lemma 1.8. Let $\pi : E \longrightarrow B$ be a smooth vector bundle over a smooth compact manifold B with a smooth section s transverse to 0. Then $e(E; \mathbb{Z}/2\mathbb{Z})$ is Poincaré dual to $[s^{-1}(0)] \in H^*(B; \mathbb{Z}/2\mathbb{Z})$.

If E and B are oriented then $s^{-1}(0)$ has the intersection orientation and the above lemma makes sense in this case over any coefficient field Λ.

Our goal is to compute the cohomology of $Gr_n(\mathbb{R}^\infty)$ and so we must continue....

Definition 1.9. Let $\pi : E \longrightarrow B$ be a rank n vector bundle. The **projective bundle** $\mathbb{P}(E) \longrightarrow B$ is the fiber bundle whose fiber at a point $b \in B$ is $\mathbb{P}(\pi^{-1}(b))$ (I.e. the set of lines inside $\pi^{-1}(b)$).

Lemma 1.10. Let $\pi : E \longrightarrow B$ be a rank n vector bundle. The natural map $H^*(B) \longrightarrow H^*(\mathbb{P}(E))$ is injective. In fact $H^*(\mathbb{P}(E)) \cong H^*(\mathbb{RP}^{n-1}) \otimes H^*(B)$ and the natural map $H^*(B) \longrightarrow H^*(\mathbb{P}(E))$ is the inclusion map into the first factor.

Proof. Now $P\mathbb{P}(E)$ as a canonical line bundle γ_E whose fiber at a point $x \in E$ is the line l passing through x inside $\pi^{-1}(\pi(x))$. Let $f : \mathbb{P}(E) \longrightarrow \mathbb{RP}^\infty$ be the classifying map for this line bundle. Recall that $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[a]$ where $a \in H^2(\mathbb{RP}^\infty) - 0$. We will also write $H^*(\mathbb{RP}^{n-1}) = (\mathbb{Z}/2\mathbb{Z})[b]/(b^n)$. Recall that our fiber F is equal to $\mathbb{RP}^{n-1}$. Since γ restricted to each fiber is $\mathcal{O}(-1)$, we get that f^*a restricted to the fiber $F = \mathbb{RP}^{n-1}$ is b. Hence $f^*(a^m)|_F = b^m$ which implies that that map $H^*(E) \longrightarrow H^*(F)$ is surjective. Hence by the Leray-Hirsch theorem, $H^*(\mathbb{P}(E)) \cong H^*(\mathbb{RP}^{n-1}) \otimes H^*(B)$ and the natural map $H^*(B) \longrightarrow H^*(\mathbb{P}(E))$ is the inclusion map into the first factor. $\square$

Definition 1.11. Let $\pi : E \longrightarrow B$ be a real vector bundle. A **splitting map** for E is a map $f : B' \longrightarrow B$ so that $f^*E \cong \oplus_{i=1}^n \gamma_i$ where γ_i are line bundles over B' and where $f^* : H^*(B) \longrightarrow H^*(B')$ is injective.

Lemma 1.12. Let $\pi : E \longrightarrow B$ be a vector bundle. Let $P : \mathbb{P}(E) \longrightarrow B$ be the associated projective bundle. Then there is a line subbundle $\gamma \subset P^*E$.

Proof. Here γ is defined to be the line in P^*E which sends a point $x \in \mathbb{P}(E)$ to the corresponding line in E. $\square$

Lemma 1.13. Every real vector bundle $\pi : E \longrightarrow B$ of rank n has a splitting map.

Proof. Suppose (inductively) we have constructed a map $P_k : B_k \longrightarrow B$ for some $0 \leq k < n$ so that $P_k^*(E) = V \bigoplus \oplus_{i=1}^k \gamma_i$ where V is a vector bundle and γ_i are line bundles and so that $P_k^* : H^*(B) \longrightarrow H^*(B_k)$ is injective. Define $B_{k+1} \equiv \mathbb{P}(V)$ and let $p : \mathbb{P}(V) \longrightarrow B'$ be the natural map. Then by Lemma 1.12 we have that $p^*(V) \cong V' \oplus \gamma_{k+1}$ where γ_{k+1} is a line subbundle of V. Define

$$P_{k+1} : B_{k+1} \longrightarrow B, \quad P_{k+1} \equiv P_k \circ p.$$

Then $P_{k+1}^* E = V' \bigoplus \oplus_{i=1}^{k+1} \gamma_i$. Also $p^* : H^*(B_k) \longrightarrow H^*(B_{k+1})$ is injective by Lemma 1.10. Hence $P_{k+1}^* : H^*(B) \longrightarrow H^*(B_{k+1})$ is injective. Therefore we are done by induction.

□

Definition 1.14. A polynomial $p(a_1, \cdots, a_n) \in (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n]$ is **symmetric** if $p(a_1, \cdots, a_n) = p(a_{\sigma(1)}, \cdots, a_{\sigma(n)})$ for any permutation σ of $\{1, \cdots, n\}$.

The n**th symmetric function** $\sigma_i \in (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n]$ is the polynomial

$$\sum_{0 \leq j_1 < j_2 < \cdots < j_i \leq n} \prod_{k=1}^{i} a_{j_k}.$$

We have the following lemma (which we won't prove):

Lemma 1.15. The subring $R^\sigma \subset R \equiv (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n]$ of symmetric polynomials is freely generated by elementary symmetric functions $\sigma_1, \cdots, \sigma_n$. Hence

$$R \cong (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_n] \subset (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n].$$

Theorem 1.16. Let

$$h_n : (\mathbb{RP}^\infty)^n \longrightarrow Gr_n(\mathbb{R}^\infty)$$

be the classifying map for the rank n bundle $\oplus_{i=1}^n \gamma_1^\infty$. Then

$$h_n^* : H^*(Gr_n(\mathbb{R}^\infty)) \longrightarrow H^*((\mathbb{RP}^\infty)^n) \cong (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n]$$

is injective and its image is the free algebra generated by the elementary symmetric functions $\sigma_1, \cdots, \sigma_n$.

Hence

$$H^*(Gr_n(\mathbb{R}^\infty)) \cong (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_n]$$

for natural classes

$$\sigma_1 \in H^1(Gr_n(\mathbb{R}^\infty)), \cdots, \sigma_n \in H^n(Gr_n(\mathbb{R}^\infty)).$$

Proof. First of all the natural map $h_n^* : H^*(Gr_n(\mathbb{R}^\infty)) \longrightarrow H^*((\mathbb{RP}^\infty)^n)$ is injective for the following reason:

Let $f : B \longrightarrow Gr_n(\mathbb{R}^\infty)$ be the splitting map. Let $g : B \longrightarrow (\mathbb{RP}^\infty)^n$ be the corresponding classifying map for $f^*\gamma_n^\infty$. Then since $(g \circ h_n)^*\gamma_n^\infty \cong f^*\gamma_n^\infty$ and since $Gr_n(\mathbb{R}^\infty)$ is a classifying space, we can homotope f so that $f = g \circ h_n$. Since $f^* : H^*(Gr_n(\mathbb{R}^\infty)) \longrightarrow H^*(B)$ is injective, we get that $h_n^* : H^*(Gr_n(\mathbb{R}^\infty)) \longrightarrow H^*((\mathbb{RP}^\infty)^n)$ is injective.

The image of the map must be contained inside $(\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_n]$ since permuting linear bundles does not change the isomorphism type of their direct sum decomposition. This means that if we compose h_n with a map permuting the factors inside $(\mathbb{RP})^n$, we get a map which is homotopic to h_n.

Hence it is sufficient for us to show that σ_i is in the image of h_n^* for all i. This is done in the following way: We have that $h_n^*(e(\gamma_n^\infty)) = e(\oplus_{i=1}^n \gamma_1^\infty) = \prod_{i=1}^n a_i = \sigma_n$. Hence $\sigma_n \in \mathrm{Im}(h_n^*)$.

We have: $H^*((\mathbb{RP}^{n-1})^{n-1}) = (\mathbb{Z}/2\mathbb{Z})[a'_1, \cdots, a'_{n-1}]$. Let $\sigma'_k \in H^*((\mathbb{RP}^{n-1})^{n-1})$ be the kth symmetric function in $a'_1, \cdots, a'_{n-1}$.

Now suppose (by induction) that the image of

$$h_{n-1} : H^*(Gr_{n-1}(\mathbb{R}^\infty)) \longrightarrow H^*(\mathbb{RP}^\infty)^{n-1}$$

contains σ'_k for all k.

Consider the commutative diagram:

$$\begin{array}{ccc} H^*((\mathbb{RP}^\infty)^{n-1}) & \xleftarrow{h^*_{n-1}} & H^*(Gr_{n-1}(\mathbb{R}^\infty)) \\ \uparrow{\scriptstyle \iota^*_{n-1}} & & \uparrow \\ H^*((\mathbb{RP}^\infty)^n) & \xleftarrow{h^*_n} & H^*(Gr_n(\mathbb{R}^\infty))) \end{array}$$

Consider the restricted map:

$$A' \equiv \iota^*_{n-1}|_{(\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_{n-1}]} : (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_{n-1}] \longrightarrow (\mathbb{Z}/2\mathbb{Z})[\sigma'_1, \cdots, \sigma'_{n-1}].$$

This is an isomorphism since $\iota^*_{n-1}(a_n) = 0$. Since $\sigma'_k \in \mathrm{Im}(h^*_{n-1})$ we then get that $\sigma_k \in \mathrm{Im}(h^*_n)$ by looking at the above commutative diagram and the fact that A' is an isomorphism. Hence by induction we have that $\mathrm{Im}(h^*_n) = (\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_{n-1}, \sigma_n]$.

□