Classifying Spaces

To make our lives much easier, all topological spaces from now on will be homeomorphic to CW complexes.

Fact: All smooth manifolds are homeomorphic to CW complexes.

Kirby and Siebenmann show all topological manifolds in dimension $\neq 4$ are homeomorphic to CW complexes.

Definition 1.1. A **topological group** is a group G whose group operations

$$\mu : G \times G \to G,\ (g_1, g_2) \to g_1 g_2 \quad \text{and} \quad G \to G,\ g \to g^{-1}$$

are continuous. Let X be a topological space. If G is a group, we define G^{op} to be the opposite group with multiplication $\mu^{op}(g,h) = \mu(h,g), \quad \forall g,h \in G$. A **left G-space** is a group homomorphism $\mu : G \to Homeo(X)^{op}$ so that the map:

$$\widetilde{\mu} : G \times X \to X, \quad \widetilde{\mu}(g,x) = \mu(g)(x)$$

is continuous. Usually if we have a G-space on X, we will write $x.g$ instead of $\widetilde{\mu}(g,x)$. Also we will just call a left G-space a **G-space or continuous G-action.** Also the map $\widetilde{\mu} : G \times X \to X$ is called a **continuous G-action**.

A morphism between two G-spaces μ, μ' of G on X and X' respectively is a continuous map $f : X \to X'$ so that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mu} : G \times X & \longrightarrow & X \\ \Big\downarrow (id_G, f) & & \Big\downarrow f \\ \widetilde{\mu'} : G \times X' & \longrightarrow & X' \end{array}$$

Such a morphism is an **isomorphism** if f is a homeomorphism. (Exercise: show that an isomorphism has an inverse morphism).

A **G-subspace** $U \subset X$ of a G-space X is a subset U satisfying $g.U = U$ for all $g \in G$. This is a G-space so that the inclusion map $U \hookrightarrow X$ is a morphism of G-spaces.

The **principal homogeneous space** or **G-torsor** is a G-space

$$G \subset Homeo(G), \quad g \to (h \to gh).$$

If X is a G-space then the **stabilizer of** $x \in X$ (denoted by G_x) is the set of $g \in G$ so that $g.x = x$. This is a subgroup of G. A G-action on X is **free** if $G_x = id$ for all $x \in X$. In other words, every non-trivial g in G sends each $x \in X$ to a different point.

A G-action is **transitive** if for every $x, y \in X$, there is a $g \in G$ so that $g.x = y$.

Exercise: show that a G-action on X is transitive and free if and only if it is a G-torsor.

Here are some examples of topological groups.

Example 1.2. All matrix groups are topological groups such as $GL(\mathbb{R}^k), SO(k), SU(k), GL(\mathbb{C}^k)$.

We also have infinite dimensional groups. Let X be a hausdorff topological space. Then the group $\mathrm{Homeo}(X)$ has a natural topology called the **compact open topology**. This is the topology with basis given by sets

$$U(V, K) \subset \mathrm{Homeo}(K)$$

where $V \subset G$ is open and $K \subset G$ is compact and $U(V, K)$ is the set of homeomorphisms mapping V inside K.

Here we give a definition of a fiber bundle with structure group given by a **topological group** G:

Definition 1.3. Let F be a G-space.

A **fiber bundle with structure group** G **and fiber** F is a map $\pi : V \longrightarrow B$ between topological spaces, an open cover $(U_i)_{i \in S}$ of B, and homeomorphisms $\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times F$ called **trivializations** satisfying the following properties:

(1) Let $\pi_{U_i} : U_i \times F \longrightarrow U_i$ be the natural projection. Then $\pi|_{\pi^{-1}(U_i)} = \pi_i \circ \tau_i$. In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\tau_i} & U_i \times F \\ & {}_{\pi|_{\pi^{-1}(U_i)}} \searrow \quad \swarrow {}_{\pi_{U_i}} & \\ & U_i & \end{array}$$

(2) The **transition maps**

$$\tau_i \circ \tau_j^{-1} : (U_i \times U_j) \times F \longrightarrow (U_i \times U_j) \times F$$

are smooth maps satisfying:

$$\tau_i \circ \tau_j^{-1}(x, z) = (x, \Phi_{ij}(x).z)$$

where

$$\Phi_{ij} : U_i \cap U_j \longrightarrow G$$

is a *continuous* map.

We will call the maps $\Phi_{ij} : U_i \cap U_j \longrightarrow G$ **transition data**. Here the **structure group** should be thought of as the transition data $\Phi_{ij}, i, j \in S$ along with the G-action on F.

Note that the maps Φ_{ij} are part of the data *defining* such a bundle. Also note that each fiber of E is a G-space. Also E itself is a G-space where the action preserves each fiber and $x \in \pi^{-1}(\pi(x))$ gets sent to $g.x \in \pi^{-1}(\pi(x))$ for all $x \in E$.

Main Problem: Classify fiber bundles with structure group G and fiber F up to isomorphism.

A **fiber bundle with fiber** F is just a fiber bundle with structure group $Homeo(F)$.

Suppose we have another fiber bundle $p' : E' \to B'$ with structure group G and fiber F' then a **Morphism covering a continuous map** $f : B' \to B$ is a continuous map $\tau : E' \to E$ satisfying $f \circ \pi' = \pi \circ \tau'$ which is also a homomorphism of G-spaces E and E'.

Here is a silly example: For any topological group G, we have the trivial G-action then we have a fiber bundle with structure group G given by $B \times F$. So a trivial fiber bundle can have structure group G for any topological group G.

Definition 1.4. Let F be a G-torsor. A **principal G-bundle** is a fiber bundle $\pi : E \to B$ with structure group G and fiber F.

Because F is homeomorphic to G, we quite often assume that the fiber of π is G.

Definition 1.5. A G-space is X **locally trivial** if for every $x \in X$, there is an open G-subspace $U \subset X$ containing $x \in X$ so that U is isomorphic to the G-space $B \times G$ for some B where $g \in G$ sends $(b,h) \in B \times G$ to (b, hg) for all $(b,h) \in B \times G$.

Here is an alternative way of describing principal bundles when the structure group is G (which can be very useful).

Lemma 1.6. Let X be a locally trivial G-space. Then $X \longrightarrow X/G$ is a principal G-bundle.

Also for every principal G-bundle $\pi : E \longrightarrow B$, there is a locally trivial G-space X so that E is isomorphic to the principal bundle $X \longrightarrow X/G$.

Exercise. (Hint: the action on E preserves each fiber and the restriction of this action to each fiber is the corresponding G-torsor.)

We now give an important example of a principal G-bundle.

Example 1.7. Let $p : V \to B$ be a vector bundle of rank k. Let $\pi : Fr(V) \to B$ be the fiber bundle whose fiber at $b \in B$ is the set of bases of $\pi^{-1}(B)$. This is a principal $GL(\mathbb{R}^k)$ bundle called the **frame bundle of V.**

We will now generalize this notion of frame bundle to every fiber bundle with structure group G.

Suppose that we have a principal G-bundle $\pi : E \to B$ and suppose that we have a G-space X. Then we can form a new fiber bundle with fiber X as follows: Let $\tau_i : E|_{U_i} \to U_i \times F,\ i \in S$ be the trivializations defining E and let

$$\Phi_{ij} : U_i \cap U_j \to G$$

be the transition data for this principal bundle. Then we can define a new bundle $E \times_G X$ with fiber X exactly the same transition data. In other words, the transition maps are:

$$\tau_h^X \circ (\tau_i^X)^{-1} : U_i \cap U_j \times X \longrightarrow U_i \cap U_j \times X, \quad \tau_h^X \circ (\tau_i^X)^{-1}(b,x) = (b, \Phi_{ij}(x).x).$$

This is called the **bundle associated to the G-action μ.**

Another way of describing this is as follows. We have a diagonal G-action on $E \times X$. We define $E \times_G X \equiv (E \times X)/G$. Then we have another G action on $E \times_G X$ sending (e,x) to $(g.e, x)$ and this is the locally trivial G-space corresponding to $E \times_G X$.

Conversely, suppose that we have a a G-space F and a fiber bundle $p : V \to B$ with structure group G and fiber F. Then we can construct a principal G-bundle called the **Frame bundle** as follows: Let $\Phi_{ij} : U_i \cap U_j \to G$ be the transition data for the fiber bundle p. Then the **frame bundle** of $p : V \to B$ is the fiber bundle with fiber G defined using the exactly the same transition data.

Exercise: show that this definition of the frame bundle is identical to the definition of frame bundle in the above example. Note that here a vector bundle is a fiber bundle with structure group $GL(\mathbb{R}^k)$.

Lemma 1.8. Let F be a G-space. The above discussion gives us a 1-1 correspondence between principal G bundles over B and fiber bundles with structure group G and fiber F. This correspondence sends a principal bundle to the bundle with associated G-action $G \subset Homeo(X)$ and it inverse map sends a fiber bundle with structure group $G \to Homeo(X)$ to its associated frame bundle.

Proof: Exercise.

Everything can be done in the smooth category as well.

As a result of this discussion, classifying fiber bundles up to isomorphism is the same as classifying the corresponding frame bundles up to isomorphism.

Definition 1.9. A principal bundle is **Universal** if its total space is contractible. We usually write $EG \to BG$ for such a bundle. Here BG is called the **Classifying space of** G. The group $H^*(BG)$ is called the **Group Cohomology of** G.

Theorem 1.10. (Milnor) Every topological group G has a universal principal bundle.

Proof. idea: We will prove this is the case that G is a CW complex and the multiplication map and inverse map are cellular maps (i.e. send the k-skeleton to the k-skeleton for each $k \in \mathbb{N}_{\geq 0}$) and also when it is defined only using a *countable number of cells*. Examples of such groups include $SO(n), GL(n;\mathbb{R}), GL(n;\mathbb{C}), U(n), O(n)$. We can also consider limits of such groups, such as $\lim_n GL(n;\mathbb{R})$.

Whiteheads theorem says that a continuous map of CW complexes inducing an isomorphism in π_k for all k is a homotopy equivalence. The Hurewicz theorem tells us that for all $k \geq 1$, $H_k(X) = 0$ if $\pi_m(X) = 0$ for all $m \leq k$. Combining the Hurewicz theorem with Whiteheads theorem, we have that if a continuous map of CW complexes is an isomorphism on π_1 and also induces an isomorphism on homology, then it is a homotopy equivalence.

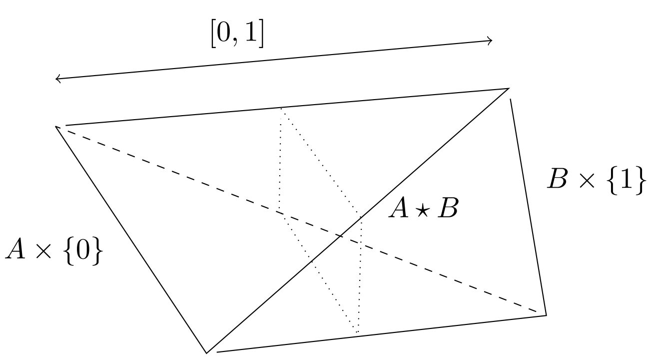
This means that a simply connected CW complex whose homology groups all vanish then it is contractible. Therefore it is sufficient to construct a ∞-universal bundle as ∞-bundles are equivalent to universal bundles. In fact we will construct n-universal bundles E_n along with morphisms $E_k \to E_{k+1} \to \cdots$. Then $EG \to BG$ will be the direct limit of these bundles.

To do this we need the following definition:

Definition 1.11. Let A, B be two topological spaces. Then the **join** $A \star B$ is defined as:

$$A \star B \equiv A \times B \times [0,1]/\sim$$

$$(a,b,0) \sim (a,b',0) \text{ and } (a,b,1) \sim (a,b',1) \quad \forall\, a,a' \in A \quad \forall\, b,b' \in B.$$



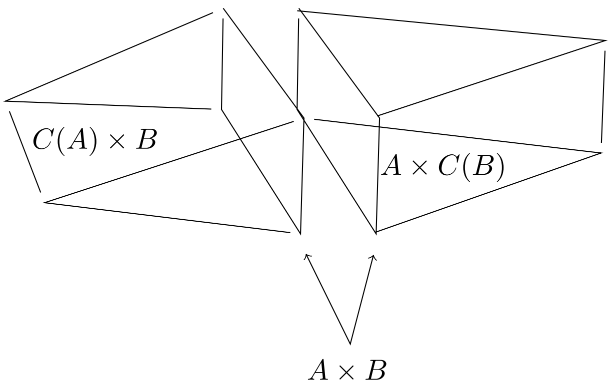
The **cone of** A $C(A)$ is the set

$$C(A) \equiv A \times [0,1]/\sim, \quad (a,1) \sim (a',1) \quad \forall\, a,a' \in A.$$

Note that the cone is contractible.

We have the following homeomorphism:

$$A \star B \cong (C(A) \times B) \cup_{A \times B} (A \times C(B)). \tag{1}$$



If A is a CW complex then $C(A)$ is also a CW complex. The point is that the cone of each n-cell is an n-cell with an $n+1$ cell attached along the n-cell. One then creates $C(A)$ by gluing these pairs of cells together. Also the product of two CW complexes is a CW complex. This means that the join $A \times B$ is also a CW complex by (1).

Lemma 1.12. $A \star A$ is simply connected if A is path connected.

Proof. Equation (1) tells us $A \star A \cong (C(A) \times A) \cup_{A \times A} (A \times C(A))$. We use van Kampens theorem.

So

$$\pi_1(A \star A) = \pi_1(C(A) \times A) \star_{\pi_1(A \times A)} \pi_1(A \times C(A)).$$

This is trivial since the maps

$$\alpha : \pi_1(A \times A) \to \pi_1(C(A) \times A) = \pi_1(A),$$

$$\beta : \pi_1(A \times A) \to \pi_1(A \times C(A)) = \pi_1(A)$$

are the natural projection maps to the first and second factors respectively and since the map $\alpha \times \beta : \pi_1(A \times A) \to \pi_1(A) \times \pi_1(B)$ is an isomorphism.

□

Lemma 1.13. If $H_*(A) = 0$ for $* \leq k$ then $H_*(A \star A) = 0$ for $* \leq k+1$.

Proof. Equation (1) tells us $A \star A \cong (C(A) \times A) \cup_{A \times A} (A \times C(A))$. We now have the Mayor-Vietoris sequence:

$$H_{k+1}(A \times A) \xrightarrow{a} H_{k+1}(C(A) \times A) \oplus H_{k+1}(A \times C(A))$$

$$\to H_{k+1}(A \star A) \to H_k(A \times A).$$

We have that $H_k(A \times A) = 0$ by the Künneth formula and also the map a is an isomorphism. Hence $H_{k+1}(A \star A) = 0$.

□

Lemma 1.14. Suppose that B is a G-space. Then $B \times G$ with the product G action is isomorphic to $B \times G$ where G acts only on the second fact by left multiplication.

Proof. The isomorphism is the map $(b, g) \to (g^{-1}.b, g)$. □

Lemma 1.15. Let X_1, X_2 be locally trivial G-spaces. Then $X_1 \star X_2$ is a locally trivial G-space. Here G acts on $X_1 \star X_2$ by $g.(x_1, x_2, t) = (g.x_1, g.x_2, t)$.

Proof. It is locally trivial for the following reason: Let $U_1 \subset X_1$ and $U_2 \subset X_2$ be G invariant open subsets isomorphic to $B_1 \times G$ and $B_2 \times G$ where G acts trivially on B_i and by left multiplication on G. Then

$$C(U_1) \times U_2 \subset X_1 \star X_2, \quad U_1 \times C(U_2) \subset X_1 \star X_2$$

are isomorphic to product trivializations by Lemma 1.14. □

We will now continue with the proof of Theorem 1.10. We let $E_n = G \star G .. \star G$ multiplied n times. This is a locally trivial G-space by Lemma 1.15. Also $\pi_1(E_n) = 0$ for $n \geq 2$ and $H_k(E_n) = 0$ for all $0 < k < n$ by Lemmas 1.12 and 1.13. Let $EG \equiv \lim_{n\to\infty} E_n$. This is a contractible locally trivial G-space since it is simply connected and all of its reduced homology groups are 0.

□

Definition 1.16. A principal G-bundle is **trivial** if it isomorphic to a product $B \times G$.

Lemma 1.17. A principal bundle $\pi : E \to B$ is trivial if and only if it admits a section.

Proof. If it is trivial then it has the section given by id.

Conversely suppose E has a section σ. For each $x \in E$ there is a unique $g_x \in G$ so that $x = g.\sigma(\pi(x))$. Hence we have a bundle isomorphism

$$\phi : E \to B \times G, \quad \phi(x) = (\pi(x), g_x).$$

Exercise: show that the above map is a bundle isomorphism. □

Definition 1.18. Let $\pi : E \to B$ be a fiber bundle and let $f : A \to B$ be a continuous map. A **lift** if f is a continuous map $\widetilde{f} : A \to E$ satisfying $\pi \circ \widetilde{f} = f$.

A continuous map $\pi : E \to B$ satisfies the **homotopy lifting property** if for any map $F : [0,1] \times B' \to B$ and a lift $\widetilde{F_0} : B' \to E$ of $F|_{\{0\}\times B'}$, there is a lift $\widetilde{F} : [0,1] \times B' \to B$ of F satisfying $\widetilde{F}|_{\{0\}\times B'} = \widetilde{F_0}$.

Lemma 1.19. Every fiber bundle satisfies the homotopy lifting property (assuming the base space and total space are CW complexes).

Proof. Let $\widetilde{F_0}$,B, B' and F be as in the above definition. It is sufficient for us to replace F^*E with E. Hence we can assume that F is the identity map and that $\widetilde{F_0}$ is a section of $E|_{\{0\}\times B}$. To finish our lemma it is sufficient to extend this section to all of $[0,1] \times B$.

We do this cell by cell on B. We can ensure that this cell decomposition is fine enough (by subdividing) so that the restriction of E to each cell is isomorphic to a trivial bundle.

Therefore it is sufficient to prove the following: Suppose that B is the n-ball and that we have a section σ of $E|_N$ where $N = ([0,1] \times \partial B) \cup (\{0\} \times B)$. Then we wish to construct a new section σ' of E satisfying $\sigma'|_N = \sigma|_N$.

Since E is trivial, the section σ corresponds to a continuous map $\sigma : N \to F$ and any section σ' as above corresponds to a map $\sigma' : B \to F$ satisfying $\sigma'|_N = \sigma_N$. We have that $[0,1] \times B$ deformation retracts onto N. Let $r : [0,1] \times B \to N$ be the corresponding retraction (I.e. $r \circ r = r$). Then our section is the map $\sigma' \equiv \sigma \circ r$. □

Corollary 1.20. If $\pi : E \to B$ is a fiber bundle with contractible base (again E, B are CW complexes). Then it is isomorphic to a trivial bundle.

Proof. Let $\pi_{Fr} : Fr(E) \to B$ be the corresponding frame bundle. Then E is trivial if and only if $Fr(E)$ admits a section.

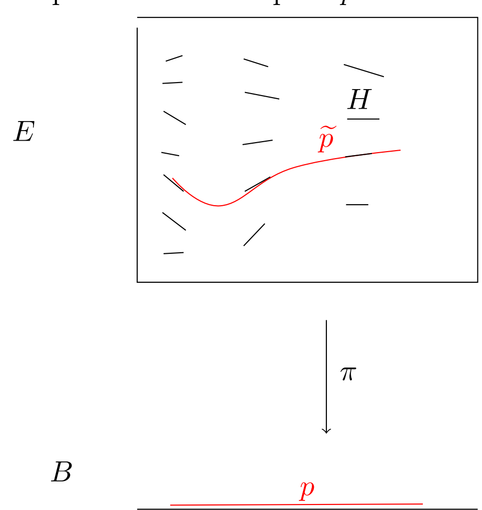
Let $C : [0,1] \times B \to B$ be our deformation retraction to a point $\star \in B$. Let $\widetilde{\star} \in C^*Fr(E)|_{\{(1,\star)\}}$. Let $\widetilde{C}_0 : [0,1] \times B \to C^*Fr(E)$ send (t,b) to $\widetilde{\star}$. Then since $Fr(E)$ satisfies the homotopy lifting property, we get a map $\widetilde{C} : [0,1] \times B \to C^*Fr(E)$ satisfying $(C^*\pi_{Fr}) \circ \widetilde{C} = id$. Hence $\widetilde{C}|_{\{0\}\times B}$ is a section of $C^*Fr(E)|_{\{0\}\times B}$ which is isomorphic to $Fr(E)$ due to the fact that $C|_{\{0\}\times B}$ is the identity map. □

The homotopy lifting property is related to Ehresmann connections in the following way:

Definition 1.21. Let $\pi : E \to B$ be a smooth fiber bundle. An **Ehresmann connection** is a subbundle $H \subset TE$ of the tangent bundle of E so that $d\pi|_{H_x} : H_x \to T_{\pi(x)}B$ is an isomorphism for all $x \in E$.

A path $p : [0,1] \to E$ is **horizontal** if $\dot{p}(t) \in H_{p(t)}$ for all $t \in [0,1]$.

By uniqueness of ODE's, for every smooth path $p : [0,1] \to B$ and point $e \in \pi^{-1}(p(0))$, there is at most one horizontal path $\widetilde{p} : [0,1] \to E$ satisfying $\pi(\widetilde{p}(t)) = p(t)$ and $\widetilde{p}(0) = e$. If F is compact then such a path $\widetilde{p}$ exists as well.



Also if F is compact then for any smooth path $p : [0,1] \to B$ we have an associated **parallel transport map** $P_{H,p} : E|_{p(0)} \to E|_{p(1)}$ sending $e \in E|_{p(0)}$ to $\widetilde{p}_e(1)$ where $\widetilde{p}_e$ is the unique horizontal lift of p satisfying $\widetilde{p}_e(0) = e$.

Lemma 1.22. Every smooth fiber bundle has an Ehresmann connection.

Proof. Let $T^vE \to E$ be the **vertical tangent bundle of** E corresponding to vectors which are tangent to the fibers of E. Ehresmann connections correspond to non-zero sections s of the bundle $Hom(TE, T^vE)$ with the property that $s|_{T^vE} : T^vE \to T^vE$ is the identity map because $H_x = ker(s(x))$ is horizontal.

Now since E is locally trivial, we have such sections s_i of $E|_{U_i}$ for some open cover $(U_i)_{i\in S}$. Choose a partition of unity $(\rho_i)_{i\in S}$ subordinate to this cover. Then $\sum_{i\in S} \lambda_i.s_i$ is the section we want. □

Now lets look at the homotopy lifting property in the above context. We have a smooth fiber bundle $\pi : E \to B$ with compact fibers and an Ehresmann connection H. Let $F : B' \times [0,1] \to B$ be a smooth map and let $\widetilde{F}_0 : B' \to B$ be a lift of $F|_{B'\times\{0\}}$. Let

$$P_{b',t} : \pi^{-1}(F(b',0)) \to \pi^{-1}(F(b',t))$$

be the parallel transport map along the path

$$F_{b'} : [0,t] \to B,\ F_{b'}(s) = F(b', s)$$

for all $b' \in B'$. Then

$$\widetilde{F} : B' \times [0,1] \to E, \quad \widetilde{F}(b', t) = P_{b',t}(\widetilde{F}_0(b'))$$

is a lift of F which is equal to $\widetilde{F}_0$ along $B' \times \{0\}$.

Definition 1.23. A **family of principal bundles** is a principle bundle $\widetilde{E}$ over $[0,1] \times B$. Here we define $\widetilde{E}_t$ to be its restriction to $\{t\} \times B = B$.

A family of principle bundles as above **joins** $\pi : E \to B$ **with** $\pi' : E' \to B$ if $\widetilde{E}_0$ is isomorphic to E and $\widetilde{E}_1$ is isomorphic to E'.

Definition 1.24. Let $\pi : E \to B$ and $\pi' : E' \to B$ be principal G bundles. Recall that E and E' are locally trivial G-spaces.

We define $E \times_G E'$ to be the quotient $(E \times E')/G$ where G acts diagonally (i.e. $g.(e, e') = (g.e, g.e')$). This has a natural map $p : E \times_G E' \to B$ sending (e, e') to $\pi(e) = \pi'(e')$.

Lemma 1.25. $p : E \times_G E' \to B$ is a principal G bundle. Also it admits a section if and only if E is isomorphic to E'

Proof. If we have trivializations $\tau_i : E|_{U_i} \to U_i \times G$ and $\tau'_i : E'|_{U_i} \to U_i \times G$ then we get a trivialization

$$E \times_G E'|_{E_i} \to G \times G/G \cong G, \quad x \to (\tau_i(x), \tau'_i(x)).$$

This makes $E \times_G E'$ into a principal bundle.

If we have an isomorphism $\phi : E \to E'$ then we have a section $\sigma : B \to E \times_G E'$ sending b to $(x, \phi(x))$ for any choice of $x \in E|_b$ (it does not matter what x we choose since $(g.x, \phi(g.x)) = (g.x, g\phi(x)) = (x, \phi(x))$).

Conversely if $E \times_G E'$ has a section σ, then we have an isomorphism $E \to E'$ sending $x \in E$ to $\phi(x) \in E'$ where $\phi(x)$ satisfies $(x, \phi(x)) = \sigma(x)$. □

Lemma 1.26. If a family of principle bundles joins $\pi : E \to B$ with $\pi' : E' \to B$ then E is isomorphic to E'.

Proof. Here we use the homotopy lifting property. Let $\widetilde{E}$ be our family of principal bundles. Let pr^*E be the pullback of E to $[0,1] \times B$ via the natural projection map $pr : [0,1] \times B \to B$. It is sufficient to construct an isomorphism $\widetilde{E} \to pr^*E$. Hence it is sufficient to construct a section of $A \equiv \widetilde{E} \times_G pr^*E$.

Since $\widetilde{E}|_{\{0\} \times B}$ is isomorphic to $pr^*E|_{\{0\} \times B}$, we have a section of $A|_{\{0\} \times B}$. By the homotopy lifting property this extends to a section of A. □

Proposition 1.27. Let $\pi_G : EG \to BG$ be any universal G-bundle. For any principal G bundle $\pi : E \to B$, there is a continuous map $f : B \to BG$ so that E is isomorphic to f^*EG.

Also f_1^*EG is isomorphic to f_2^*EG if and only if f_1 is homotopic to f_2.

Again, here we are assuming that all topological spaces in the above proposition are CW complexes to make our lives easier.

Proof. In some sense this proof is similar in spirit to Grassmannian case. Having said that instead of dealing with an arbitrary set of trivializations, we will use trivializations over each cell of the base B.

We do this cell by cell on B. We subdivide our cell decomposition so that the restriction of E to each cell is trivial.

Let B^i be the i-skeleton of B. First of all the bundle E is trivial over B^0 and hence we have a constant map $f^0 : B^0 \to BG$.

Now suppose that we have constructed $f^i : B^i \to BG$ so that $E|_{B_i}$ is isomorphic to $(f^i)^*EG$. We wish to extend f^i to a new map f' over an $i+1$-cell $D \subset B$ which is attached to B^i.

By Lemma 1.25, the isomorphism $E|_{\partial D} \cong (f^i)^*EG|_{\partial D}$ corresponds to a section σ of

$$E \times_G (f^i)^*EG|_{\partial D}.$$

Also since $E|_D$ is trivial, we get by Corollary 1.20 isomorphisms:

$$E \times_G (f^i)^*EG|_{\partial D} \cong (D \times G) \times_G (f^i)^*EG|_{\partial D} \cong (f^i)^*EG|_{\partial D}. \tag{2}$$

Under these isomorphisms, the section σ corresponds to a map $\sigma' : \partial D \to EG$ satisfying $\pi_{EG} \circ s' = f$. Since EG is contractible, the map σ' extends to a map $\sigma'' : D \to EG$.

We define

$$f' : D \to BG, \quad f' \equiv \pi_{EG} \circ s''.$$

This is equal to f^i along ∂D. Also under the identification (2), σ'' corresponds to a section $\widetilde{\sigma}$ of $E \times_G (f^i)^*EG|_{\partial D}$ extending the section σ. Hence the isomorphism $E|_{\partial D} \cong (f^i)^*EG|_{\partial D}$ extends to an isomorphism $E|_D \cong (f')^*EG|_D$.

Hence by induction we have constructed a map $f : B \to BG$ so that $E \cong f^*EG$.

We now need to prove uniqueness: Suppose we have two maps $f_1, f_2 : B \to BG$ so that $f_1^*EG \cong f_2^*EG$. We wish to show that f_1 is homotopic to f_2. Again we do this cell by cell. Suppose by induction, we have homotoped f_1 and f_2 so that $f_1|_{B^i} = f_2|_{B^i}$. We now wish to homotope $f_1|_D$ to $f_2|_D$ relative to ∂D over an $i+1$-cell $D \subset B$ which is attached to B^i.

Since f_1^*EG and f_2^*EG are trivial over D by Corollary 1.20 as D is contractible, we get maps

$$\sigma_1 : D \to EG, \quad \sigma_2 : D \to EG$$

satisfying $\pi_{EG} \circ \sigma_i = f_i$ for $i = 1, 2$ and $\sigma_1|_{\partial D} = \sigma_2|_{\partial D}$.

Since EG is contractible and $\sigma_1|_{\partial D} = \sigma_2|_{\partial D}$, we have that σ_1 is homotopic to σ_2 relative to ∂D. Hence $f_1|_D = \pi_{EG} \circ \sigma_1$ is homotopic to $f_2|_D = \pi_{EG} \circ \sigma_2$ relative to ∂D. Hence we are done by induction. □

Theorem 1.28. Let $p : E \longrightarrow B$ be a fibration with fiber F. (More generally a continuous map satisfying the homotopy lifting property). Then we have a long exact sequence of homotopy groups:

$$\pi_k(F) \longrightarrow \pi_k(E) \longrightarrow \pi_k(B) \longrightarrow \pi_{k-1}(F) \longrightarrow$$

We won't prove this theorem. The connecting map

$$\pi_k(B) \longrightarrow \pi_{k-1}(F)$$

is constructed as follows: Let $\star_B \in B$ and $\star_F \in F \subset E$ be the basepoints of B and F respectively so that $\pi(\star_F) = \star_B$. Let $b : S^k \longrightarrow B$ be an element of $\pi_k(B)$. We have a homeomorphism

$$S^{k-1} \times [0,1]/\sim \cong S^k$$

where $(x, i) \sim (x', i)$ for all $x, x' \in S^{k-1}$ and $i = 0, 1$. Hence we have natural surjection

$$\sigma : S^{k-1} \times [0,1] \twoheadrightarrow S^k$$

so that $S^{k-1} \times \{i\}$ maps to a point $\star_i$ for $i = 0, 1$. We will assume that $\star_0 = \star_B$. Now let $p : [0,1] \longrightarrow S^k$ be a path joining $\sigma(S^{k-1} \times \{1\})$ with $\sigma(S^{k-1} \times \{0\})$. Hence we have map

$$b \circ \sigma : S^{k-1} \times [0,1] \longrightarrow B.$$

We also have a lift

$$\widetilde{b}_0 : S^{k-1} \times \{0\} \longrightarrow E, \quad \widetilde{b}_0(x) = \star_F.$$

By the homotopy lifting property this means we have a lift $\widetilde{b} : S^{k-1} \times [0,1] \longrightarrow E$ of b. Hence we have an element

$$\widetilde{b}|_{S^{k-1} \times \{1\}} : S^{k-1} \longrightarrow \pi^{-1}(\star_1) \cong \pi^{-1}(\star_0) \cong F$$

of $\pi_{k-1}(F)$.

Corollary 1.29. Let G be a topological group and $\pi : EG \longrightarrow BG$ be a universal principal G bundle. Then $\pi_k(EG) = \pi_{k-1}(G)$.

The above corollary combined enables us to classify principal G bundles on spheres (assuming that we know the homotopy groups of G). In particular, if F is a fixed G space, then there is a 1-1 correspondence between fiber bundles over S^m with structure group G and fiber F and elements of $\pi_{m-1}(G)$.

Such fiber bundles can also be constructed explicitly using the **clutching construction**. Let $b : S^{m-1} \longrightarrow G$ represent an element of $\pi_{m-1}(G)$. Since S^m is given by two balls glued together along their boundary, we have open subsets $U_1, U_2 \subset S^m$ diffeomorphic to open m-balls so that $U_1 \cap U_2$ deformation retracts on to $S^{m-1} \subset S^m$. Hence we have a map $\Phi_{12} : U_1 \cap U_2 \to G$ whose restriction to S^{m-1} is b. This gives the transition data for the fiber bundle representing $[b] \in \pi_{m-1}(G)$.

To show how practical this is, let us classify rank 3 vector bundles over S^3. In this case the structure group is $GL(3)$ which is homotopic to $O(3)$. We can calculate $\pi_2(O(3))$ as follows:

The connected component of $O(3)$ containing id is $SO(3)$ so it is sufficient to compute $\pi_2(SO(3))$. Each element of $SO(3)$ is a rotation about some axis. This means that we have a fibration $SO(3) - id \to \mathbb{RP}^2$ sending a $\rho \in SO(3)$ to the corresponding unique axis of rotation of ρ. The fiber is $SO(2) - id \cong (0, 1)$. So we have the following homotopy log exact sequence:

$$\pi_k((0,1)) \longrightarrow \pi_k(SO(3) - id) \longrightarrow \pi_k(\mathbb{RP}^2) \longrightarrow \pi_k((0,1))$$

which implies that $\pi_k(SO(3) - id) = \pi_k(\mathbb{RP}^2)$.

Now $\pi_2(\mathbb{RP}^2) = \pi_2(S^2) = \mathbb{Z}$ and hence $\pi_2(SO(3) - id) = \mathbb{Z}$. The generator of $\pi_2(\mathbb{RP}^2)$ is the double covering map $b : S^2 \longrightarrow \mathbb{RP}^2$ and hence the generator of $\pi_2(SO(3) - id)$ is a lift $\widetilde{b} : S^2 \longrightarrow SO(3) - id$ of b to $SO(3) - id$. Here $\widetilde{b}(x)$ is defined as a clockwise rotation of angle $\theta \in (0, 2\pi)$ about the axis x. If we choose θ to be small then this is a small sphere near the point $id \in SO(3)$. This means that $\widetilde{b}(S^2)$ is contained in a small chart of $SO(3)$ and hence is contractible. Hence the map

$$\beta : \pi_2(SO(3) - id) \longrightarrow \pi_2(SO(3))$$

is 0. Also since $SO(3)$ is a three dimensional manifold, the map β is surjective as any map $h : S^2 \longrightarrow SO(3)$ can be perturbed to a map $\widehat{h} : S^2 \longrightarrow SO(3) - id$. Hence $\pi_2(SO(3)) = 0$. Therefore every rank 3 vector bundle on S^3 is trivial.

Grassmannian

To show that the Grassmannian $Gr_n(\mathbb{R}^\infty)$ classifies vector bundles, all we need to do is show that its frame bundle is contractible. The frame bundle EG is the set of linearly independent sets of n vectors $v_1, \cdots, v_n$ in $\mathbb{R}^\infty$. Hence EG is an open subset of $(\mathbb{R}^\infty - 0)^n$. Let $D : \mathbb{R}^\infty - 0 \longrightarrow \mathbb{N}$ send a vector $(x_1, \cdots, x_k, 0, 0, 0, 0, 0, 0, \cdots)$ where $x_k \neq 0$ to k. Define

$$\widehat{D} : (\mathbb{R}^\infty - 0)^n \longrightarrow (\mathbb{R}^\infty - 0)^n, \quad \widehat{D}(v_1, \cdots, v_n) \equiv \max(D(v_1), \cdots, D(v_n)).$$

Let $T : \mathbb{R}^\infty - 0 \longrightarrow \mathbb{R}^\infty - 0$ be the right shift operator sending $(x_1, x_2, \cdots)$ to $(0, x_1, x_2, \cdots)$ and define

$$\widehat{T} : (\mathbb{R}^\infty - 0)^n \longrightarrow (\mathbb{R}^\infty - 0)^n, \quad \widehat{T}(v_1, \cdots, v_n) \equiv (T(v_1), \cdots, T(v_n).$$

Then we get a new operator

$$W : (\mathbb{R}^\infty - 0)^n \longrightarrow (\mathbb{R}^\infty - 0)^n, \quad W(v) \equiv \widehat{T}^{\widehat{D}(v)}(v).$$

In other words, W shifts the n vectors in $(\mathbb{R}^\infty - 0)^n$ so far to the right that they do not interact anymore.

Let $e_k = (0, 0, \cdots, 0, 1, 0, 0, \cdots,)$ be the kth standard basis vector. Let $v_0 \equiv (e_1, \cdots, e_n) \in EG$. We will construct a deformation retraction from EG to the point v_0. To do this we first show that W is homotopy equivalent to the identity map. This homotopy is the map

$$\alpha : [0, 1] \times EG \longrightarrow EG, \quad \alpha(t, v) \equiv (1 - t)v + tW(v).$$

Note that $\alpha(t, v)$ is always a set of n linearly independent vectors since W shifts very far to the right. The following is a homotopy from W to the constant map $EG \longrightarrow \{v_0\}$:

$$\beta : [0, 1] \times EG \longrightarrow EG, \quad \beta(t, v) \equiv tv_0 + (1 - t)W(v).$$

Again this is a well defined map since v_0 and $W(v)$ do not interact in any way. Hence EG is contractible. Therefore the frame bundle of $Gr_n(\mathbb{R}^\infty)$ is a classifying space and hence $BG = Gr_n(\mathbb{R}^\infty)$.