1. Pontryagin Classes

Before we start defining Pontryagin classes, we need a few more lemmas concerning Chern classes.

Definition 1.1. The **complexification** of a real vector bundle V is the complex vector bundle $V \otimes_{\mathbb{R}} \mathbb{C}$.

This bundle has a complex structure J and it sends the real subbundle $V \subset V \otimes_{\mathbb{R}} \mathbb{C}$ to JV. We have that $V \cap JV$ is the zero section and $V + JV = V \otimes_{\mathbb{R}} \mathbb{C}$. Hence we have a canonical isomorphism $V \oplus JV \cong V$. Also the map $J|_V : V \longrightarrow JV$ is a bundle isomorphism, hence we have a canonical isomorphism

$$\Phi : V \oplus V \longrightarrow V \otimes_{\mathbb{R}} \mathbb{C}, \quad (x, y) \longrightarrow x + Jy.$$

In particular the complex structure on $V \oplus V$ sends (x, y) to $(-y, x)$.

Definition 1.2. If $\pi : E \longrightarrow B$ is a complex vector bundle with complex structure J. The **conjugate** $\overline{\pi} : \overline{E} \longrightarrow B$ of E is the complex vector bundle $(E, -J)$. Equivalently if $\Phi_{ij} : U_i \cap U_j \longrightarrow GL(n, \mathbb{C})$ are the transition data, where $GL(n, \mathbb{C})$ acts in the usual way on $\mathbb{C}^n$ then the its conjugate has exactly the same transition data Φ_{ij} but $GL(n, \mathbb{C})$ acts as follows:

$$GL(n, \mathbb{C}) \times \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad (A, z) \longrightarrow \overline{A(\overline{z})}.$$

Lemma 1.3. If V is a real vector bundle then $V \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic as a complex vector bundle to its conjugate.

Proof. Under the canonical ismorphism $V \oplus V \longrightarrow V \otimes_{\mathbb{R}} \mathbb{C}$, our isomorphism sends $(x, y) \in V \oplus V$ to $(x, -y)$. □

Lemma 1.4. If $\pi : E \longrightarrow B$ is a complex vector bundle, we have that $c_k(\overline{E}) = (-1)^k c_k(E)$.

Proof. We will first prove this for γ^1_∞ over $\mathbb{CP}^\infty$. Define

$$\iota : \mathbb{CP}^\infty \longrightarrow \mathbb{CP}^\infty, \quad [z] \longrightarrow [\overline{z}].$$

This is a homeomorphism and $\iota^*\gamma^1_\infty$ is isomorphic to $\overline{\gamma^1_\infty}$ as a complex vector bundle. Hence $c_k(\overline{\gamma^1_\infty}) = \iota^*(c_k(\gamma^1_\infty))$. Now $\iota|_{\mathbb{CP}^1}$ sends $\mathbb{CP}^1$ to itself. It is the reflection map and hence orientation reversing. Therefore $\iota^* u = -u$. Hence $c_1(\overline{\gamma^1_\infty}) = -c_1(\gamma^1_\infty) = -u$.

We will now prove our theorem for the canonical bundle γ^n_∞ of $Gr_n(\mathbb{C}^\infty)$. Let $h_n : (\mathbb{CP}^\infty)^n \longrightarrow Gr_n(\mathbb{C}^\infty)$ be the classifying map of $\oplus_{i=1}^n p_i^* \gamma^1_\infty$ where $p_i : (\mathbb{CP}^\infty)^n \longrightarrow \mathbb{CP}^\infty$ is the ith projection map. Since $h_n^* : H^*(Gr_n(\mathbb{C}^\infty)) \longrightarrow H^*((\mathbb{CP}^\infty)^n)$ is injective and since

$$h_n^*(\overline{\gamma^n_\infty}) = \overline{\oplus_{i=1}^n p_i^* \gamma^1_\infty} = \mathbb{Z}[u_1, \cdots, u_n],$$

it is sufficient for us to prove our lemma for $\oplus_{i=1}^n p_i^* \gamma^1_\infty$. Now

$$c_k(\oplus_{i=1}^n p_i^* \gamma^1_\infty) \sum_{I \subset \{1, \cdots, n\},\ |I| = k} \cup_{j \in I} c_1(p_i^* \gamma^1_\infty) = \sum_{I \subset \{1, \cdots, n,\ |I| = k} \prod_{j \in I} u_j$$

by the Whitney product theorem. Therefore

$$c_k(\overline{\oplus_{i=1}^n p_i^* \gamma^1_\infty}) = \sum_{I \subset \{1, \cdots, n,\ |I| = k} \prod_{j \in I} (-u_j) = (-1)^k c_k(\oplus_{i=1}^n p_i^* \gamma^1_\infty).$$

Finally we prove our lemma in general. Let $f : B \longrightarrow Gr_k(\mathbb{C}^\infty)$ be the classifying map for E. Then $\overline{E} \cong f^*(\overline{\gamma^n_\infty})$

$$c_k(\overline{E}) = f^*(c_k(\overline{\gamma^n_\infty})) = f^*((-1)^k c_k(\gamma^n_\infty) = (-1)^k c_k(E).$$

□

Corollary 1.5. $2c_k(V \otimes_{\mathbb{R}} \mathbb{C}) = 0$ for all odd k.

Proof. Since $V \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic as a complex vector bundle to $\overline{V \otimes_{\mathbb{R}} \mathbb{C}}$, we get that $c_k(V \otimes_{\mathbb{R}} \mathbb{C}) = c_k(\overline{V \otimes_{\mathbb{R}} \mathbb{C}}) = -c_k(V \otimes_{\mathbb{R}} \mathbb{C})$. □

Therefore ignoring these odd Chern classes, we have the following definition:

Definition 1.6. Let $V \longrightarrow B$ be real vector bundle then the i**th Pontryagin class** is

$$p_i(V) \equiv c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(B).$$

The **total Pontryagin class** of V is the class $p(V) \equiv p_0(V) + p_1(V) + \cdots$.

If X is a complex manifold then we define $p_i(X) \equiv p_i(TX)$.

Lemma 1.7. If $f : B' \longrightarrow B$ is a continuous map and $\pi : V \longrightarrow B$ is a real vector bundle then $f^*(p_i(V)) = p_i(f^*(V))$.

Proof. This follows immediately from the naturality property of Chern classes. □

Theorem 1.8. Let $V, V' \longrightarrow B$ be two vector bundles over the same base then $p(V \oplus V')$ is equal to $p(V)p(V')$ mod 2. In other words, $2(p(V \oplus V') - p(V)p(V')) = 0$.

Proof. Since $2c_k(V \otimes_{\mathbb{R}} \mathbb{C}) = 2c_k(V' \otimes_{\mathbb{R}} \mathbb{C}) = 0$ for all odd k,

$$2(p(V \oplus V') - p(V)p(V')) = 2(c(V \otimes_{\mathbb{R}} \mathbb{C} \oplus V' \otimes_{\mathbb{R}} \mathbb{C}) - c(V \otimes_{\mathbb{R}} \mathbb{C})c(V' \otimes_{\mathbb{R}} \mathbb{C})) = 0$$

by the Whitney sum formula. □

Lemma 1.9. (Exercise:) Let $\pi : E \longrightarrow B$ be a complex vector bundle and let $E(\mathbb{R})$ be its underlying real vector bundle. Then $E(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic as a complex vector bundle to $E \oplus \overline{E}$.

Proposition 1.10. For any complex n bundle $\pi : E \longrightarrow B$, the Chern classes of E determine the Pontryagin classes of E by the following formula:

$$1 - p_1(E) + p_2(E) - \cdots + (-1)^n p_n(E) =$$
$$(1 - c_1(E) + c_2(E) - \cdots + (-1)^n c_n(E))(1 + c_1(E) + \cdots + c_n(E).$$

Hence $p_k(E)$ is equal to:

$$c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + \cdots + 2(-1)^{j-k}c_{k-j}(E)c_{k+j}(E) + \cdots +$$
$$2(-1)^k c_{2k-1}(E)c_1(E) + 2(-1)^{k+1}c_{2k}(E).$$

Proof. Since $E(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic as a complex vector bundle to $E \oplus \overline{E}$, our theorem follow from the Whitney sum formula and the fact that $c_k(\overline{E}) = (-1)^k c_k(E)$. □

Let us compute the Pontryagin classes of $\mathbb{CP}^n$. We will use the above proposition to do this. Before we do this we need to compute the Chern classes of $\mathbb{CP}^n$ first using the following Lemma.

Lemma 1.11. We have that $c(T\mathbb{CP}^n) = (1-u)^{n+1}$ where $u \in H^2(\mathbb{CP}^n)$ is Poincaré-dual to $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^n)$.

Proof. Let $\omega \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ be the orthogonal complement of γ_n^1. *Claim*: (Exercise) $T\mathbb{CP}^n \cong Hom_{\mathbb{C}}(\gamma_n^1, \omega)$ (using the same reasoning as with $\mathbb{RP}^n$).

Let $\underline{\mathbb{C}}$ be the trivial $\mathbb{C}$ bundle $\mathbb{CP}^n \times \mathbb{C}$. Since $\underline{\mathbb{C}} \cong Hom_{\mathbb{C}}(\gamma_n^1, \gamma_n^1)$. Then

$$T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong Hom_{\mathbb{C}}(\gamma_n^1, \omega \oplus \gamma_n^1) = Hom_{\mathbb{C}}(\gamma_n^1, \oplus_{j=1}^n \underline{\mathbb{C}}) \cong ((\gamma_n^1)^*)^n.$$

Since $(\gamma_n^1)^*|_{\mathbb{CP}^1} = \mathcal{O}_{\mathbb{CP}^1}(-1)$, we get that $c_1((\gamma_1^1)^*) = -c_1(\gamma_1^1)$ and so $c_1((\gamma_\infty^1)^*) = -c_1(\gamma_\infty^1)$.
Hence $c(T\mathbb{CP}^n) = (1 - c_1(\gamma_\infty^1))^{n+1} = (1-u)^{n+1}$. □

Hence by the above lemma and the above proposition, we have:

$$1 - p_1(\mathbb{CP}^n) + p_2(\mathbb{CP}^n) - \cdots = c(\overline{T\mathbb{CP}^n}) \oplus c(T\mathbb{CP}^n) = (1+u)^{n+1}(1-u)^{n+1} = (1-u^2)^{n+1}.$$

Therefore

$$p_k(\mathbb{CP}^n) = \binom{n+1}{k} u^{2k}.$$

E.g.

$$p(\mathbb{CP}^5) = 1 + 6u^2 + 15u^4.$$

Lemma 1.12. Let $\pi : V \longrightarrow B$ be an oriented rank n vector bundle. Then the real $2n$-plane bundle $(V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ (I.e the real structure underlying $V \otimes_{\mathbb{R}} \mathbb{C}$) is isomorphic to $V \oplus V$ and the natural orientation on $(V \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ coming from the complex structure gets sent to natural sum orientation on $V \oplus V$ if and only if $n(n-1)/2$ is even.

Proof. Let J be the natural complex structure on $V \oplus_{\mathbb{R}} \mathbb{C}$. If $v_1, \cdots, v_n$ is an oriented basis for a fiber of V then $v_1, Jv_1, v_2, Jv_2, \cdots, v_n, Jv_n$ is an oriented real basis for the corresponding fiber of $(E \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}}$ and

$$v_1 \oplus 0, v_2 \oplus 0, \cdots, v_n \oplus 0, 0 \oplus v_1, \cdots, 0 \oplus v_n$$

is an oriented basis for the corresponding fiber of $V \oplus V$. The orientations of these bases agree if and only if $n(n-1)/2$ is even. □

We have the following immediate corollary:

Corollary 1.13. If V is an oriented rank $2n$ vector bundle. Then $p_n(V)$ is equal to the square of the Euler class $e(V)$.

Let $\widetilde{Gr}_n(\mathbb{R}^\infty)$ be the oriented Grassmannian. I.e. the space parameterizing oriented n-planes inside $\mathbb{R}^\infty$. Let $\widetilde{\gamma}_\infty^n$ be the corresponding canonical oriented bundle over this Grassmannian. We have the following theorem which we won't prove (see Theorem 15.9 in Milnor and Stasheff's Characteristic classes book.)

Theorem 1.14. If Λ is an integral domain containing $\frac{1}{2}$, then $H^*(\widetilde{Gr}_{2k+1}(\mathbb{R}^\infty))$ over Λ is generated by the Pontryagin classes

$$p_1(\widetilde{\gamma_\infty^{2k+1}}), \cdots p_k(\widetilde{\gamma_\infty^{2k+1}})$$

and $H^*(\widetilde{Gr}_{2k}(\mathbb{R}^\infty))$ is generated by

$$p_1(\widetilde{\gamma_\infty^{2k}}), \cdots p_{k-1}(\widetilde{\gamma_\infty^{2k}}), e(\widetilde{\gamma_\infty^{2m}}).$$